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Lexicographic products of r-uniform hypergraphs and some applications to hypergraph Ramsey theory

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Abstract

Lexicographic products of graphs have been studied by many authors due to their useful properties involving independent sets and cliques. In this article, we show that analogues of such properties can be extended to lexicographic products of r-uniform hypergraphs and we use them to prove a generalization of Abbott's Theorem and a new multicolor inequality for hypergraph Ramsey numbers, which generalizes a theorem by Xiaodong, Zheng, Exoo and Radziszowski. We conclude by showing that our results, along with a construction due to Exoo, imply the following lower bounds for t-colored 3-uniform Ramsey numbers: $R^t(5;3) \geq 81^{2^{t-2}} + 1$.

1 Introduction

The lexicographic product of graphs stands out among other products because of the predictable nature in which its clique numbers and independence numbers behave. For graphs G_1 and G_2 (having vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$, respectively), the lexicographic product $G_1[G_2]$ is defined to have vertex set $V(G_1) \times V(G_2)$ and edge set

$$\{(a_1, b_1)(a_2, b_2) \mid a_1a_2 \in E(G_1) \text{ or } (a_1 = a_2 \text{ and } b_1b_2 \in E(G_2))\}.$$

It is clear that $G_1[G_2]$ is not commutative, hence the unusual product notation.

For example, consider the following graphs G_1 and G_2.

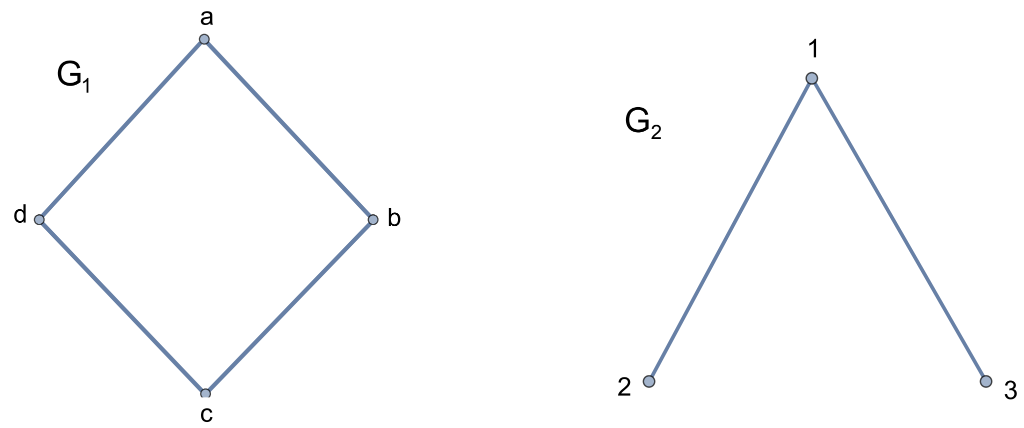


Figure 1 shows the lexicographic product $G_1[G_2]$. Observe that the product may be viewed as replacing the vertices of G_1 with copies of G_2.

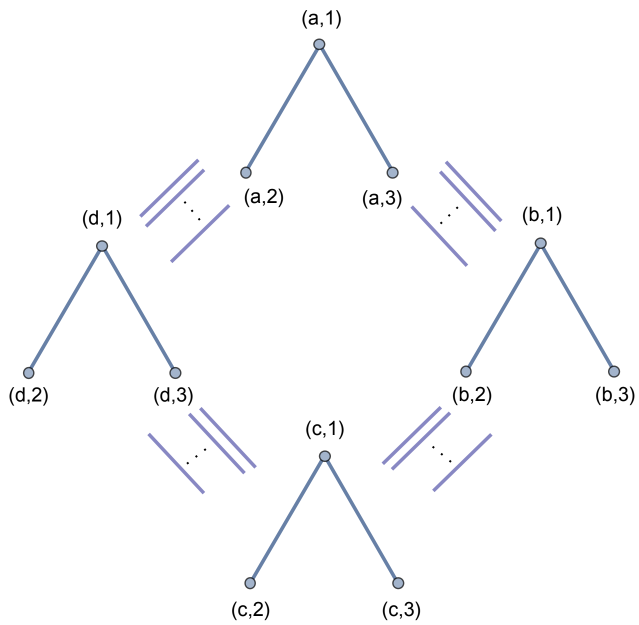


Figure 1: The lexicographic product $G_1[G_2]$.

