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Set-valued tableaux and generalized Catalan numbers

Paul Drube

Department of Mathematics and Statistics
Valparaiso University
Valparaiso, Indiana, U.S.A.
paul.drube@valpo.edu

Abstract

Standard set-valued Young tableaux are a generalization of standard Young tableaux in which cells contain (possibly empty) sets of positive integers, with the added conditions that every integer at position (i, j) must be smaller than every integer at positions $(i, j + \delta)$ and $(i + \delta, j)$ for all $\delta \geq 1$. This paper explores the combinatorics of standard set-valued Young tableaux with two-rows, and how those tableaux may be used to provide new combinatorial interpretations of generalized Catalan numbers. The paper begins by drawing a bijection between arbitrary classes of two-row standard set-valued Young tableaux and collections of two-dimensional lattice paths that lie weakly below a unique maximal path. That bijection is then used to derive new combinatorial interpretations for the two-parameter Fuss-Catalan numbers (Raney numbers), the rational Catalan numbers, and the solution to the so-called "generalized tennis ball problem". The paper closes by introducing a general methodology for the enumeration of standard set-valued Young tableaux, prompting explicit formulas for the general two-row case.

1 Introduction

For a non-increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, a Young diagram Y of shape λ is a left-justified array of cells with exactly λ_i cells in its i^{th} row. If Y is a Young diagram of shape λ with $\sum_i \lambda_i = n$, a Young tableau of shape λ is an assignment of the integers $[n] = \{1, \ldots, n\}$ to the cells of Y such that every integer is used precisely once. A Young tableau in which integers increase from top to bottom down every column and increase from left to right across every row is said to be a standard Young tableau. For a comprehensive introduction to Young tableaux, see Fulton [4].

Let Y be a Young diagram of shape λ, and let $\rho = \{\rho_{i,j}\}$ be a collection of non-negative integers such that $\sum_{i,j} \rho_{i,j} = m$. A **set-valued tableau** of shape λ

and density ρ is a function from $[m]$ to the cells of Y such that the cell at position (i, j) receives a set of $\rho_{i,j}$ integers. A set-valued tableau is said to be a **standard set-valued Young tableau** if we additionally require that every integer at position (i, j) is smaller than every integer at positions $(i + \delta, j)$ and $(i, j + \delta)$ for all $\delta \geq 1$. In analogy with standard Young tableaux, we refer to these added conditions as "column-standardness" and "row-standardness". We denote the set of all standard set-valued Young tableaux of shape λ and density ρ by $\mathbb{S}(\lambda, \rho)$. See Figure 1 for a basic example.

1 2	3 4		1 2	3 5		1 2	3 6		1 2	4 5		1 2	4 6		1 2	5 6		1 3	2 5	$\notin \mathbb{S}(\lambda, \rho)$
5 6	7 8		4 6	7 8		4 5	7 8		3 6	7 8		3 5	7 8		3 4	7 8		4 6	7 8	

Figure 1: The six elements of $\mathbb{S}(\lambda, \rho)$ when $\lambda = (2, 2)$ and $\rho_{i,j} = 2$ for all i, j. On the right is a set-valued tableau of the same shape and density that isn't row-standard.

Set-valued tableaux were introduced by Buch [2] in his investigation of the K-theory of Grassmannians. More directly influencing this paper is the work of Heubach, Li and Mansour [8], who argued that the cardinality of $\mathbb{S}(n^2, \rho)$ with row-constant density $\rho_{1,j} = k - 1$, $\rho_{2,j} = 1$ equalled the k-Catalan number C_n^k. For a more recent appearance of set-valued tableaux see Reiner, Tenner and Yong [11], who investigated so-called "barely set-valued tableaux" with a single non-unitary density $\rho_{i,j} = 2$ (not necessarily located at a fixed position i, j).

Currently, the central difficulty in studying standard set-valued Young tableaux is the lack of a closed formula for enumerating general $\mathbb{S}(\lambda, \rho)$: there is no known set-valued analogue of the celebrated hook-length formula for standard Young tableaux. Reiner, Tenner and Yong [11] utilize a modified insertion algorithm to enumerate "barely set-valued tableaux", but theirs is an atypically tractable case and cannot be modified to the enumeration of sets $\mathbb{S}(\lambda, \rho)$ with a fixed density at each position.

The goal of this paper is to present a thorough exploration of standard set-valued Young tableaux of shape $\lambda = n^2$. In Section 2, we draw a bijection between two-row standard set-valued tableaux of arbitrary density and certain classes of two-dimensional integer lattice paths with "east" $E = (1, 0)$ and "north" $N = (0, 1)$ steps. In particular, $\mathbb{S}(n^2, \rho)$ with $\rho_{1,j} = a_j$ and $\rho_{2,j} = b_j$ is placed in bijection with all such lattice paths that lie weakly below the lattice path $P = E^{a_1} N^{b_1} E^{a_2} N^{b_2} \ldots$ (Theorem 2.2). In Section 3, we utilize standard set-valued Young tableaux to provide new combinatorial interpretations for various generalizations of the Catalan numbers. In particular, we show how standard set-valued Young tableaux of shape $\lambda = n^2$ and various densities are enumerated by the rational Catalan numbers (Theorem 3.2), the Raney numbers (two-parameter Fuss-Catalan numbers, Theorem 3.5), and the solution to the "(s, t)-tennis ball problem" of Merlini, Sprugnoli, and Verri [10] (Theorem 3.6). See Figure 2 for an overview of the various densities needed to achieve our combinatorial interpretations. In that figure, and in all that follows, we use a Young diagram with parenthesized entries to specify a density ρ with the exhibited cell densities $\rho_{i,j}$. In Section 4, we introduce a technique for enumerating two-row standard set-valued Young tableaux that we refer to as density shifting. Closed formulas are presented for the number of such tableaux of any density (Theorem 4.2).

k-Catalan numbers $C_n^k = \frac{1}{kn+1}\binom{kn+1}{n}$ for $\lambda = n^2$

(1)	...	(1)
$(k-1)$	...	$(k-1)$

Raney numbers $R_{k,r}(n) = \frac{r}{kn+r}\binom{kn+r}{n}$ for $\lambda = (n+1)^2$

(1)	...	(1)	(1)
$(r-1)$	...	$(k-1)$	$(k-1)$

Rational Catalan numbers $C(a,b) = \frac{1}{a+b}\binom{a+b}{a}$ for $\lambda = a^2$

(1)	...	(1)	(1)
$(\lfloor \frac{ab}{a} \rfloor - \lfloor \frac{(a-1)b}{a} \rfloor)$	...	$(\lfloor \frac{2b}{a} \rfloor - \lfloor \frac{b}{a} \rfloor)$	$(\lfloor \frac{b}{a} \rfloor - \lfloor 0 \rfloor)$

Solution to (s,t)-tennis ball problem for $\lambda = (n+1)^2$ (after n turns)

$(s-t)$	...	$(s-t)$
(t)	...	(t)

Figure 2: Densities ρ for which $|\,\mathbb{S}(\lambda,\rho)|$ yields various combinatorial interpretations.

