A lower bound for the minimal counter-example to Frankl’s conjecture

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Abstract

Frankl’s conjecture, from 1979, states that any finite union-closed family, containing at least one non-empty member set, must have an element which belongs to at least half of the member-sets. In this paper we show that if the minimum cardinality of \( \bigcup A \) over all counter-examples is \( q \), then any counter-example family must contain at least \( 4q + 1 \) sets. As a consequence, we show that a minimal counter-example must contain at least 53 sets.

1 Introduction

A family of sets \( A \) is said to be union-closed if the union of any two member sets is also a member of \( A \). Frankl’s conjecture (or the union-closed sets conjecture) states that if \( A \) is finite, then some element must belong to at least half of the sets in \( A \), provided at least one member set is non-empty. Although the origin of this conjecture is not explicit, it is generally attributed to Frankl (1979) following [5]. A detailed discussion and current standing of the conjecture can be found in [1].

In [3], Roberts and Simpson showed that if \( q \) is the minimum cardinality of \( \bigcup A \) over all counter-examples, then any counter-example \( A \) must satisfy the inequality \( |A| \geq 4q - 1 \). In this paper, we expand the ideas presented in [3] to find an improved lower bound \( 4q + 1 \). In [4], it was proved that a minimal counter-example must contain at least 13 elements in \( \bigcup A \). Hence we show that the minimal counter-example family must contain at least 53 sets.
2 Main results

2.1 Preliminary lemmas

Throughout this paper, \( \mathcal{A} \) will denote a minimal counter-example with \(|\bigcup \mathcal{A}| = q\), the minimum number of constituent elements across all counter-examples. Here \(|\mathcal{A}|\) must be odd, because if it is even we can remove a basis set (a set that cannot be obtained by the union of any two other sets of \( \mathcal{A} \)) to generate a counter-example with \(|\mathcal{A}| - 1\). Let \(|\mathcal{A}| = 2n + 1\).

We denote the family of sets in \( \mathcal{A} \) containing an element \( x \) by \( \mathcal{A}_x \).

The universal set for \( \mathcal{A} \) is defined by \( S := \bigcup \mathcal{A} \). Thus \(|S| = q\).

We define \( \mathcal{A}_x := \{ A \in \mathcal{A} : x \not\in A \} \). Let \( C_x := \bigcup \mathcal{A}_x \). We denote the family containing all such \( C_x \) by \( \mathcal{C} \):

\[
\mathcal{C} := \{ C_x : x \in S \}.
\]

For any \( x \) we define the family \( \mathcal{D}_x \) to be

\[
\mathcal{D}_x := \mathcal{A}_x \setminus \{ S \} \setminus \mathcal{C}.
\]

We now define and note the difference between the terms abundant and abundance. We call an element \( x \) abundant in a family \( \mathcal{F} \) if \( 2|\mathcal{F}_x| \geq |\mathcal{F}| \). (By definition, our counterexample \( \mathcal{A} \) cannot contain any abundant element.) On the other hand, we define abundance of \( x \) in \( \mathcal{F} \) simply as \(|\mathcal{F}_x|\).

Next, we define and distinguish the terms mutually dominant and dominant. We say that two elements \( a \) and \( b \) are mutually dominating if \( a \) and \( b \) always appear together in the member sets of \( \mathcal{A} \). We say \( a \) dominates \( b \) if \( \mathcal{A}_b \subset \mathcal{A}_a \) and \(|\mathcal{A}_a| > |\mathcal{A}_b|\). Our counter-example family \( \mathcal{A} \) cannot contain any mutually dominating pair of elements, since they can be replaced by a single element which in turn would violate the minimality of \( q \). Therefore, for any \( a, b \in S \), if \( a \neq b \), then \( C_a \neq C_b \). However, \( \mathcal{A} \) may contain elements which dominate other elements.

**Definition 1.** We define the sets \( I \) and \( J \) by:

\[
I := \{ a \in S : a \text{ is NOT dominated by any other element in } S \};
\]

\[
J := \{ b \in S : b \text{ is dominated by some other element in } S \}.
\]

If an element is present in \( n \) sets of \( \mathcal{A} \), then it cannot be dominated by any other element. Hence they must be present in \( I \). We know from [2] that \( \mathcal{A} \) must contain at least three elements with abundance \( n \). Thus \(|I| \geq 3\). Note that every non-empty set in \( \mathcal{A} \) must contain at least one element from \( I \).