As usual, we denote the complement of a graph G by $\overline{G}$ (i.e., the graph with vertex set $V(\overline{G}) = V(G)$ and edge set $E(\overline{G}) = \{ab \mid ab \notin G\}$). One can easily check that the complement $\overline{G_1[G_2]}$ contains the edges

$$\{(a,i)(c,j),(b,i)(d,j) \mid i,j=1,2,3\}$$

and

$$\{(a,2)(a,3),(b,2)(b,3),(c,2)(c,3),(d,2)(d,3)\},$$

from which it can be confirmed that

$$\overline{G_1[G_2]} = \overline{G_1}[\overline{G_2}].$$

For any graph G, let $\beta(G)$ and $\omega(G)$ denote the independence number and clique number of G, respectively. In 1974, Geller and Stahl [6] proved that

$$\beta(G_1[G_2]) = \beta(G_1)\beta(G_2). \tag{1}$$

From Property (1) and the observation that $\overline{G_1[G_2]} = \overline{G_1}[\overline{G_2}]$, it follows that

$$\omega(G_1[G_2]) = \omega(G_1)\omega(G_2). \tag{2}$$

Properties (1) and (2) elucidate the utility of lexicographic products in Ramsey theory.

Define the t-color Ramsey number $R(k_1, k_2, \dots, k_t)$ to be the least natural number n such that every arbitrary coloring of the edges of K_n (the complete graph of order n) using t colors results in a subgraph isomorphic to K_{k_i} in some color i $(1 \le i \le t)$. In the case where $k_1 = k_2 = \cdots = k_t$, we write $R^t(k_1)$ for the corresponding t-color Ramsey number. In 1965, Abbott (Theorem 2.3.2, [1]) proved that

$$R^t(pq+1) > (R^t(p+1)-1)(R^t(q+1)-1).$$

In fact, the 2-color version of this result,

$$R(pq+1, pq+1) > (R(p+1, p+1)-1)(R(q+1, q+1)-1),$$

is easily seen to follow directly from properties (1) and (2) of the lexicographic product.

In this article, we will consider lexicographic products for r-uniform hypergraphs and show that they satisfy many similar properties to that of their 2-uniform counterparts (ie., graphs). Recall that an r-uniform hypergraph H consists of a nonempty set $V(H)$ of vertices and a set $E(H)$ of different unordered r-tuples of distinct vertices, called hyperedges. For an overview of hypergraph products and some of their basic properties, the reader is referred to [8]. Let H_1 and H_2 be r-uniform hypergraphs. Then the lexicographic product of H_1 and H_2, denoted $H_1[H_2]$, is the r-uniform hypergraph with vertex set $V(H_1) \times V(H_2)$ and hyperedge set

$$\left\{ (a_1,b_1)(a_2,b_2)\dots(a_r,b_r) \;\middle|\; \begin{array}{l} a_1a_2\dots a_r \in E(H_1) \text{ or} \\ a_1 = a_2 = \cdots = a_r \text{ and } b_1b_2\dots b_r \in E(H_2) \end{array} \right\}.$$

It should be noted that whenever we consider a hyperedge $a_1a_2\dots a_r$, in any r-uniform hypergraph, it is assumed that the a_i's are distinct. As with the case of graphs, it is easily confirmed that the lexicographic product for r-uniform hypergraphs is not commutative.