We pause to introduce a foundational result that, in the case of rectangular λ, serves as a set-valued analogue of the Schützenberger involution for standard Young tableaux. In the case of densities ρ that are constant across each row, notice that Proposition 1.1 manifests as invariance under a vertical reflection of those densities.

Proposition 1.1. *For rectangular $\lambda = n^m$ and any density $\rho = \{\rho_{i,j}\}$, let $r(\rho) = \{\rho_{m-i+1,n-j+1}\}$. Then $|\,\mathbb{S}(\lambda,\rho)| = |\,\mathbb{S}(\lambda, r(\rho))|$.*

Proof. One may define a bijection $f : \mathbb{S}(\lambda,\rho) \to \mathbb{S}(\lambda, r(\rho))$ such that $f(T) \in \mathbb{S}(\lambda, r(\rho))$ is obtained by reversing the alphabet of $T \in \mathbb{S}(\lambda,\rho)$ and rotating the resulting tableau by 180-degrees. □

2 Set-Valued Tableaux and Two-Dimensional Lattice Paths

As it will provide a framework for many of the combinatorial interpretations from Section 3, we begin by drawing a general bijection between $\mathbb{S}(\lambda,\rho)$ with $\lambda = n^2$ and various classes of two-dimensional lattice paths. This requires a consideration of all integer lattice paths from $(0,0)$ to (a,b) that use only east $E = (1,0)$ and north $N = (0,1)$ steps, which we refer to as N-E lattice paths of shape (a,b). For reasons that will become evident in the proof of Theorem 2.2, we eventually restrict our attention to ρ for which there does not exist $1 \leq j \leq n-1$ with $\rho_{2,j} = \rho_{1,j+1} = 0$. If ρ avoids such a pair of zero density cells, we refer to ρ as a "reduced density".

So fix $\lambda = n^2$, and consider the density ρ where $\rho_{1,j} = a_j$ and $\rho_{2,j} = b_j$. For the remainder of this section let $\sum_j a_j = a$ and $\sum_j b_j = b$. We may then define a map $\psi_\rho : \mathbb{S}(\lambda,\rho) \to \mathcal{P}$, where $\mathcal{P}$ is the set of N-E lattice paths of shape (a,b), by associating entries in the top row of $T \in \mathbb{S}(\lambda,\rho)$ to east steps in $\psi_\rho(T)$ and associating entries in the bottom row of T to north steps in $\psi_\rho(T)$. The map ψ_ρ is always an injection, but its image is dependent upon the choice of ρ. See Figure 3 for a quick example of ψ_ρ.

To characterize $\mathrm{im}(\psi_\rho)$, we introduce a partial order on $\mathcal{P}$. For $P_1, P_2 \in \mathcal{P}$, define $P_1 \geq P_2$ if P_1 lies weakly above P_2 across $0 \leq x \leq a$. Notice that this poset is

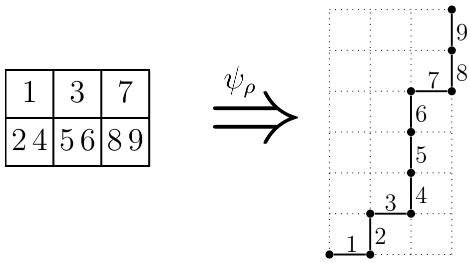

Figure 3: A standard set-valued Young tableau $T \in \mathbb{S}(\lambda, \rho)$ with $\lambda = n^2$ and $\rho_{1,j} = 1, \rho_{2,j} = 2$ for all j, alongside the corresponding N-E lattice path $\psi_\rho(T)$ of shape $(3, 6)$.

isomorphic to a finite order ideal of Young's lattice via the map that takes a path to the Young diagram lying above its conjugate. Our map ψ_ρ respects this partial order in the following sense.

Lemma 2.1. *For fixed $\lambda = n^2$ and ρ, take $P_1, P_2 \in \mathcal{P}$ such that $P_1 \geq P_2$. If $P_1 \in \operatorname{im}(\psi_\rho)$, then $P_2 \in \operatorname{im}(\psi_\rho)$.*

Proof. We prove the statement for when P_1 directly covers P_2. This corresponds to the situation where P_2 may be obtained from P_1 by replacing a single NE subsequence with an EN subsequence at the same position. Assume that this $NE \mapsto EN$ replacement occurs at the i and $i+1$ steps of both P_1 and P_2. By assumption there exists $T_1 \in \mathbb{S}(\lambda, \rho)$ with $\psi_\rho(T_1) = P_1$. It must be the case that the integer i appears in the second row of T_1 and $i+1$ appears in the first row of T_1. Since T_1 is column-standard, this also implies that i and $i+1$ cannot appear in the same column of T_1. Then define T_2 to be the tableau obtained by flipping the positions of i and $i+1$ in T_1. As i and $i+1$ are consecutive integers, and since i and $i+1$ cannot appear in the same column of T_1, this new tableau T_2 is both row- and column-standard. Thus $T_2 \in \mathbb{S}(\lambda, \rho)$ and $\psi_\rho(T_2) = P_2$. $\square$

For any λ and ρ, there exists a tableau $T_{max} \in \mathbb{S}(\lambda, \rho)$ that is referred to as the column superstandard tableau of that shape and density. This is the unique standard set-valued Young tableau such that, for all j, every integer in the j^{th} column of T_{max} is smaller than every integer in the $(j+1)^{st}$ column of T_{max}. In terms of our map ψ_ρ, this is precisely the tableau such that $\psi_\rho(T_{max}) = E^{a_1}N^{b_1}\ldots E^{a_n}N^{b_n}$. The tableau T_{max} is important because, in the case of reduced densities ρ, the order ideal generated by $\psi_\rho(T_{max})$ precisely corresponds to $\operatorname{im}(\psi_\rho)$:

Theorem 2.2. *Fix $\lambda = n^2$, and take any reduced density ρ with $\rho_{1,j} = a_j$ and $\rho_{2,j} = b_j$ for all j. For $P_{max} \in \mathcal{P}$ defined by $P_{max} = E^{a_1}N^{b_1}\ldots E^{a_n}N^{b_n}$, the set $\mathbb{S}(\lambda, \rho)$ is in bijection with $I = \{P \in \mathcal{P} \mid P \leq P_{max}\}$.*

Proof. As $P_{max} = \psi_\rho(T_{max})$ and $P_{max} \in \operatorname{im}(\psi_\rho)$, Lemma 2.1 gives $I \subseteq \operatorname{im}(\psi_\rho)$. Since ψ_ρ is known to be injective, it is only left to show that $\operatorname{im}(\psi_\rho) \subseteq I$.

