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# Outside perfect 8-cycle systems

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*In memory of Anne Penfold Street*

**Abstract**

The two 4-cycles $(a, b, c, d)$ and $(e, f, g, h)$ are called the outside 4-cycles of the 8-cycle $(a, b, c, d, e, f, g, h)$. Given an 8-cycle system, if we can form a 4-cycle system by choosing two outside 4-cycles from each 8-cycle in the system, then the 8-cycle system is called outside perfect.

In this paper we prove that an outside perfect maximum packing of $K_n$ with 8-cycles of order $n$ exists for all $n \geq 8$, except $n = 9$, for which no such system exists.

## 1 Introduction

An $m$-cycle system of order $n$ is a pair $(\mathcal{X}, \mathcal{C})$ where $\mathcal{X}$ is a finite set and $\mathcal{C}$ is a collection of edge-disjoint $m$-cycles which partitions the edge-set of the complete undirected graph $K_n$ with vertex set $\mathcal{X}$. The spectrum for $m$-cycle systems (that is, the set of all $n$ such that an $m$-cycle system of order $n$ exists) is the set of all $n$ such that [13]

(i) $n \geq m \geq 3$,

(ii) $n \equiv 1 \pmod 2$ and

(iii) $n(n-1)/2m$ is an integer.

An $m$-cycle, $C_m$, with edges $a_i a_{i+1}$ for $1 \leq i \leq m-1$ and $a_1 a_m$ will be denoted by $(a_1, a_2, \ldots, a_m)$. In this paper we will study $m$-cycle systems when $m = 8$ and consider the following property. For each 8-cycle $(a, b, c, d, e, f, g, h)$ we will call the two 4-cycles $(b, c, d, e)$ and $(a, f, g, h)$ as outside 4-cycles of the 8-cycle and denote the 8-cycle by $(a - b, c, d, e - f, g, h)$. Note that for every 8-cycle there are 4 different possible pairs of such 4-cycles. Given an 8-cycle system $(\mathcal{X}, \mathcal{C})$, let $8(4)C$ be the collection of outside 4-cycles of the 8-cycles in $C$ (two 4-cycles from each 8-cycle in $\mathcal{C}$). If $8(4)C$ is a 4-cycle system, $C$ is said to be outside perfect. We can extend the existence of outside perfect 8-cycle systems problem to maximum packings of $K_n$ with 8-cycles (see [8] and [9]) by constructing for every $n \geq 8$; a maximum packing $(\mathcal{X}, \mathcal{C}, \mathcal{L})$ of $K_n$ with 8-cycles so that $(\mathcal{X}, 8(4)C, \mathcal{L})$ is also a packing, where $8(4)C$ is a collection of outside 4-cycles of the 8-cycles of $\mathcal{C}$, and $\mathcal{L}$ is the leave of the maximum packing. We call such a maximum packing an outside perfect maximum packing with 8-cycles.

In this paper we will construct outside perfect maximum packings of $K_n$ with 8-cycles for all $n \geq 8$, except $n = 9$, for which no such system exists.

The existence of outside perfect maximum packing of $K_n$ with 6-cycles has been recently introduced and solved by Lindner, Meszka and Rosa in [12]. Studies on different properties of perfect 8-cycle systems and different forms of related systems can be found in [1, 2, 3, 4, 5, 6, 7, 10, 11].

The following table lists leaves that are considered for the maximum packings with 8-cycles.

| Spectrum for maximum packing with 8-cycles | Leave |
|---|---|
| 1 (mod 16) | $\emptyset$ |
| 3 (mod 16) | $C_3$ |
| 5 (mod 16) | $K_5$ |
| 7 (mod 16) | $C_5$ |
| 9 (mod 16), $n \neq 9$ | $C_4$ |
| 11 (mod 16) | $C_3 + C_4$ |
| 13 (mod 16) | $C_6$ |
| 15 (mod 16) | $C_4 + C_5$ |
| 0, 2, 8, 10 (mod 16) | 1-factor |
| 4, 6, 12, 14 (mod 16) | $K_4$ + a 1-factor on the remaining vertices |

Table 1: Maximum packings with 8-cycles

## 2 Preliminary results and the main construction

We will start by introducing the necessary construction and lemmas that we will use throughout the paper.