We now prove slightly modified versions of two lemmas from [3].

**Lemma 1.** Let \( a \) be an element of \( S \). If \( a \notin I \) then \( I \subseteq C_a \), and if \( a \in I \) then \( I \setminus \{ a \} \subseteq C_a \).
Proof. When \( a \notin I \), let \( y \in I \). Since \( a \) cannot dominate \( y \), there must exist a set containing \( y \) but not \( a \). So \( y \in C_a \).

When \( a \in I \), let \( z \in I \) and \( z \neq a \). Since \( a \) cannot dominate \( z \), there must exist a set containing \( z \) but not \( a \). So \( z \in C_a \). But \( a \notin C_a \) because \( \bigcup A_\pi \) cannot contain \( a \).

So we conclude that if \( a \in I \), then it must be present in \( q-1 \) sets of \( C \).

**Lemma 2.** For any \( a \), \( C_a \) cannot be a basis set of \( A \).

*Proof.* Let \( C_a \) be a basis. So we can remove \( C_a \) to get a new union-closed \( A' \) with \(|A'| = |A| - 1|.

If \( a \notin I \), then \( I \subseteq C_a \) (Lemma 1). Since \( I \) must contain all elements with abundance \( n \), removing \( C_a \) would generate another counter-example \( A' \) with \(|A'| < |A|\), which is a contradiction.

If \( a \in I \), then \( I \setminus \{a\} \subseteq C_a \) (Lemma 1). Let \( B_a \) be a basis set containing \( a \). Removing \( B_a \) and \( C_a \) from \( A \) we get \( A'' \) with \(|A''| = |A| - 2 = 2n - 1|\), and no element is contained in more than \( n-1 \) sets. Hence \( A'' \) is also a counter-example, which is again a contradiction.

**Definition 2.** We say that elements \( a \) and \( b \) are mutually abundant if \( 2|A_a \cap A_b| \geq |A_a| \) and \( 2|A_a \cap A_b| \geq |A_b| \).

**Definition 3.** For every element \( a \), we define the sets \( H_a \) and \( L_a \) as follows:

\[
H_a := \{ b \in S : b \text{ is abundant in } A_a \};
\]

\[
L_a := \{ c \in S : c \text{ is abundant in } A_a \}.
\]

We now prove a few lemmas which will be used repeatedly in the next section.

**Lemma 3.** If \( a, b \in I \), \( b \in H_a \) and \( D_a \cap D_b \neq \emptyset \), then \(|A| \geq 4q + 3\).

*Proof.* Since \( b \in H_a \), it must be present in at least \((n+1)/2\) sets of \( A_\pi \). Also \( b \in S \) and \( b \) must be in \( q-2 \) sets of \( C \setminus \{C_a\} \). It must also be present in at least one set of \( D_a \), since \( D_a \cap D_b \neq \emptyset \). So we have

\[
\frac{(n+1)}{2} + 1 + (q-2) + 1 \leq n,
\]

which yields \(|A| \geq 4q + 3\).

**Lemma 4.** If \(|A_x| = |A_y| = n, x \neq y\), then \( y \in H_x \) or \( y \in L_x \), but \( y \notin H_x \cap L_x \).

*Proof.* Suppose \( y \notin H_x \) and \( y \notin L_x \). Let us assume that \( n \) is even (say \( n = 2k \)). Since \( y \notin L_x \), we have \(|A_x \cap A_y| \leq k - 1\). Since \( y \notin H_x \), we have \(|A_x \cap A_y| \leq k\). So \(|A_y| \leq k - 1 + k = n - 1\), a contradiction.
On the other hand, if \( n \) is odd (say \( n = 2k + 1 \)), since \( y \notin L_x \), we have \( |A_x \cap A_y| \leq k \). Since \( y \notin H_x \), we have \( |A_x \cap A_y| \leq k \). So \( |A_y| \leq k + k = n - 1 \), a contradiction again.

The case \( y \in H_x \cap L_x \) is not possible because it will render \( y \) abundant in \( A \).

Lemma 5. If \( |A_x| = |A_y| = n \) and \( y \in H_x \), then \( x \in H_y \).