Assume by contradiction there exists $T \in \mathbb{S}(\lambda, \rho)$ with $\psi_\rho(T) \not\leq P_{max}$. Then there is a smallest index i such that the i^{th} steps of both $\psi_\rho(T)$ and P_{max} begin at the same point, the i^{th} step of $\psi_\rho(T)$ is a N step, and the i^{th} step of P_{max} is an E step. This means that the subtableaux of T and T_{max} consisting only of $\{1,\ldots,i-1\}$ have the same shape and density, but the integer i lies in the first row of T_{max} and in the second row of T. So assume that i lies at position $(1,j)$ of T_{max}. The construction of T_{max} implies that every integer in the $(j-1)^{st}$ columns of both T and T_{max} is smaller than i. In T, this means that i must lie in the leftmost cell $(2,j')$ satisfying both $j' \geq j$ and $\rho_{2,j'} \neq 0$. If $\rho_{2,j} \neq 0$, then i lies in the $(2,j)$ cell of T and there must exist an entry at position $(1,j)$ of T that is larger than i. If $\rho_{2,j} = 0$, since ρ is a reduced density it must be the case that i lies at a cell $(2,j')$ such that $\rho_{1,j'} \neq 0$. In this case, there once again must be an entry at $(1,j')$ of T that is larger than i. We may then conclude that T is not column-standard. □

Example 2.3. *For any pair of integers $a,b>0$, there exists a unique greatest element $Q = N^bE^a$ and a unique least element $Q' = E^aN^b$ in the poset of N-E lattice paths of shape (a,b). Using the notation of Theorem 2.2, Q and Q' serve as P_{max} for the choice of reduced densities ρ and ρ' shown below.*

ρ :

(0)	(a)
(b)	(0)

ρ' :

(a)
(b)

By Theorem 2.2, $\mathbb{S}(2^2,\rho)$ is in bijection with all N-E lattice paths of shape (a,b), whereas $\mathbb{S}(1^2,\rho')$ is in bijection with the singlet set $\{Q'\}$. This agrees with the basic enumerations $|\mathbb{S}(2^2,\rho)| = \binom{a+b}{a}$ and $|\mathbb{S}(1^2,\rho')| = 1$.

Since there exist reduced densities with cells satisfying $\rho_{i,j} = 0$, it is possible for distinct sets $\mathbb{S}(\lambda,\rho)$, $\mathbb{S}(\lambda,\rho')$ of the same shape to be associated with the same maximal path $P_{max} \in \mathcal{P}$. Thus, it is possible for distinct sets $\mathbb{S}(\lambda,\rho)$, $\mathbb{S}(\lambda,\rho')$ to lie in bijection with the same set of N-E lattice paths. See Figure 4 for an example.

Also observe that, if ρ is not a reduced density, it may be the case that $I \subsetneq \mathrm{im}(\psi_\rho)$. This only requires the existence of an index j where $\rho_{2,j} = \rho_{1,j+1} = 0$, $\rho_{1,j} \neq 0$, and $\rho_{2,j+1} \neq 0$. For such a non-reduced density, begin with T_{max} and permute a nonzero number of entries between the cells at $(1,j)$ and $(2,j+1)$. The resulting tableaux T still lies in $\mathbb{S}(\lambda,\rho)$ but satisfies $\psi_\rho(T) \not\leq P_{max}$. Luckily, non-reduced densities will not appear in the combinatorial interpretations of Section 3, allowing us to forego a full characterization of $\mathrm{im}(\psi_\rho)$ in this case.

ρ_1 :

(1)	(0)	(1)
(1)	(2)	(1)

ρ_2 :

(1)	(0)	(1)
(2)	(1)	(1)

Figure 4: A pair of distinct densities ρ_1, ρ_2 for shape $\lambda = 3^2$ such that $\mathbb{S}(\lambda,\rho_1)$ and $\mathbb{S}(\lambda,\rho_2)$ are both in bijection with N-E lattice paths of shape $(2,4)$ lying below $P_{max} = E^1N^3E^1N^1$.

3 Generalized Catalan Numbers and Set-Valued Tableaux

Before proceeding to our new combinatorial interpretations, we briefly summarize known results about the k-Catalan numbers. For any $k \geq 1$, the k-Catalan numbers are given by $C_n^k = \frac{1}{kn+1}\binom{kn+1}{n}$ for all $n \geq 0$. Notice that the k-Catalan numbers specialize to the usual Catalan numbers when $k = 2$.

See Hilton and Pedersen [7] or Heubach, Li and Mansour [8] for various combinatorial interpretations of the k-Catalan numbers. Relevant to our work is the standard result that C_n^k enumerates the set $\mathcal{D}_n^k$ of k-good paths of length kn: the subset of N-E lattice paths of shape $(n, (k-1)n)$ that stay weakly below the line $y = (k-1)x$. This set of k-good paths is obviously in bijection with k-ary paths of length kn: integer lattice paths from $(0, 0)$ to $(nk, 0)$ that use steps $u = (1, \frac{1}{k-1})$, $d = (1, -1)$ and stay weakly above $y = 0$. The results of Section 2 justify our preference for k-good paths over k-ary paths.

Heubach, Li and Mansour [8] showed that standard set-valued Young tableaux of shape $\lambda = n^2$ and row-constant density $\rho_{1,j} = k - 1$, $\rho_{2,j} = 1$ are counted by C_n^k. This was done by placing such tableaux in bijection with k-ary paths of length kn. An equivalent result quickly follows from Theorem 2.2, yielding the combinatorial interpretation for the k-Catalan numbers shown in the top-left of Figure 2.

Proposition 3.1. *Take any $k \geq 1, n \geq 0$. Then $C_n^k = |\,\mathbb{S}(n^2, \rho)|$ for the density ρ with $\rho_{1,j} = 1$ and $\rho_{2,j} = k - 1$ for all $1 \leq j \leq n$.*

Proof. The set $\mathcal{D}_n^k$ of k-good paths of length kn are precisely those N-E lattice paths of shape $(n, (k-1)n)$ that lie weakly below $P_{max} = (EN^{k-1})^n$. The result then follows from the bijection of Theorem 2.2. □

3.1 Rational Catalan numbers

The first generalization of the Catalan numbers for which we provide a new combinatorial interpretation are the rational Catalan numbers. For relatively prime positive integers a and b, there exists a rational Catalan number $C(a, b) = \frac{1}{a+b}\binom{a+b}{a}$. As originally shown by Bizley [1] and extended by Grossman [6], the rational Catalan number $C(a, b)$ equals the number of rational Dyck paths of shape (a, b). By rational Dyck paths of shape (a, b) we mean N-E lattice paths of shape (a, b) that lie weakly below the line of rational slope $y = \frac{b}{a}x$. Observe that, for any $k \geq 1$ and $n \geq 1$, the integers n and $(k-1)n + 1$ are relatively prime and the rational Catalan numbers specialize to the k-Catalan numbers as $C(n, (k-1)n + 1) = C_n^k$.