**Lemma 2.1** *There exists an outside perfect* 8*-cycle decomposition of* $K_{4t,4s}$*, for all* $t, s \in \mathbb{Z}^+$.

**Proof** Let $X = \{x_0, x_1, x_2, x_3\}$ and $Y = \{y_0, y_1, y_2, y_3\}$ be the parts of $K_{4,4}$. Consider $\mathcal{C} = \{(x_2 - y_2, x_3, y_3, x_0 - y_0, x_1, y_1), (y_2 - x_0, y_1, x_3, y_0 - x_2, y_3, x_1)\}$ to get an outside perfect 8-cycle decomposition of $K_{4,4}$.

Next let $\mathcal{X} = \{x_0, x_1, \ldots, x_{4t-1}\}$ and $\mathcal{Y} = \{y_0, y_1, \ldots, y_{4s-1}\}$ be the parts of $K_{4t,4s}$, where $X_i = \{x_{4i}, x_{4i+1}, x_{4i+2}, x_{4i+3}\}$, $Y_j = \{y_{4j}, y_{4j+1}, y_{4j+2}, y_{4j+3}\}$ for $i = 0, 1, 2, \ldots, t-1$, $j = 0, 1, \ldots, s-1$ and $t, s \in \mathbb{Z}^+$. Placing an outside perfect 8-cycle decomposition of $K_{4,4}$ on the vertex set $X_i \cup Y_j$ for each pair $i$, $j$ gives us an outside perfect 8-cycle decomposition of $K_{4t,4s}$. □

**Lemma 2.2** *There exists an outside perfect* 8*-cycle decomposition of* $K_{4t,4s+2}$*, for all* $t, s \in \mathbb{Z}^+$.

**Proof** Let $X = \{x_0, x_1, x_2, x_3\}$ and $Y = \{y_0, y_1, y_2, y_3, y_4, y_5\}$ be the parts of $K_{4,6}$. Consider $\mathcal{C} = \{(x_2 - y_2, x_3, y_3, x_0 - y_4, x_1, y_1), (x_2 - y_4, x_3, y_5, x_0 - y_0, x_1, y_3), (y_2 - x_0, y_1, x_3, y_0 - x_2, y_5, x_1)\}$ to get an outside perfect 8-cycle decomposition of $K_{4,6}$.

Next let $\mathcal{X} = \{x_0, x_1, \ldots, x_{4t-1}\}$ and $\mathcal{Y} = \{y_0, y_1, \ldots, y_{4s+1}\}$ be the parts of $K_{4t,4s+2}$, where $X_i = \{x_{4i}, x_{4i+1}, x_{4i+2}, x_{4i+3}\}$, $Y_0 = \{y_0, y_1, y_2, y_3, y_4, y_5\}$, $Y_j = \{y_{4j+2}, y_{4j+3}, y_{4j+4}, y_{4j+5}\}$ for $i = 0, 1, \ldots, t-1$, $j = 1, 2, \ldots, s-1$ and $t, s \in \mathbb{Z}^+$. Placing an outside perfect 8-cycle decomposition of $K_{4,4}$ on the vertex set $X_i \cup Y_j$ for each pair $i$, $j$ and $j \geq 1$, and an outside perfect 8-cycle decomposition of $K_{4,6}$ on the vertex set $X_i \cup Y_0$ for each $i = 0, 1, \ldots, t-1$ gives us an outside perfect 8-cycle decomposition of $K_{4t,4s+2}$. □

**Lemma 2.3** *If there exist an outside perfect* 8*-cycle system of order* $r + 1$ *and an outside perfect* 8*-cycle decomposition of* $K_{r,s}$*, then there exists an outside perfect* 8*-cycle decomposition of* $K_{r+s+1} \setminus K_{s+1}$.