Proof. Since \( y \in H_x \), we have \( y \notin L_x \) from Lemma 4. So \( x \) and \( y \) cannot be mutually abundant (because \( |A_x| = |A_y| = n \)). Hence \( x \notin L_y \). Thus, from Lemma 4, we have \( x \in H_y \).

Definition 4. For any \( x, y \in S \), we define
\[
A_{xy} := A_x \cap A_y, \quad E_{xy} := \cup A_{xy}.
\]

Note that \( A_{xy} \) is union-closed.

Lemma 6. If \( x, y \in I \), then \( E_{xy} \notin C \).

Proof. From Lemma 1, any \( C_a \in C \) must contain either \( I \) or \( I \setminus \{a\} \). But \( E_{xy} \) can contain at most \( I \setminus \{x\} \setminus \{y\} \). Hence \( E_{xy} \notin C \).

As a corollary to the above lemma, note that \( A_{xy} \) cannot contain any set from \( C \) when \( x, y \in I \). Also \( S \notin A_{xy} \), since \( S \) must contain both \( x \) and \( y \).

Now we prove our central result, \( |A| \geq 4q + 1 \). To do so, we divide the proof into the following two cases.

2.2 The case when \( C_x \neq S \setminus \{x\} \) for some \( x \)

Theorem 1. If there exists \( x \in I \) such that \( |A_x| < n \), then \( |A| \geq 4q + 1 \).

Proof. We have \( |A_x| \geq n + 2 \). There must exist \( y \in I \) abundant in \( A_x \) (for if \( y \) is dominated by some \( z \), then \( z \) would also be abundant in \( A_x \) and we would then choose \( z \) instead of \( y \)). Hence \( y \) must be in at least \( (n+2)/2 \) sets of \( A_x \). Since \( y \in I \), \( y \) must be in \( q - 2 \) sets of \( C \setminus \{C_x\} \). Also \( y \in S \). So we have
\[
\frac{n + 2}{2} + (q - 2) + 1 \leq n
\]
which yields \( |A| \geq 4q + 1 \).

Theorem 2. If \( |A_x| = n \) for all \( x \in I \), then \( |A| \geq 4q + 1 \).
Proof. Let $y \in I$ and $y \in H_x$. If $\mathcal{D}_x \cap \mathcal{D}_y \neq \emptyset$, then we immediately have $|A| \geq 4q + 3$ from Lemma 3. So let $\mathcal{D}_x \cap \mathcal{D}_y = \emptyset$. Then $|A_{xy}| = q$ (since $|\{S\}| = 1$, $|C| = q$, $|\mathcal{D}_x| = |\mathcal{D}_y| = n - q$).

Since $A_{xy}$ is union closed, there must exist some $z \in I$ abundant in $A_{xy}$. We choose $z$ as the element with maximum abundance in $A_{xy}$. If $z$ is present in all $q$ sets of $A_{xy}$, then we have $|A_z| \geq 2q$ (since $z$ must be in $q$ sets of $C \cup \{S\}$). This yields $|A| \geq 4q + 1$.

So let $z$ be present in at most $q - 1$ sets of $A_{xy}$. Hence there must exist $s \in I$ present in $A_{xy} \setminus A_z$. Consequently, there exists $G_s \in A_{xy}$ such that $s \in G_s$ and $z \notin G_s$. Since $z$ is maximal in $A_{xy}$, $s$ must also be present in at most $q - 1$ sets of $A_{xy}$. So there must exist $G_z \in A_{xy}$ such that $z \in G_z$ and $s \notin G_z$. Also, since $A_{xy}$ is union-closed, there exists $G_{zs} \in A_{xy}$ such that $z \in G_{zs}$ and $s \in G_{zs}$. We summarize this as follows.

$$
\begin{align*}
    z & \in G_z \quad \text{and} \quad s \notin G_z; \\
    s & \in G_s \quad \text{and} \quad z \notin G_s; \\
    s & \in G_{zs} \quad \text{and} \quad z \in G_{zs};
\end{align*}
$$

where $G_z, G_s, G_{zs} \in A_{xy}$.

Our set-up is depicted in Figure 1.