In Section 2, we consider the independence and clique numbers for lexicographic products in the setting of r-uniform hypergraphs. The primary hurdle when $r \geq 3$ is the fact that a clique in an r-uniform hypergraph may not correspond to a (strong) independent set in the hypergraph's complement. So, Properties (1) and (2) do not readily extend to this setting. Theorems 1-4 in Section 2 are analogues of well-known results on graphs, extended to r-uniform hypergraphs. In particular, we address the issue of determining clique numbers and apply these results in to proving a generalization of Abbott's Theorem in Section 3. We then conclude with several consequences of this result, including a new multicolor inequality for hypergraph Ramsey numbers (generalizing a result of Xiaodong, Zheng, Exoo, and Radziszowski [10]) and a proof that

$$R^t(5;3) \geq 81^{2^{t-2}} + 1,$$

where

$$R^t(5;3) := R(\underbrace{5, 5, \dots, 5}_{t\ copies}; 3)$$

is the t-colored 3-uniform Ramsey number. Our motivation for focusing on 3-uniform Ramsey numbers stems from the "Stepping-up" Lemma, usually attributed to Erdős and Hajnal (see [7]). This theorem gives powerful lower bounds for Ramsey numbers of arbitrary uniformity given 3-uniform bounds. However, the search for these bounds has been extremely difficult (cf. [3]).

2 Independent Sets and Cliques

For an r-uniform hypergraph H, a clique is a subset $K \subseteq V(H)$ of vertices such that the subhypergraph induced by K contains all possible hyperedges of H, i.e., all r-element subsets of K. Using this definition, every collection of $r-1$ (or fewer) vertices of K forms a clique. Equivalently, a subset of n vertices $\{x_1, x_2, \dots, x_n\}$ forms a clique if the induced subhypergraph contains $\binom{n}{r}$ hyperedges. Since $i\binom{n}{r} = 0$ whenever $n < r$, no hyperedges are needed to form a clique on $r-1$ (or fewer) vertices. The clique number $\omega(H)$ is then defined to be the cardinality of a maximal clique in H.

A (strong) independent set of vertices in an r-uniform hypergraph is a subset of nonadjacent vertices (no two vertices are included in a common hyperedge). Note that in Chapter 2, Section 4 of [2], such sets are called strongly stable sets. We denote the cardinality of a maximal independent set of vertices in H by $\beta(H)$. In the case of graphs, one has that $\beta(G) = \omega(\overline{G})$, but this property does not extend to higher values of r. Observe that for $r \geq 3$, an independent set of cardinality $n \geq r$ in H forms a clique in the complement $\overline{H}$, but the converse is not necessarily true.

Here, $\overline{H}$ has vertex set $V(\overline{H}) = V(H)$ and hyperedge set

$$E(\overline{H}) = \{a_1a_2 \dots a_r \mid a_1a_2 \dots a_r \notin E(H)\}.$$

For example, consider the 4-uniform hypergraph $K_5^{(4)} - e$ formed by removing a single hyperedge from the complete 4-uniform hypergraph $K_5^{(4)}$ of order 5. In this case, $\beta(K_5^{(4)} - e) = 1$ since every pair of vertices are adjacent. However, $\omega(\overline{K_5^{(4)} - e}) = 4$, corresponding to the single hyperedge. In general,

$$\beta(H) \leq \omega(\overline{H})$$

is the strongest general statement we can make for hypergraphs.

For each vertex $a \in V(H_1)$, define the set

$$a[H_2] = \{(a, b_1)(a, b_2) \dots (a, b_r) \mid b_1b_2 \dots b_r \in E(H_2)\},$$

which is isomorphic to H_2. Similarly, for $b \in V(H_2)$, define the set

$$H_1[b] = \{(a_1, b)(a_2, b) \dots (a_r, b) \mid a_1a_2 \dots ar \in E(H_1)\},$$

which is isomorphic to H_1. For $i = 1, 2$, the projection mappings

$$\text{proj}_i : H_1[H_2] \longrightarrow H_i$$

are defined by

$$\text{proj}_i(a, b) = \begin{cases} a & \text{if } i = 1 \\ b & \text{if } i = 2. \end{cases}$$

For $(a_1, b_1), (a_2, b_2) \in V(H_1[H_2])$, one can define the relation $(a_1, b_1) \equiv (a_2, b_2)$ if and only if

$$\text{proj}_1(a_1, b_1) = a_1 = a_2 = \text{proj}_1(a_2, b_2).$$

Using these notations, we prove the following property of independence numbers for $H_1[H_2]$, generalizing Theorem 1 from Geller and Stahl's paper [6].