**Proof** Let $\mathcal{X} = \{\infty\} \cup \{x_1, x_2, \ldots, x_r\} \cup \{y_1, y_2, \ldots, y_s\}$. Placing an outside perfect 8-cycle system of order $r + 1$ on $\{\infty\} \cup \{x_1, x_2, \ldots, x_r\}$ and an outside perfect 8-cycle decomposition of $K_{r,s}$ on $\{x_1, x_2, \ldots, x_r\} \cup \{y_1, y_2, \ldots, y_s\}$ gives an outside perfect 8-cycle decomposition of $K_{r+s+1} \setminus K_{s+1}$, where the vertex set of $K_{s+1}$ is $\{\infty\} \cup \{y_1, y_2, \ldots, y_s\}$. □

**Lemma 2.4** *If there exist outside perfect maximum* 8*-cycle packings of order* $r$ *and of order* $s$ *with a* 1*-factor leave and an outside perfect* 8*-cycle decomposition of* $K_{r,s}$*, then there exists an outside perfect maximum* 8*-cycle packing of order* $r + s$ *with a* 1*-factor leave.*

**Proof** Let $\mathcal{X} = \{x_1, x_2, \ldots, x_r\} \cup \{y_1, y_2, \ldots, y_s\}$. Placing an outside perfect maximum 8-cycle packing of order $r$ on $\{x_1, x_2, \ldots, x_r\}$ and $s$ on $\{y_1, y_2, \ldots, y_s\}$ and an outside perfect 8-cycle decomposition of $K_{r,s}$ on $\{x_1, x_2, \ldots, x_r\} \cup \{y_1, y_2, \ldots, y_s\}$ gives an outside perfect maximum 8-cycle packing of order $r+s$ with a 1-factor leave. □

**The main construction**

Let $X = H \cup \{(i, j) \mid 1 \leq i \leq k, 1 \leq j \leq 16\}$, where $|H| \in \{0, 1, \ldots, 15\}$ and $k \in \mathbb{Z}^+$.

(1) On $H \cup \{(1, j) \mid 1 \leq j \leq 16\}$, place an outside perfect maximum 8-cycle packing of order $16 + h$, where $h = |H|$.

(2) On each set $H \cup \{(i, j) \mid 1 \leq j \leq 16\}$, for $2 \leq i \leq k$, place an outside perfect 8-cycle decomposition of $K_{16+h} \setminus K_h$.

(3) For each $x, y = 1, 2, \ldots, k$, $x < y$ place an outside perfect 8-cycle decomposition of $K_{16,16}$ on $\{(x, j) \mid 1 \leq j \leq 16\} \cup \{(y, j) \mid 1 \leq j \leq 16\}$.

If the necessary systems in (1), (2) and (3) exist, this construction gives an outside perfect maximum 8-cycle packing of order $16k + h$. □

## 3 Small examples

**Example 3.1** *An outside perfect maximum* 8*-cycle packing of order* 8*.*

Let $\mathcal{X} = \mathbb{Z}_8$, $\mathcal{L} = \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}\}$ and $\mathcal{C} = \{(2 - 1, 3, 4, 6 - 5, 7, 0)$, $(5 - 1, 7, 4, 2 - 6, 0, 3)$, $(1 - 6, 3, 7, 2 - 5, 0, 4)\}$. □

**Example 3.2** *An outside perfect maximum* 8*-cycle packing of order* 10*.*

Let $\mathcal{X} = \mathbb{Z}_{10}$, $\mathcal{L} = \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{8, 9\}\}$ and $\mathcal{C} = \{(3 - 4, 6, 5, 7 - 0, 2, 1)$, $(0 - 3, 5, 1, 4 - 8, 2, 9)$, $(2 - 6, 8, 7, 9 - 5, 0, 4)$, $(7 - 4, 9, 3, 8 - 0, 6, 1)$, $(5 - 2, 7, 3, 6 - 9, 1, 8)\}$. □

**Example 3.3** *An outside perfect maximum* 8*-cycle packing of order* 11*.*

Let $\mathcal{X} = \mathbb{Z}_{11}$, $\mathcal{L} = \{(0, 1, 2), (3, 4, 5, 6)\}$ and $\mathcal{C} = \{(1 - 4, 2, 5, 7 - 6, 0, 3)$, $(6 - 1, 5, 8, 9 - 10, 0, 4)$, $(9 - 4, 8, 10, 1 - 7, 0, 5)$, $(6 - 10, 3, 2, 7 - 9, 0, 8)$, $(9 - 1, 8, 3, 7 - 4, 10, 2)$, $(6 - 9, 3, 5, 10 - 7, 8, 2)\}$. □