Our combinatorial interpretation, shown in the bottom-left of Figure 2, again follows from Theorem 2.2:

Theorem 3.2. *Take positive integers a, b such that $\gcd(a, b) = 1$. Then $C(a, b) = |\,\mathbb{S}(a^2, \rho)|$ for the density ρ with $\rho_{1,j} = 1$ for all $1 \leq j \leq a$ and $\rho_{2,j} = \lfloor \frac{bj}{a} \rfloor - \lfloor \frac{b(j-1)}{a} \rfloor$.*

Proof. Consider the lattice path $P_{(a,b)} = E^1 N^{c_1} \ldots E^1 N^{c_a}$ of shape (a, b), where $c_i = \lfloor \frac{bi}{a} \rfloor - \lfloor \frac{b(i-1)}{a} \rfloor$. As $\sum_{i=1}^{k} c_i = \lfloor bk \rfloor a$ for all $1 \leq k \leq a$, the path $P_{(a,b)}$ has a northwest corner at the first integer lattice point below the intersection of $y = \frac{b}{a}x$ with $x = k$, for every $1 \leq k \leq a$. This implies that the set of rational Dyck paths of shape (a, b) are in bijection with N-E lattice paths of shape (a, b) that lie weakly below $P_{(a,b)}$. By Theorem 2.2, such lattice paths are in bijection with $\mathbb{S}(a^2, \rho)$ for the given density. □

For $(a, b) = (n, (k-1)n+1)$, it may be shown that Theorem 3.2 gives $\rho_{2,j} = k-1$ for all $1 \leq j \leq n-1$. As rational Dyck paths of shape $(n, (k-1)n+1)$ are enumerated by the k-Catalan number C_n^k, this implies that Theorem 3.2 may be viewed as a direct generalization of Proposition 3.1.

Example 3.3. *By Theorem 3.2,* $|\mathbb{S}(7^2, \rho)| = C(7, 9) = 715$ *for the density* ρ *below*

ρ :

(1)	(1)	(1)	(1)	(1)	(1)	(1)
(1)	(1)	(1)	(2)	(1)	(1)	(2)

3.2 Raney numbers

The next generalization of the Catalan numbers for which we will present a new combinatorial interpretation are the Raney numbers, also known as the two-parameter Fuss-Catalan numbers. Unlike the case with both the k-Catalan numbers and rational Catalan numbers, this combinatorial interpretation will require a more involved approach than a straightforward citation of Theorem 2.2.

For any $k \geq 1$ and $r \geq 1$, the Raney numbers are given by $R_{k,r}(n) = \frac{r}{kn+r}\binom{kn+r}{n}$ for all $n \geq 0$.[1] The Raney numbers specialize to the k-Catalan numbers both as $R_{k,1}(n) = C_n^k$ and as $R_{k,k}(n-1) = C_n^k$. For numerous identities involving the Raney numbers, see Hilton and Pederson [7]. The most pertinent result from Hilton and Pederson [7] is the fact that the Raney numbers are calculable from the k-Catalan numbers as

$$R_{k,r}(n) = \sum_{(i_1,\ldots,i_r)\vdash n} C_{i_1}^k C_{i_2}^k \ldots C_{i_r}^k \tag{1}$$

Here $(i_1, \ldots, i_r) \vdash n$ denotes all weak compositions of n, meaning that one or more of the i_j may be zero. The summation of (1) is useful in that it allows one to construct combinatorial interpretations for the Raney numbers as ordered r-tuples of pre-existing interpretations for the k-Catalan numbers:

Proposition 3.4. *Fix* $k, r \geq 1$, $n \geq 0$. *Then* $R_{k,r}(n)$ *equals the number of ordered* r*-tuples* $(T_1, \ldots, T_r)$ *with* $T_j \in \mathbb{S}(i_j^2, \rho)$, *where* ρ *is the row-constant density* $\rho_{1,j} = 1$, $\rho_{2,j} = k-1$ *and* $i_1 + \cdots + i_r = n$.

Our goal is to replace the ordered r-tuples of Proposition 3.4 with a single set-valued tableau of shape $\lambda = (n+1)^2$. This utilizes a technique that we refer to

[1] Hilton and Pedersen [7] use the alternative notation $d_{qk}(p) = \frac{p-q}{pk-q}\binom{pk-q}{k-1}$ for their two-parameter generalization. The two notations are related via the change of variables $R_{p,p-q}(k-1) = d_{qk}(p)$.

"horizontal tableaux concatenation", whereby the entries of the ordered r-tuple are continuously reindexed and a new column with density $\rho_{1,1} = 1, \rho_{2,1} = r - 1$ is added to the front of the resulting tableau. This additional column carries the information needed to recover the original partition of the tableau into r pieces.

So fix $n \geq 0$ and take any two-row rectangular shape $\lambda = (n + 1)^2$. To ease notation, for any $k, r \geq 1$ we temporarily define the density $\rho(k, r) = \{\rho_{i,j}\}$ by $\rho_{1,j} = 1$ for all $1 \leq j \leq n$, $\rho_{2,1} = r - 1$, and $\rho_{2,j} = k - 1$ for all $2 \leq j \leq n$. Notice that $\rho(k, r)$ is the density shown in the top-right of Figure 2.

Theorem 3.5. *Take any $k, r \geq 1$, $n \geq 0$, and define $\rho(k, r)$ as above. Then $R_{k,r}(n) = |\,\mathbb{S}((n + 1)^2, \rho(k, r))|$.*

Proof. Let S denote the set of ordered r-tuples of set-valued tableaux $(T_1, \ldots, T_r)$ with $T_j \in \mathbb{S}(i_j^2, \rho)$ and $i_1 + \cdots + i_r = n$, where ρ is the row-constant density $\rho_{1,j} = 1, \rho_{2,j} = k - 1$. We provide a pair of injective functions $\phi_1 : S \to \mathbb{S}((n + 1)^2, \rho(k, r))$ and $\phi_2 : \mathbb{S}((n + 1)^2, \rho(k, r)) \to S$. For an example illustrating both functions, see Figure 5.

For ϕ_1, take any $(T_1, \ldots, T_r) \in S$. Observe that the entries of each T_j are in bijection with $[ki_j]$, and that a total of kn entries appear across all of the T_j. Also notice that, if $i_j = 0$, then T_j is the empty tableau with zero columns. We begin by constructing an intermediate tableau D as follows

1. For every $1 \leq j \leq r$, add an empty column in front of T_j. For T_1, fill the top cell of that new column with 0. For all other T_j, fill the bottom cell of that new column with 0. We refer to the resulting tableaux of shape $(i_j + 1)^2$ as the $\widetilde{T_j}$.

2. For every $1 \leq j \leq r$, reindex the entries of $\widetilde{T_j}$ (including the 0 entry) by $x \mapsto x + j + k(i_1 + \cdots + i_{j-1})$. As each of the $\widetilde{T_j}$ contains precisely $ki_j + 1$ entries, this implies that every integer of $[kn + r]$ is used precisely once across our reindexed tableaux.