Theorem 1. *For $r \geq 2$ and r-uniform hypergraphs H_1 and H_2,*

$$\beta(H_1[H_2]) = \beta(H_1)\beta(H_2).$$

Proof. Let S_1 be an independent set of H_1, S_2 be an independent set of H_2, and consider the Cartesian product $S_1 \times S_2$. From the definition of the lexicographic product, it is easily confirmed that $S_1 \times S_2$ is an independent set in $H_1[H_2]$ as the existence of a hyperedge $(a_1, b_1)(a_2, b_2) \dots (a_r, b_r)$ using vertices from $S_1 \times S_2$ would require that either $a_1a_2 \dots a_r$ is a hyperedge in H_1 or $b_1b_2 \dots b_r$ is a hyperedge in H_2. Thus, $H_1[H_2]$ contains an independent set with cardinality $\beta(H_1)\beta(H_2)$:

$$\beta(H_1[H_2]) \geq \beta(H_1)\beta(H_2).$$

Now, let S be an independent set in $H_1[H_2]$. The relation $\equiv$ defines a partition

$$S = S_1 \cup S_2 \cup \cdots \cup S_n,$$

where each S_i is an independent set of vertices all having the same projection under proj_1. So, $|S_i| \leq \beta(H_2)$ for each i. If we let a_i be the first common coordinate of the vertices in S_i, then $\{a_1, a_2, \ldots, a_n\}$ must be an independent set in H_1. Hence, $n \leq \beta(H_1)$. It follows that

$$\beta(H_1[H_2]) \leq \beta(H_1)\beta(H_2),$$

from which equality holds. □

Geller and Stahl [6] also used their result on independence numbers to imply an analogous result for vertex covers. For r-uniform hypergraphs with $r \geq 3$, the connection between the independence number and the vertex covering number is not as clear, leading us to consider a new definition. For $1 \leq k \leq r-1$, define a k-fold vertex cover of an r-uniform hypergraph H to be a set of vertices $S \subseteq V(H)$ such that every hyperedge in H includes at least k vertices from S. The k-fold vertex covering number $\alpha_k(H)$ is the minimal cardinality of a k-fold vertex cover. Note that α_1 is the usual vertex covering number. We obtain the following relationship between β and α_{r-1}.

Theorem 2. *If H is an r-uniform hypergraph and $I \subseteq V(H)$ is an independent set, then $V(H) - I$ is an $(r-1)$-fold vertex cover. Conversely, if $C \subseteq V(H)$ is an $(r-1)$-fold vertex cover, then $V(H) - C$ is an independent set.*

Proof. Let $I \subseteq V(H)$ be an independent set. Then every hyperedge in H contains at most one vertex from I. It follows that every hyperedge in H must contain at least $r-1$ vertices from $V(H)-I$. Conversely, suppose that $C \subseteq V(H)$. Then every hyperedge in H contains at least $r-1$ vertices from C. Thus, every hyperedge in H contains at most one vertex from $V(H) - C$. □

From this theorem, we see that for all r-uniform hypergraphs of order n,

$$\beta(H) = n - \alpha_{r-1}(H).$$

Thus, Theorem 1 implies that if n_1 and n_2 are the orders of H_1 and H_2, respectively, then

$$\begin{aligned}
\alpha_{r-1}(H_1[H_2]) &= n_1 n_2 - \beta(H_1[H_2]) \\
&= n_1 n_2 - \beta(H_1)\beta(H_2) \\
&= n_1 n_2 - (n_1 - \alpha_{r-1}(H_1))(n_2 - \alpha_{r-1}(H_2)) \\
&= n_1 \alpha_{r-1}(H_2) + n_2 \alpha_{r-1}(H_1) - \alpha_{r-1}(H_1)\alpha_{r-1}(H_2).
\end{aligned}$$

Now we turn our attention to clique numbers.