By the hypothesis of this theorem, we have $|A_x| = |A_y| = |A_z| = n$. Therefore, applying Lemma 4, we have the following three sub-cases:
(a) $z \in H_x$: 

We consider the family $\mathcal{A}_{xy}$. There exists a basis $B_x$, where $x \in B_x$ and $s \notin B_x$, since $s$ cannot dominate $x$. Since $D_x \cap D_y = \emptyset$, $y \notin B_x$. Hence $B_x \in \mathcal{A}_{xy}$. Since $G_z \in \mathcal{A}_{xy}$, $y \notin G_z$. Also $s \notin G_z$. Therefore $G_z \in \mathcal{A}_{xy}$.

Since $B_x$ and $G_z$ are in $\mathcal{A}_{xy}$, we have $x, z \in E_{sy}$. From Lemma 6, $E_{sy} \notin \mathcal{C}$. Hence $E_{sy} \in D_x \cap D_z$. Thus, since $D_x \cap D_z \neq \emptyset$ and $z \in H_x$, we have $|A| \geq 4q + 3$ from Lemma 3.

(b) $z \in H_y$:

The proof is similar to case (a), but with the roles of $x$ and $y$ reversed.

(c) $z \in L_x$ and $z \in L_y$:

Here $z \in L_x$ implies $x \in L_z$, since $|A_x| = |A_z| = n$. Similarly, since $z \in L_y$, we have $y \in L_z$. Therefore, we have $x, y \notin H_z$ from Lemma 4. Since $x, y \notin H_z$, let $r \in I$ be an element of $H_z$.

If $r$ is present in any set of $\mathcal{A}_{xy}$, then we have a set $G_r \in \mathcal{A}_{xy}$ containing both $r$ and $z$, since $\mathcal{A}_{xy}$ is union-closed. Since $G_r \notin \mathcal{C}$, we have $G_r \in D_r \cap D_z$. Therefore we have $|A| \geq 4q + 3$ from Lemma 3, since $r \in H_z$ and $D_r \cap D_z \neq \emptyset$.

Let us assume that $r$ is not in any sets of $\mathcal{A}_{xy}$. So $D_r \subset D_x \cup D_y$. Since $r$ cannot be dominated by $s$, there must exist a basis $B_r$ such that $r \in B_r$ and $s \notin B_r$.

If $B_r \in D_x$, then $B_r \in \mathcal{A}_{xy}$ (because $y \notin B_r$, since $D_x \cap D_y = \emptyset$). Also, $G_z \in \mathcal{A}_{xy}$. So $z, r \in E_{sy} \notin \mathcal{C}$.

If $B_r \in D_y$, then $B_r \in \mathcal{A}_{xy}$ (because $x \notin B_r$, since $D_x \cap D_y = \emptyset$). Also, $G_z \in \mathcal{A}_{xy}$. So $z, r \in E_{sx} \notin \mathcal{C}$.

So at least one of $E_{sx}$ and $E_{sy}$ must be present in $D_r \cap D_z$. Therefore, we have $|A| \geq 4q + 3$ from Lemma 3, since $r \in H_z$ and $D_r \cap D_z \neq \emptyset$.

2.3 The case when $C_x = S \setminus \{x\}$ for all $x$

In this case, no element can be dominated by any other element. Thus all elements must be present in $q - 1$ sets of $\mathcal{C}$.

Theorem 3. If there exists $x$ such that $|A_x| < n$, then $|A| \geq 4q + 1$.

Proof. The proof is similar to that of Theorem 1. We have $|A_x| \geq n + 2$. Let $y \in H_x$. So $y$ must be in at least $(n + 2)/2$ sets of $\mathcal{A}_y$. It must be in $q - 2$ sets of $\mathcal{C} \setminus \{C_x\}$. Also, $y \in S$. So $(n + 2)/2 + (q - 2) + 1 \leq n$, which yields $|A| \geq 4q + 1$.

Theorem 4. If for all $x$, $|A_x| = n$, then $|A| \geq 4q + 1$.

Proof. Since $|A_x| = n$ for all $x$, no element can dominate any other element. Therefore $I = S$. Since, in the proof of Theorem 2, we did not consider any element from $J$, this just becomes a special case of Theorem 2.
Corollary 1. The minimal counter-example to Frankl’s conjecture must contain at least 53 sets.

Proof. Combining Theorems 1, 2, 3 and 4, we obtain $|A| \geq 4q + 1$. Since it is shown in [4] that $q \geq 13$, we have $|A| \geq 53$.

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References


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