**Example 3.4** *Outside perfect maximum* 8*-cycle packings of orders* 12, 14, 20 *and* 22*.*

Following the construction in Lemma 2.4 considering $s = 4$ and: $r = 8$ for the order 12, $r = 10$ for the order 14, $r = 16$ for the order 20 and $r = 18$ for the order 22, setting $\{y_1, y_2, y_3, y_4\}$ as the vertex set of $K_4$ in the leave, gives outside perfect maximum 8-cycle packings of orders 12, 14, 20 and 22. The leave is a $K_4$ and a set of independent edges saturating the remaining elements. □

**Example 3.5** *An outside perfect maximum* 8*-cycle packing of order* 13.

Let $\mathcal{X} = \mathbb{Z}_{13}$, $\mathcal{L} = \{(0, 1, 2, 3, 4, 5)\}$ and $\mathcal{C} = \{(6 - 7, 8, 9, 10 - 11, 12, 0), (7 - 10, 12, 9, 11 - 6, 8, 0), (11 - 8, 10, 5, 3 - 0, 2, 4), (3 - 8, 2, 5, 11 - 0, 4, 9), (1 - 3, 7, 2, 6 - 10, 0, 9), (3 - 12, 7, 9, 5 - 1, 4, 10), (6 - 3, 11, 2, 12 - 5, 8, 1), (12 - 6, 9, 2, 10 - 1, 7, 4), (5 - 6, 4, 8, 12 - 1, 11, 7)\}$. □

**Example 3.6** *An outside perfect maximum* 8*-cycle packing of order* 15.

Let $\mathcal{X} = \mathbb{Z}_{15}$, $\mathcal{L} = \{(6, 11, 12, 13, 14), (7, 8, 9, 10)\}$ and $\mathcal{C} = \{(4 - 5, 8, 10, 12 - 0, 1, 2), (12 - 5, 9, 11, 4 - 0, 14, 7), (3 - 10, 2, 6, 13 - 0, 8, 1), (13 - 10, 1, 11, 3 - 0, 2, 9), (9 - 14, 8, 2, 11 - 0, 5, 1), (11 - 14, 12, 3, 9 - 0, 10, 5), (6 - 1, 4, 3, 13 - 5, 7, 0), (7 - 3, 2, 12, 8 - 4, 14, 1), (6 - 5, 14, 2, 13 - 1, 12, 4), (3 - 8, 6, 9, 4 - 7, 2, 5), (6 - 7, 13, 8, 11 - 10, 14, 3), (10 - 6, 12, 9, 7 - 11, 13, 4)\}$. □

**Example 3.7** *Outside perfect maximum* 8*-cycle packings of order* 16 *and* 18.

In Lemma 2.4, considering $r = 8$ and: $s = 8$ for the order 16 and $s = 10$ for the order 18 gives outside perfect maximum 8-cycle packings of orders 16 and 18 with a 1-factor leave. □

**Example 3.8** *An outside perfect* 8*-cycle system of order* 17.

Let $\mathcal{X} = \{\infty\} \cup \{1, 2, \ldots, 16\}$. Place a copy of an outside perfect 8-cycle decomposition of $K_{4,6}$ where the groups of size 4 and 6 are $\{1, 2, 3, 4\}$ and $\{11, 12, 13, 14, 15, 16\}$, and a copy of an outside perfect 8-cycle maximum packing of order 13 on $\{\infty\} \cup \{5, 6, \ldots, 16\}$ being sure that the 6-cycle leave is $(5, 6, 7, 8, 9, 10)$. Then place the following five outside perfect 8-cycles $(1 - 2, 3, 5, 6 - 7, 4, \infty)$, $(2 - 5, 10, 4, 1–8, 3, \infty)$, $(8 - 7, 2, 4, 6 - 1, 3, 9)$, $(1 - 5, 4, 9, 2 - 6, 3, 10)$, $(1 - 7, 3, 4, 8 - 2, 10, 9)$ on the uncovered edges. □