Theorem 3. *For $r \geq 3$, and r-uniform hypergraphs H_1 and H_2,*

$$\omega(H_1[H_2]) = \max(\omega(H_1), \omega(H_2)).$$

Proof. If $\{a_1, a_2, \ldots, a_m\}$ is a clique in H_1, then

$$\{(a_1, y), (a_2, y), \ldots, (a_m, y)\}$$

is a clique in $H_1[H_2]$ for any $y \in V(H_2)$. Similarly, if $\{b_1, b_2, \ldots, b_n\}$ is a clique in H_2, then

$$\{(x, b_1), (x, b_2), \ldots, (x, b_n)\}$$

is a clique in $H_1[H_2]$ for any $x \in V(H_1)$. Thus,

$$\omega(H_1[H_2]) \geq \max(\omega(H_1), \omega(H_2)).$$

To prove the other direction, suppose that

$$K = \{(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)\}$$

is a clique in $H_1[H_2]$. If all of the a_i are distinct, then $a_{i_1}a_{i_2}\ldots a_{i_r} \in E(H_1)$ for all distinct $i_1, i_2, \ldots i_r \in \{1, 2, \ldots, k\}$. In this case, we find that H_1 has a clique with cardinality k. Now consider the case in which at least two elements in K have the same projections under proj_1. If some other element K has a different projection, then there will have to be a hyperedge in $H_1[H_2]$ that contains these three elements, which cannot occur by the definition of the lexicographic product. Hence, all elements in K must have the same projection under proj_1 if any two do. Then $\{b_1, b_2, \ldots, b_k\}$ must form a clique in H_2, from which the inequality

$$\omega(H_1[H_2]) \leq \max(\omega(H_1), \omega(H_2))$$

follows. □

If we wish to make use of the clique numbers of lexicographic products of hypergraphs to prove Ramsey number results analogous to Abbott's work [1], we must also consider the clique number for $\overline{H_1[H_2]}$. From the definition, we find that the hyperedges

$$(a_1, b_1)(a_2, b_2)\ldots(a_r, b_r) \in E(\overline{H_1[H_2]})$$

can be partitioned into three classes:

- (A) r-edges such that $a_1a_2\ldots a_r \in E(\overline{H_1})$ and any b_j's, which may not be distinct;
- (B) r-edges such that $a_1 = a_2 = \cdots = a_r$ and $b_1b_2\ldots b_r \in E(\overline{H_2})$;
- (C) r-edges such that at least two a_i values are equal, but not all are equal, and any b_j's.

The r-edges in (A) and (B) are precisely those in $E(\overline{H_1[H_2]})$. From this description of $E(\overline{H_1[H_2]})$, we obtain the following theorem.

Theorem 4. *For $r \geq 3$ and r-uniform hypergraphs H_1 and H_2,*

$$\omega(\overline{H_1[H_2]}) = \omega(\overline{H_1})\omega(\overline{H_2}).$$

Proof. Let S_1 be a clique in $\overline{H_1}$ and S_2 be a clique in $\overline{H_2}$. From the three classes of hyperedges in $\overline{H_1[H_2]}$ listed above, one can verify that $S_1 \times S_2$ is a clique in $\overline{H_1[H_2]}$. Note that this observation is true even if S_1 or S_2 (or both) have cardinalities less than r. Hence,

$$\omega(\overline{H_1[H_2]}) \geq \omega(\overline{H_1})\omega(\overline{H_2}).$$

Now suppose S is a clique in $\overline{H_1[H_2]}$. Partition the elements in S into subsets based on the distinct values of their projections proj_2:

$$S = S_1 \cup S_2 \cup \cdots \cup S_m.$$

A fixed S_i has the form

$$S_i = \{(a_1, b_i), (a_2, b_i), \ldots, (a_n, b_i)\},$$

where $a_1, a_2, \ldots a_n$ are distinct. In order for any r-tuple of elements in S_i to form a hyperedge in $\overline{H_1[H_2]}$, they must be r-edges in class (A) listed above. Thus, $|S_i| \leq \omega(\overline{H_1})$ for all $1 \leq i \leq m$. Also, since S is a clique in $\overline{H_1[H_2]}$, we find that