3. Concatenate the reindexed tableaux in the given order, producing a tableau D of shape $\lambda_D = (n + r)^2$.

The tableau D is row- and column-standard by construction, and contains precisely r cells of density 0. Also observe that the top row of D contains a total of $n + 1$ integers, while the bottom row of D contains a total $(k - 1)(n + 1)$ integers. The reindexing of step #2 also ensures that the map $(T_1, \ldots, T_r) \mapsto D$ is injective, as distinct choices for $(T_1, \ldots, T_r)$ will result is distinct collections of integers appearing across the top row of D.

To obtain $\phi_1(T_1, \ldots, T_r) = T$ from D, we shift all entries of D to the left until each cell contains the number of entries proscribed by $\rho(k, r)$. Any zero density columns at the right are then deleted, producing a tableau T of shape $\lambda = (n + 1)^2$ and density $\rho(k, r)$. This step $D \mapsto T$ is clearly injective, as both tableaux contain the same collection of integers across their top rows. This implies that the overall map ϕ_1 is injective. It only remains to be shown that $T \in \mathbb{S}((n + 1)^2, \rho(k, r))$.

As D is row-standard, so is T. To see that T is column-standard, notice that top row entries of D that were originally associated with T_j are shifted left by precisely $j-1$ cells as we pass from D to T. Alternatively, bottom row entries in D that were originally associated with T_j are shifted left by at least $j-1$ cells as we pass from D to T. This latter observation follows from the fact that $r-1$ integers are needed to fill the $(2,1)$ cell of T, whereas only $j-1 \leq r-1$ bottom row entries had been added to the left of T_j as we constructed D (those being the 0 entries of $\widetilde{T}_2, \ldots, \widetilde{T}_j$). As bottom row entries of D are shifted at least as far to the left as top row entries of D, column-standardness of D implies that T is also column-standard.

Now for ϕ_2, take any $T \in \mathbb{S}((n+1)^2, \rho(k,r))$. We begin by adding an empty column at the right of T and shifting all entries in the bottom row to the right until the $(2,1)$ cell is empty, the cells at $(2,2), \ldots, (2, n+1)$ each contain $k-1$ entries, and the cell at $(2, n+2)$ contains $r-1$ entries. We then work through the columns of the resulting tableau $\widetilde{T}$ from left to right as follows

1. Identify the least integer j such that the smallest entry a_j at $(2,j)$ is smaller than the sole entry at $(1,j)$. Then insert an empty column immediately before the j^{th} column.

2. Consider all bottom row entries greater than or equal to a_j, and shift those integers one entry to the left (so that the cell at $(2,j)$ receives a_j, all subsequent bottom row cells that aren't the bottom-rightmost cell receive $k-1$ entries, and the bottom-rightmost cell is left with one fewer entry than previously).

3. Repeat steps #1-#2 until the resulting tableaux D' is column-standard.

Due to how $\widetilde{T}$ was constructed from T, this procedure terminates after precisely $r-1$ new columns have been added to $\widetilde{T}$. It follows that D' has shape $\lambda_D = (n+r)^2$. Also notice that the identification of j in step #1 guarantees that the sole entry in each of the $r-1$ new columns of D' is smaller than every entry in the column immediately at their right. Since T and D' feature identical sets of integers across their top rows, the procedure $T \mapsto D'$ is injective.

To produce the ordered r-tuple $\phi_2(T)$ from D', we essentially reverse the three-step procedure from ϕ_1 that was used to obtain D. Begin by using the $r-1$ columns with an empty top row cell to divide D' into r tableaux $\widetilde{T}_1, \ldots, \widetilde{T}_r$, with the leftmost column of $\widetilde{T}_j$ corresponding to the j^{th} column of D' with an empty top row cell for each $j \geq 2$. Each $\widetilde{T}_j$ will contain $ki_j + 1$ entries for some positive integer i_j, where $i_1 + \cdots + i_r = kn + r$. Then delete the leftmost column of each $\widetilde{T}_j$ and reindex its remaining entries by $x \mapsto x - j - k(i_1 + \cdots + i_{j-1})$. This will yield an ordered tuple $\phi_2(T) = (T_1, \ldots, T_r)$ of tableaux such that $T_j \in \mathbb{S}(i_j^2, \rho)$ for density $\rho_{1,j} = 1, \rho_{2,j} = k-1$. The procedure that takes D' to $\phi_2(T)$ is injective, because distinct D' will correspond to distinct sequences of integers across the top rows of the $T_1, \ldots, T_r$. It follows that the overall map ϕ_2 is also injective, as required. □

Notice that our interpretation of $R_{k,r}(n)$ as the cardinality of $\mathbb{S}((n+1)^2, \rho(k,r))$ immediately recovers the k-Catalan specialization $R_{k,k}(n-1) = C_n^k$ of Heubach, Li

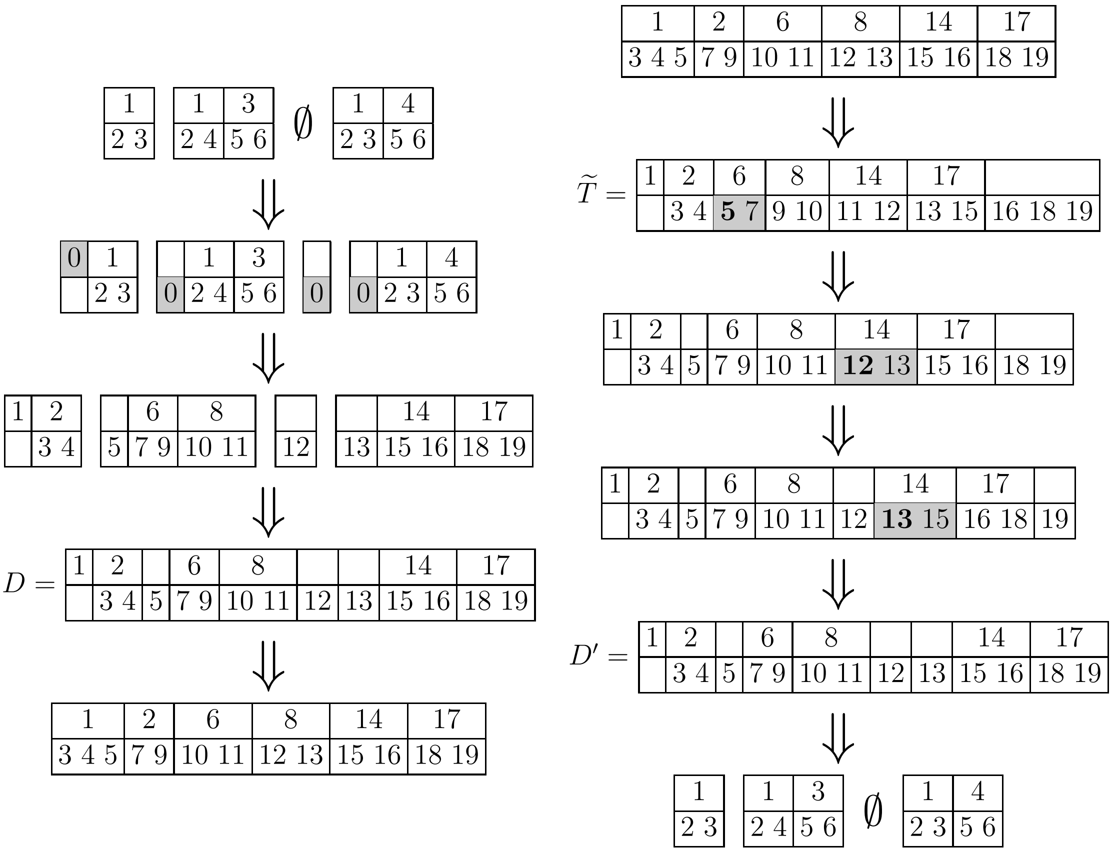

Figure 5: Transforming an r-tuple $(T_1, \ldots, T_r)$ of set-valued tableaux into a single set-valued tableau of density $\rho(k, r)$ via horizontal concatenation, alongside the inverse procedure.