**Example 3.9** *An outside perfect maximum* 8*-cycle packing of order* 19.

Let $\mathcal{X} = \{\infty_1, \infty_2, \infty_3\} \cup \{x_1, x_2, \ldots, x_8\} \cup \{y_1, y_2, \ldots, y_8\}$. Place a copy of an outside perfect maximum 8-cycle packing of order 11 on $\{\infty_1, \infty_2, \infty_3\} \cup \{x_1, x_2, \ldots, x_8\}$ and on $\{\infty_1, \infty_2, \infty_3\} \cup \{y_1, y_2, \ldots, y_8\}$ being sure that the 3-cycle in the leave is $(\infty_1, \infty_2, \infty_3)$ in both systems and the 4-cycles in the leave are $(x_1, x_2, x_3, x_4)$ and $(y_1, y_2, y_3, y_4)$. Then place an outside perfect maximum 8-cycle packing of order 8 on $\{x_1, x_2, x_3, x_4\} \cup \{y_1, y_2, y_3, y_4\}$ being sure the 1-factor leave is $\{\{x_1, x_3\}, \{x_2, x_4\}, \{y_1, y_3\}, \{y_2, y_4\}\}$ and an outside perfect 8-cycle decomposition of $K_{4,4}$ on $\{x_1, x_2, x_3, x_4\} \cup \{y_5, y_6, y_7, y_8\}$, on $\{x_5, x_6, x_7, x_8\} \cup \{y_1, y_2, y_3, y_4\}$ and on $\{x_5, x_6, x_7, x_8\} \cup \{y_5, y_6, y_7, y_8\}$. Then the leave of the system will be $(\infty_1, \infty_2, \infty_3)$. □

**Example 3.10** *An outside perfect maximum* 8*-cycle packing of order* 21*.*

Considering $r = 16$ and $s = 4$ in Lemma 2.3 gives an outside perfect 8-cycle decomposition of $K_{21} \setminus K_5$, therefore an outside perfect 8-cycle packing of order 21 with leave a $K_5$ on $\{\infty\} \cup \{y_1, y_2, y_3, y_4\}$. □

**Example 3.11** *An outside perfect maximum* 8*-cycle packing of order* 23*.*

Let $\mathcal{X} = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup \{x_1, x_2, \ldots, x_8\} \cup \{y_1, y_2, \ldots, y_{10}\}$. Place a copy of an outside perfect maximum 8-cycle packing of order 15 on $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup \{y_1, y_2, \ldots, y_{10}\}$ with leave a 5 cycle $(\infty_1, \infty_2, \infty_3, \infty_4, \infty_5)$ and a 4-cycle $(y_1, y_2, y_3, y_4)$. Place a copy of an outside perfect maximum 8-cycle packing of $K_{13} \setminus K_5$ on $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup \{x_1, x_2, \ldots, x_8\}$ where the vertex set of the hole $K_5$ is $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$, the leave 4-cycle is $(x_1, x_2, x_3, x_4)$. Then place an outside perfect maximum 8-cycle packing of order 8 on $\{x_1, x_2, x_3, x_4\} \cup \{y_1, y_2, y_3, y_4\}$ being sure that the 1-factor leave is $\{\{x_1, x_3\}, \{x_2, x_4\}, \{y_1, y_3\}, \{y_2, y_4\}\}$, an outside perfect 8-cycle decomposition of $K_{4,4}$ on $\{x_5, x_6, x_7, x_8\} \cup \{y_1, y_2, y_3, y_4\}$, an outside perfect 8-cycle decomposition of $K_{4,6}$ on $\{x_1, x_2, x_3, x_4\} \cup \{y_5, y_6, y_7, y_8, y_9, y_{10}\}$ and on $\{x_5, x_6, x_7, x_8\} \cup \{y_5, y_6, y_7, y_8, y_9, y_{10}\}$. The eight 8-cycles of an outside perfect maximum 8-cycle packing of $K_{13} \setminus K_5$ is $(\infty_1 - x_1, \infty_2, x_2, \infty_3 - x_3, \infty_4, x_4)$, $(x_1 - \infty_3, x_4, \infty_2, x_3 - \infty_1, x_2, \infty_4)$, $(\infty_1, x_5, \infty_2, x_6, \infty_3 - x_7, \infty_4, x_8)$, $(x_7 - \infty_1, x_6, \infty_4, x_5 - \infty_3, x_8, \infty_5)$, $(x_1 - \infty_5, x_5, x_3, x_6 - x_8, \infty_2, x_7)$, $(x_1 - x_8, x_7, x_4, x_6 - \infty_5, x_2, x_5)$, $(x_3 - x_7, x_2, x_6, x_5 - x_8, x_4, \infty_5)$, $(x_3 - x_8, x_2, x_4, x_5 - x_7, x_6, x_1)$. □