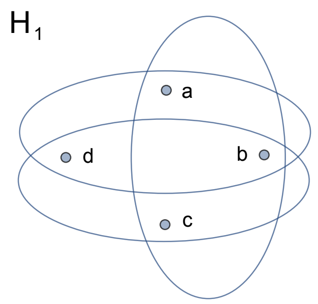
$$(a_1, b_{i_1})(a_1, b_{i_2}) \ldots (a_1, b_{i_r}) \in E(\overline{H_2}),$$

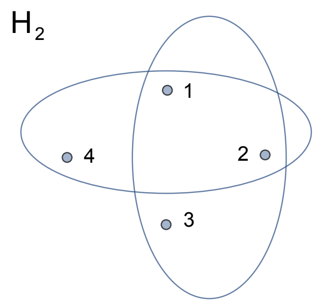
for any distinct $i_1, i_2, \ldots, i_r \in \{1, 2, \ldots, m\}$. So, S_2 forms a clique in $\overline{H_2}$. Thus, we have

$$\omega(\overline{H_1[H_2]}) \leq \omega(\overline{H_1})\omega(\overline{H_2}),$$

from which the theorem follows. □

As an example, consider the 3-uniform hypergraphs H_1 and H_2 given below.





The hyperedges contained in $H_1[H_2]$ fall into two subsets:

$$E_1 = \{(a_1, b_1)(a_2, b_2)(a_3, b_3) \mid a_1a_2a_3 \in E(H_1) \text{ and } b_1, b_2, b_3 \in V(H_2)\}$$

and

$$E_2 = \{(a, b_1)(a, b_2)(a, b_3) \mid a \in V(H_1) \text{ and } b_1b_2b_3 \in E(H_2)\}.$$

Hence, $|E_1| = 192$ and $|E_2| = 8$. We saw in the proof of Theorem 3 that if

$$K = \{(a_1, b_1), (a_2, b_2), \ldots, (a_r, b_r)\}$$

is a clique in $H_1[H_2]$, then either all of the a_i are distinct (in which case, $K \subseteq E_1$) or $a_1 = a_2 = \cdots = a_r$ (in which case, $K \subseteq E_2$). In this example, we find that E_1 and E_2 both have clique number 3, and hence, $\omega(H_1[H_2]) = 3$.

This example becomes more interesting when we consider the complement $\overline{H_1[H_2]}$. As we saw before Theorem 4, the hyperedges in $\overline{H_1[H_2]}$ fall into three classes. The hyperedges that fall into classes (A), (B), and (C) form the sets

$$E_{(A)} = \{(a, y_1)(c, y_2)(d, y_3) \mid y_1, y_2, y_3 \in \{1, 2, 3, 4\}\}$$

$$E_{(B)} = \{(x, 1)(x, 2)(x, 3), (x, 1)(x, 2)(x, 4) \mid x \in \{a, b, c, d\}\},$$

and

$$E_{(C)} = \{(x, z_1)(x, z_2)(y, z_3) \mid x \neq y \text{ and } z_1 \neq z_2\},$$

respectively. The cardinalities of these three disjoint sets are

$$|E_{(A)}| = 64, \quad |E_{(B)}| = 8, \quad \text{and} \quad |E_{(C)}| = 288.$$

The hyperedges in $E_{(A)} \cup E_{(B)}$ are precisely those contained in $\overline{H_1}[\overline{H_2}]$. Here, we find that $\omega(\overline{H_1}) = 3 = \omega(\overline{H_2})$ and one example of a maximal clique in $\overline{H_1[H_2]}$ is given by

$$\{(b, 2), (b, 3), (b, 4), (c, 2), (c, 3), (c, 4), (d, 2), (d, 3), (d, 4)\}.$$

3 Applications to Ramsey Numbers

Using Theorems 3 and 4, we obtain the following generalization of Abbott's Theorem for r-uniform hypergraphs. Note that there is no reason to restrict ourselves to "diagonal" multicolor Ramsey numbers as Abbott did. Analogous to the definition of a Ramsey number for graphs, define the r-uniform t-color hypergraph Ramsey number $R(k_1, k_2, \ldots, k_t; r)$ to be the least natural number n such that every t-coloring of the r-edges in $K_n^{(r)}$ results in a monochromatic $K_{k_i}^{(r)}$-subhypergraph in color i for some color $i \in \{1, 2, \ldots, t\}$.