and Mansour [8] when $r = k$. Also notice the special meaning of Theorem 3.5 as it applies to the extreme case of $r = 1$, as set-valued tableaux of density $\rho(k, 1)$ have a cell of density 0 at position $(2, 1)$. In this case, there is a bijection from $\mathbb{S}((n+1)^2, \rho(k, 1))$ to $\mathbb{S}(n^2, \rho(k, k))$ that deletes the first column of $T \in \mathbb{S}((n+1)^2, \rho(k, 1))$ and re-indexes the remaining nk entries of T by $x \mapsto x - 1$. This bijection directly corresponds to the Raney number identity $R_{k,1}(n) = R_{k,k}(n-1) = C_n^k$.

3.3 Solution to the (s, t)-Tennis Ball Problem

The so-called "tennis ball problem" was introduced by Tymoczko and Henle [12] and subsequently formalized by Mallows and Shapiro [9]. The classic version of the problem (essentially) asks for the number of size n subsets of $[2n]$ that contain at least j elements of $[2j]$ for every $1 \leq j \leq n$. Working independently from Mallows and Shapiro [9], Grimaldi and Moser [5] proved that the number of such subsets is the Catalan number C_{n+1}. For a full statement of the original tennis ball problem, which involves throwing tennis balls out of a kitchen window, see Mallows and Shapiro [9].

Directly generalizing these phenomena was the (s, t)-tennis ball problem of Merlini, Sprugnoli and Verri [10]. For any pair of positive integers $t < s$, the (s, t)-tennis ball problem asks for the number of size tn subsets of $[sn]$ that contain at least tj elements of $[sj]$ for every $1 \leq j \leq n$. Notice that the original tennis ball corresponds to the specific case of $s = 2$ and $t = 1$. If we let $\mathcal{B}_{s,t}(n)$ denote the number of valid subsets in the (s,t)-tennis ball problem, Merlini, Sprugnoli and Verri [10] showed that $\mathcal{B}_{k,1}(n) = C^k_{n+1}$. Generating functions for all $\mathcal{B}_{s,t}(n)$ were later developed by de Mier and Noy [3].

Most relevant to what follows is the result of de Mier and Noy [3] that gives a combinatorial interpretation for arbitrary $\mathcal{B}_{s,t}(n)$ as the number of N-E lattice paths of shape $((s-t)n, tn)$ that stay weakly below $(N^t E^{s-t})^n$. This observation directly leads to a new combinatorial interpretation of the $\mathcal{B}_{s,t}(n)$ in terms of standard set-valued Young tableaux:

Theorem 3.6. *For any positive integers s, t such that $t < s$, $\mathcal{B}_{s,t}(n) = |\,\mathbb{S}((n+1)^2, \rho)|$ for the row-constant density $\rho_{1,j} = t$, $\rho_{2,j} = s - t$.*

Proof. The result of de Mier and Noy [3] is equivalent to saying that $\mathcal{B}_{s,t}(n)$ equals the number of N-E lattice paths of shape $((s-t)(n+1), t(n+1))$ that stay weakly below $P_{max} = E^{s-t}(N^t E^{s-t})N^t = (E^{s-t}N^t)^{n+1}$. By Theorem 2.2, such lattice paths are in bijection with standard set-valued Young tableaux of shape $(n+1)^2$ and row-constant density $\rho_{1,j} = s - t$, $\rho_{2,j} = t$. ☐

If one wishes to bypass the usage of N-E lattice paths in the proof of Theorem 2.2, it is relatively straightforward to place the size tn subsets specified by the (s,t)-tennis ball problem in direct bijection with elements of $\mathbb{S}((n+1)^2, \rho)$. This bijection f places the elements of the size tn subset S across the top row of $f(S)$, filling cells $(1,2)$ through $(1, n+1)$, places all remaining elements of $[sn]$ across the bottom row of $f(S)$, filling cells $(2,1)$ through $(2,n)$, reindexes all entries by $x \mapsto x+t$, and then places $[t]$ in the cell at $(1,1)$.

Observe that, in the specific case of $t = 1$, Theorem 3.6 places $\mathcal{B}_{s,t}(n)$ in bijection with our set-valued tableaux interpretation of the k-Catalan numbers (after an application of Proposition 1.1) and hence recovers the result of Merlini, Sprungnoli, and Verri [10] that $\mathcal{B}_{s,1}(n) = C^s_{n+1}$. Further specializing Theorem 3.6 to $s = 2, t = 1$ places our result in agreement with the standard interpretation of C_{n+1} as standard Young tableaux of shape $\lambda = (n+1)^2$.

The result of Theorem 3.6 may be further generalized to the “non-constant” tennis ball problem of de Mier and Noy [3] to give an (admittedly rather contrived) combinatorial interpretation of $|\,\mathbb{S}((n+1)^2, \rho)|$ for any density ρ without zero-density cells. If $\vec{s} = \{s_i\}_{i=1}^n$ and $\vec{t} = \{t_i\}_{i=1}^n$ are sequences of positive integers such that $t_i < s_i$ for all i, the $(\vec{s}, \vec{t})$-tennis ball problem asks for the number of size $\sum_{i=1}^n t_i$ subsets of $[\sum_{i=1}^n s_i]$ that contain at least $\sum_{i=1}^j t_i$ elements of $[\sum_{i=1}^j s_i]$ for every $1 \leq j \leq n$. If we use $\mathcal{B}_{\vec{s},\vec{t}}(n)$ to denote the number of such subsets, de Mier and Noy [3] show that $\mathcal{B}_{\vec{s},\vec{t}}(n)$ equals the number of N-E lattice paths that lie weakly below $N^{t_1}E^{s_1-t_1}N^{t_2}E^{s_2-t_2}\ldots N^{t_n}E^{s_n-t_n}$. A similar argument to Theorem 3.7 then gives

Theorem 3.7. *Let $\vec{s} = \{s_i\}_{i=1}^n$ and $\vec{t} = \{t_i\}_{i=1}^n$ be sequences of positive integers such that $t_i < s_i$ for all i. Then $\mathcal{B}_{\vec{s},\vec{t}}(n) = |\,\mathbb{S}((n+1)^2, \rho)|$, where ρ is the density below and x, y are arbitrary non-negative integers..*

(x)	(t_1)	$\dots$	(t_{n-1})	(t_n)
(s_1-t_1)	(s_2-t_2)	$\dots$	(s_n-t_n)	(y)

4 Enumeration of Two-Row Set-Valued Tableaux

Although an enumeration of $\mathbb{S}(\lambda,\rho)$ for general λ and ρ is not currently tractable, the two-row case of $\lambda=(n_1,n_2)$ is sufficiently simple that methodologies may be developed for arbitrary ρ. In this section, we present a technique for such enumeration that we refer to as "density shifting". This procedure sets up a bijection between $\mathbb{S}(\lambda,\rho)$ and a collection of sets $\mathbb{S}(\lambda',\rho'_i)$, where $\lambda'=(n_1-1,n_2-1)$ and the densities ρ'_i are determined by ρ.