**Example 3.12** *An outside perfect maximum* 8*-cycle packing of order* 25*.*

Let $\mathcal{X} = \{\infty\} \cup X_1 \cup X_2 \cup X_3 \cup Y_1 \cup Y_2$, where $X_1 = \{0, 1, 2, 3, 4, 5\}, Y_1 = \{6, 7, 8, 9, 10, 11\}$ and $|X_2| = |X_3| = |Y_2| = 4$. Place a copy of an outside perfect maximum 8-cycle packing of order 15 on $\{\infty\} \cup X_1 \cup X_2 \cup X_3$, where the leave is the 5-cycle $(0, 1, 2, 3, 4)$ and a 4-cycle in $X_2$. Now place a copy of an outside perfect maximum 8-cycle packing of order 11 on $\{\infty\} \cup Y_1 \cup Y_2$, where the leave is the 3-cycle $(6, 7, 8)$ and a 4-cycle in $Y_2$. Then place an outside perfect maximum 8-cycle packing of order 8 on $X_2 \cup Y_2$, an outside perfect 8-cycle decomposition of $K_{4,4}$ on $X_3 \cup Y_2$, and an outside perfect 8-cycle decomposition of $K_{4,6}$ on $X_2 \cup Y_1$, on $X_3 \cup Y_1$ and on $X_1 \cup Y_2$. Finally place the following 5 outside perfect 8-cycles $(3 - 2, 6, 1, 8 - 7, 0, 4)$, $(8 - 2, 9, 4, 10 - 5, 11, 0)$, $(3 - 7, 4, 11, 2 - 10, 0, 9)$, $(2 - 7, 5, 6, 8 - 3, 11, 1)$, $(5 - 8, 4, 6, 3 - 10, 1, 9)$ on the uncovered edges of $X_1 \cup Y_1$. Note that the leave of this packing is $(0, 1, 7, 6)$ □

**Example 3.13** *Outside perfect maximum* 8*-cycle packings of* $K_{22} \setminus K_6$*,* $K_{24} \setminus K_8$*,* $K_{26} \setminus K_{10}$*,* $K_{28} \setminus K_{12}$ *and* $K_{30} \setminus K_{14}$*.*

In Lemma 2.4, considering $r = 16$; and $s = 6$ for the case $K_{22} \setminus K_6$, $s = 8$ for the case $K_{24} \setminus K_8$, $s = 10$ for the case $K_{26} \setminus K_{10}$, $s = 12$ for the case $K_{28} \setminus K_{12}$ and $s = 14$ for the case $K_{30} \setminus K_{14}$ but keeping the hole of order $s$ on $\{y_1, y_2, \ldots, y_s\}$ gives the required decompositions with a 1-factor leave on $\{x_1, x_2, \ldots, x_r\}$. □

**Example 3.14** *Outside perfect 8-cycle decompositions of* $K_{23} \setminus K_7$, $K_{25} \setminus K_9$, $K_{27} \setminus K_{11}$, $K_{29} \setminus K_{13}$ *and* $K_{31} \setminus K_{15}$.

In Lemma 2.3, considering $r = 16$ and; $s = 6$ for the case $K_{23} \setminus K_7$, $s = 8$ for the case $K_{25} \setminus K_9$, $s = 10$ for the case $K_{27} \setminus K_{11}$, $s = 12$ for the case $K_{29} \setminus K_{13}$ and $s = 14$ for the case $K_{31} \setminus K_{15}$ gives an outside perfect 8-cycle decompositions of $K_{23} \setminus K_7$, $K_{25} \setminus K_9$, $K_{27} \setminus K_{11}$, $K_{29} \setminus K_{13}$ and $K_{31} \setminus K_{15}$. □

## 4 Maximum packings with outside perfect 8-cycles

We will consider ten cases for $n$ modulo 16.