It should be noted that de Mier and Noy have already derived generating functions $F(z)=\sum f_q z^q$ for the number of N-E lattice paths of shape $(q\sum_{i=1}^n s_i, q\sum_{i=1}^n t_i)$ that lie weakly below $(N^{t_1}E^{s_1-t_1}\dots N^{t_n}E^{s_n-t_n})^q$ (Theorem 2 of [3]). Via Theorem 3.7, the cardinality of $\mathbb{S}((n+1)^2,\rho)$ for arbitrary ρ corresponds to the linear coefficient f_1 of one such generating function. The difficulty in applying their result is that the generating function $F(z)$ is defined in terms of the $s-t$ unique solutions $w_1,w_2,\dots,w_{s-t}$ to $(w-1)^{s-t}-zw^s=0$ that qualify as fractional power series, where $t=\sum_{i=1}^n t_i$ and $s=\sum_{i=1}^n s_i$. Such generating functions obviously become arduous to compute when s and t become large.

To define our procedure of **density shifting**, fix $\lambda=(n_1,n_2)$ and a density ρ with $\rho_{1,j}=a_j$, $\rho_{2,j}=b_j$. We focus on the first two columns of an arbitrary $T\in\mathbb{S}(\lambda,\rho)$, and consider the relationship of the integers $\beta_1<\dots<\beta_{b_1}$ at position $(2,1)$ to the integers $\alpha_1<\dots<\alpha_{a_2}$ at position $(1,2)$. In particular, we divide the entries at $(2,1)$ into two (possibly empty) sets $A_1=\{\beta_1,\dots\beta_{m-1}\}$ and $A_2=\{\beta_m,\dots,\beta_{b_1}\}$, where every integer in A_1 is less than α_{a_2} and every integer in A_2 is greater than α_{a_2}. The elements of A_2 are moved to the cell at $(2,2)$, with the definition of A_2 ensuring that this shift does not violate column-standardness. The elements of A_2 always become the b_1-m+1 smallest integers at $(2,2)$. The elements of A_1 are then moved to the cell at $(1,2)$, where they are smaller than α_{a_2} but their relationship to $\alpha_1,\dots,\alpha_{a_2-1}$ depends upon the choice of T. With the cell at $(2,1)$ empty, the entire first column of the tableau is deleted and the remaining entries are re-indexed according to $x\mapsto x-a_1$. This produces a tableau $d(T)\in\mathbb{S}(\lambda',\rho')$ for $\lambda'=(n_1-1,n_2-1)$ and some density ρ' with $\rho'_{i,j}=\rho_{i+1,j+1}$ for $j>1$ and first column densities $\rho'_{i,1}$ determined by T. We refer to this new tableau $d(T)$ as the **density shift** of T. See Figure 6 for an example.

1 2	4 5 7	...
3 6 8 9	10 11	...

$\Rightarrow$

1 2	**3** 4 5 **6** 7	...
	8 **9** 10 11	...

$\Rightarrow$

1 2 3 4 5	...
6 7 8 9	...

Figure 6: A two-row set-valued tableau T and its density shift $d(T)$.

The map $T\mapsto d(T)$ is well-defined into $\bigcup_i \mathbb{S}(\lambda',\rho'_i)$, assuming that one appropriately determines the collection of shifted densities ρ'_i. However, the map is far from injective, as $d(T)$ does not remember which of its (non-maximal) entries at position

$(1,1)$ were shifted to that position from the $(2,1)$ cell of T. Relating $|\,\mathbb{S}(\lambda,\rho)|$ to the $|\,\mathbb{S}(\lambda',\rho'_i)|$ requires that we account for all possible positioning of those shifted entries.

In the statement of Theorem 4.1 and all that follows, notice the absolute value signs about the Young diagrams with parenthesized entries. As such, those diagrams denote the cardinality of $\mathbb{S}(\lambda,\rho)$ for ρ with the cell densities $\rho_{i,j}$ shown.

Theorem 4.1. *For any two-row shape $\lambda=(n_1,n_2)$ and density ρ as shown,*

$$|\,\mathbb{S}(\lambda,\rho)| = \left|\begin{array}{|c|c|c|c|}\hline (a_1) & (a_2) & (a_3) & \dots \\ \hline (b_1) & (b_2) & (b_3) & \dots \\ \hline\end{array}\right| = \sum_{i=0}^{b_1}\binom{a_2+i-1}{i}\left|\begin{array}{|c|c|c|}\hline (a_2) & (a_3) & \dots \\ \hline (b_1+b_2-i) & (b_3) & \dots \\ \hline\end{array}\right|$$

where the Young diagram inside the summation has shape $\lambda'=(n_1-1,n_2-1)$.

Proof. For fixed $\lambda=(n_1,n_2)$ and ρ as shown, partition $\mathbb{S}(\lambda,\rho)$ into subsets $S_0,\dots,S_{b_1}$ where S_i denotes the set of tableaux in $\mathbb{S}(\lambda,\rho)$ for which precisely i entries at $(2,1)$ are smaller than the largest entry at $(1,2)$. When restricted to a specific S_i, density shifting $T\mapsto d(T)$ defines a function $d_i: S_i\to\mathbb{S}(\lambda',\rho')$ with $\lambda'=(n_1-1,n_2-1)$ and $\rho'_{1,1}=a_2+i$, $\rho'_{2,1}=b_2+b_1-i$. For any $0\le i\le b_1$, we claim d_i is onto and is m-to-1, where $m=\binom{a_2+i-1}{i}$.

Fixing $0\le i\le b_1$, notice that the $(1,1)$ cell of any $T'\in\mathbb{S}(\lambda',\rho')$ is filled with the integers $\{1,2,\dots,a_2+i\}$. For any choice $\vec{u}=\{u_1,\dots,u_i\}$ of i integers from $[a_2+i-1]$, define a map $f_{\vec{u}}:\mathbb{S}(\lambda',\rho')\to\mathbb{S}(\lambda,\rho)$ as follows

1. For any $T'\in\mathbb{S}(\lambda',\rho')$, let $c_1<\dots<c_{a_2+i-1}$ denote the integers at position $(1,1)$ of T'. Then remove the i integers $c_{u_1},\dots,c_{u_i}$ at position $(1,1)$ as well as the b_1-i smallest integers at position $(2,1)$.