**Lemma 4.1** *For every* $n \equiv 1 \pmod{16}$ *there exists an outside perfect 8-cycle system of order* $n$.

**Proof** Let $h = 1$ in the main construction. Since there exist an outside perfect 8-cycle system of order 17 by Example 3.8 and an outside perfect 8-cycle decomposition of $K_{16,16}$ by Lemma 2.1, the result follows. □

**Lemma 4.2** *For every* $n \equiv 3 \pmod{16}$ *there exists an outside perfect maximum 8-cycle packing of order* $n$.

**Proof** There exist an outside perfect maximum 8-cycle packing of order 19 with a $C_3$ leave by Example 3.9 and therefore an outside perfect 8-cycle decomposition of $K_{19} \setminus K_3$. The result follows by the main construction considering $h = 3$. □

**Lemma 4.3** *For every* $n \equiv 5 \pmod{16}$ *there exists an outside perfect maximum 8-cycle packing of order* $n$.

**Proof** There exists an outside perfect maximum 8-cycle packing of order 21 with a $K_5$ leave by Example 3.10. The result follows by the main construction considering $h = 5$. □

**Lemma 4.4** *For every* $n \equiv 7 \pmod{16}$ *there exists an outside perfect maximum 8-cycle packing of order* $n$.

**Proof** There exist an outside perfect maximum 8-cycle packing of order 23 with a 5-cycle leave by Example 3.11 and an outside perfect 8-cycle decomposition of $K_{23} \setminus K_7$ by Example 3.14. The result follows by the main construction considering $h = 7$. □

**Lemma 4.5** *For every* $n \equiv 9 \pmod{16}$ *except* $n = 9$, *there exists an outside perfect maximum 8-cycle packing of order* $n$.

**Proof** A comprehensive computer search showed that none of the maximum 8-cycle packings of order 9 is outside perfect. So the smallest case when $n \equiv 9 \pmod{16}$ is 25. There exist an outside perfect maximum 8-cycle packing of order 25 with a 4-cycle leave and an outside perfect 8-cycle decomposition of $K_{25} \setminus K_9$ by Examples 3.12 and 3.14. The result follows by the main construction considering $h = 9$. □

**Lemma 4.6** *For every $n \equiv 11$ (mod 16) there exists an outside perfect maximum 8-cycle packing of order $n$.*

**Proof** There exist an outside perfect maximum 8-cycle packing of order 11 with a 3-cycle and a 4-cycle leave, and an outside perfect 8-cycle decomposition of $K_{27} \setminus K_{11}$ by Examples 3.3 and 3.14. The result follows by the main construction considering $h = 11$. □

**Lemma 4.7** *For every $n \equiv 13$ (mod 16) there exists an outside perfect maximum 8-cycle packing of order $n$.*

**Proof** There exist an outside perfect maximum 8-cycle packing of order 13 with a 6-cycle leave, and an outside perfect 8-cycle decomposition of $K_{29} \setminus K_{13}$ by Examples 3.5 and 3.14. The result follows by the main construction considering $h = 13$. □

**Lemma 4.8** *For every $n \equiv 15$ (mod 16) there exists an outside perfect maximum 8-cycle packing of order $n$.*

**Proof** There exist an outside perfect maximum 8-cycle packing of order 15 with a 4-cycle and a 5-cycle leave, and an outside perfect 8-cycle decomposition of $K_{31} \setminus K_{15}$ by Examples 3.6 and 3.14. The result follows by the main construction considering $h = 15$. □

**Lemma 4.9** *For every $n \equiv 0, 2, 8$ and 10 (mod 16) there exists an outside perfect maximum 8-cycle packing of order $n$.*

**Proof** There exist outside perfect maximum 8-cycle packings of order $10, 16$ and 18, outside perfect maximum 8-cycle decompositions of $K_{24} \setminus K_8$ and $K_{26} \setminus K_{10}$ by Examples 3.2, 3.7 and 3.13. The result follows by the main construction by considering $h = 0, 2, 8$ and 10, where the leave is a 1-factor. □

**Lemma 4.10** *For every $n \equiv 4, 6, 12$ and 14 (mod 16) there exists an outside perfect maximum 8-cycle packing of order $n$.*

**Proof** There exist outside perfect maximum 8-cycle packings of order $12, 14, 20$ and 22, outside perfect maximum 8-cycle decompositions of $K_{22} \setminus K_6$, $K_{28} \setminus K_{12}$ and $K_{30} \setminus K_{14}$ by Examples 3.4 and 3.13. The result follows by the main construction by considering $h = 4, 6, 12, 14$, where the leave is a $K_4$ and 1-factor on the remaining vertices. □

Combining Lemmas 4.1 to 4.10 gives the following result.

**Theorem 4.11** *There exists an outside perfect maximum 8-cycle packing of order $n$ for every $n \geq 8$, except $n = 9$ for which no such system exists.*

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