2. Append a new column to the left of T' and fill the bottom cell of that new column with the b_1 integers removed during step #1.

3. Reindex all entries in the resulting tableau by $x\mapsto x+a_1$, and then fill the cell at $(1,1)$ with $[a_1]$. The result is $f_{\vec{u}}(T)\in\mathbb{S}(\lambda,\rho)$.

The map $f_{\vec{u}}$ has been defined so that $d\circ f_{\vec{u}}(T')=T'$ for all $T'\in\mathbb{S}(\lambda',\rho')$. For any $T\in S_i$, if we let $\beta_1<\dots<\beta_{b_1}$ denote the integers at $(2,1)$ of T and let $c_1<\dots<c_{a_2+i-1}$ denote the entries at $(1,1)$ of $f(T)$, we also have $f_{\vec{u}}\circ d_i(T)=T$ for precisely those T where $\{\beta_1,\dots,\beta_i\}=\{c_{u_1},\dots,c_{u_i}\}$. Define $S_i|_{\vec{u}}$ to be the subset of S_i with this restriction upon the $\beta_1,\dots,\beta_i$. It follows that $S_i|_{\vec{u}}$ is in bijection with $\mathbb{S}(\lambda',\rho')$. Ranging over all choices of $\vec{u}$ then allows us to conclude that $|S_i|=\binom{a_2+i-1}{i}|\,\mathbb{S}(\lambda',\rho')|$.

Note that changing the density $\rho'_{1,1}$ at position $(1,1)$ does not change the size of any set $\mathbb{S}(\lambda',\rho')$. We then assume for simplicity that $\rho'_{1,1}=a_2=\rho_{1,2}$. As the S_i partition $\mathbb{S}(\lambda,\rho)$, varying $0\le i\le b_1$ yields the required summation. □

Now consider a pair of n-tuples of non-negative integers $\vec{x}=(x_1,\dots,x_n)$ and $\vec{y}=(y_1,\dots,y_n)$. One may define a dominance ordering on these tuples whereby $\vec{x}\preceq\vec{y}$ if $x_1+\dots+x_i\le y_1+\dots+y_i$ for every $1\le i\le n$. Theorem 4.1 may then be repeatedly applied to derive the following.

Theorem 4.2. *For any two-row shape $\lambda = (n_1, n_2)$ and density ρ as shown,*

$$|\,\mathbb{S}(\lambda,\rho)| \;=\; \left|\,\begin{array}{|c|c|c|c|}\hline (a_1) & (a_2) & (a_3) & \dots \\ \hline (b_1) & (b_2) & (b_3) & \dots \\ \hline \end{array}\,\right| \;=\; \sum_{(i_1,\ldots,i_{n_1-1}) \preceq (b_1,\ldots,b_{n_1-1})} \prod_{j=1}^{n_1-1} \binom{a_{j+1}+i_j-1}{i_j}$$

Proof. Begin by noticing that, if $n_2 < n_1$, we may replace $\mathbb{S}(\lambda,\rho)$ with a set $\mathbb{S}(n^2,\widetilde{\rho})$ where $\widetilde{\rho}_{n_2+1} = \ldots = \widetilde{\rho}_{n_1} = 0$. Clearly $|\,\mathbb{S}(\lambda,\rho)| = |\,\mathbb{S}(n^2,\widetilde{\rho})|$. Repeated application of Theorem 4.1 then yields

$$\left|\,\begin{array}{|c|c|c|c|}\hline (a_1) & (a_2) & (a_3) & \dots \\ \hline (b_1) & (b_2) & (b_3) & \dots \\ \hline \end{array}\,\right| \;=$$

$$\sum_{i_1=0}^{b_1}\binom{a_2+i_1-1}{i_1}\sum_{i_2=0}^{b_1+b_2-i_1}\binom{a_3+i_2-1}{i_2}\cdots\sum_{i_{n-1}=0}^{\substack{b_1+\cdots+b_{n-1}\\ -i_1-\cdots-i_{n-2}}}\binom{a_n+i_{n-1}-1}{i_{n-1}}\left|\,\begin{array}{|c|}\hline (a_n) \\ \hline (x) \\ \hline \end{array}\,\right|$$

Here the non-negative integer x depends upon the b_j and i_j but is irrelevant because $|\,\mathbb{S}(\widetilde{\lambda},\widetilde{\rho})| = 1$ for any $\widetilde{\rho}$ with $\widetilde{\lambda} = 1^2$. This allows us to reduce the right side to a nested series of summations involving only binomial coefficients. If we assume that $i_1, i_2, \ldots$ are non-negative integers, the bounds of summation on the right side imply the dominance conditions $i_1 + \cdots + i_j \leq b_1 + \cdots + b_j$ for all $1 \leq j \leq n_1 - 1$. □

Observe that the equation of Theorem 4.2 does not involve a_1, aligning with our intuition that changing the cell density at $(1,1)$ does not affect $|\,\mathbb{S}(\lambda,\rho)|$. Similarly, if $\lambda = n^2$, the equation of Thoerem 4.2 does not involve b_n because changing the cell density at (n,n) also does not affect $|\,\mathbb{S}(\lambda,\rho)|$.

Example 4.3. *For $\lambda = (n,n)$, $a_j = 1$, and $b_j = k-1$, the product of Theorem 4.2 reduces to*

$$\prod_{j=1}^{n_1-1}\binom{a_{j+1}+i_j-1}{i_j} \;=\; \prod_{j=1}^{n_1-1}\binom{i_j}{i_j} \;=\; 1$$

Thus $|\,\mathbb{S}(\lambda,\rho)|$ is the number of $(n-1)$-tuples of non-negative integers $(i_1,\ldots,i_{n-1}) \preceq (k-1,\ldots,k-1)$. If we let $y = kn - i_1 - \ldots i_{n-1}$, these tuples may be placed in bijection with the set $\mathcal{D}_n^k$ of k-good paths by $(i_1,\ldots,i_{n-1}) \mapsto EN^{i_1}EN^{i_2}\ldots EN^{i_{n-1}}EN^{y}$. Hence $|\,\mathbb{S}(\lambda,\rho)| = C_n^k$, as expected from Theorem 3.1.

Example 4.4. *More generally, for $\lambda = (n,n)$ and any density with $a_j = 1$ for all j, Theorem 4.2 places $\mathbb{S}(\lambda,\rho)$ in bijection with the number of $(n-1)$-tuples of non-negative integers $(i_1,\ldots,i_{n-1}) \preceq (b_1,\ldots,b_{n-1})$. These tuples may be placed in bijection with the set of N-E lattice paths of shape $(n, b_1+\ldots+b_n)$ that lie weakly below the path $P = EN^{b_1}EN^{b_2}\ldots EN^{b_n}$ via the same map as Example 4.3. This corresponds to a special case of Theorem 2.2.*

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