Theorem 5. *If* $r \geq 3$, *then*

$$R(\max(p_1, q_1) + 1, \max(p_2, q_2) + 1, \ldots, \max(p_{t-1}, q_{t-1}) + 1, p_tq_t + 1; r) \geq$$

$$(R(p_1 + 1, p_2 + 1, \ldots, p_t + 1; r) - 1)(R(q_1 + 1, q_2 + 1, \ldots, q_t + 1; r) - 1) + 1,$$

where each p_i *and* q_i *is* $\geq r - 1$.

Proof. Let $m = R(p_1 + 1, p_2 + 1 \ldots, p_t + 1; r)$ and $n = R(q_1 + 1, q_2 + 1, \ldots, q_t + 1; r)$. Then there exists a t-coloring $\mathcal{C}_1$ of the hyperedges in $K_{m-1}^{(r)}$ that has a maximum clique of order p_i in color i for every $i \in \{1, 2, \ldots, t\}$. Similarly, there exists a t-coloring $\mathcal{C}_2$ of the hyperedges in $K_{n-1}^{(r)}$ that has a maximum clique of order q_i in color i for every $i \in \{1, 2, \ldots, t\}$. Identify the vertices in $K_{(m-1)(n-1)}^{(r)}$ with the Cartesian product $V(K_{m-1}^{(r)}) \times V(K_{n-1}^{(r)})$. We will construct a t-coloring $\mathcal{C}$ of the hyperedges in $K_{(m-1)(n-1)}^{(r)}$ as follows. For any $j \in \{1, 2, \ldots, t-1\}$, if either $a_1 a_2 \ldots a_r \in E(K_{m-1}^{(r)})$ has color j in $\mathcal{C}_1$ or if $a_1 = a_2 = \cdots = a_r$ and $b_1 b_2 \ldots b_r \in E(K_{n-1}^{(r)})$ has color j in $\mathcal{C}_2$, then we assign the hyperedge

$$(a_1, b_1)(a_2, b_2) \ldots (a_r, b_r)$$

color j. If neither of these two conditions are satisfied, then assign it color t. For each $j \in \{1, 2, \ldots, t-1\}$, define H_j to be the subhypergraph of $K_{m-1}^{(r)}$ spanned by the hyperedges of color j in $\mathcal{C}_1$ and define H'_j to be the subhypergraph of $K_{n-1}^{(r)}$ spanned by the hyperedges of color j in $\mathcal{C}_2$. Then by definition, the subhypergraph of $K_{(m-1)(n-1)}^{(r)}$ spanned by the hyperedges of color j in $\mathcal{C}$ is isomorphic to $H_j[H'_j]$, and hence, has a maximum clique of order $max(p_j, q_j)$ by Theorem 3. Now define

$$H = \bigcup_{j \in S} H_j \qquad \text{and} \qquad H' = \bigcup_{j \in S} H'_j,$$

where $S = \{1, 2, \ldots, t-1\}$. By our construction, the subhypergraph of $K_{(m-1)(n-1)}^{(r)}$ spanned by the hyperedges using colors in S is isomorphic to $H[H']$. It follows that the subhypergraph of $K_{(m-1)(n-1)}^{(r)}$ spanned by the hyperedges of color t in $\mathcal{C}$ is isomorphic to $\overline{H[H']}$ and has a maximum clique of order $p_t q_t$ by Theorem 4. Thus, it follows that

$$R(\max(p_1, q_1)+1, \max(p_2, q_2)+1, \ldots, \max(p_{t-1}, q_{t-1})+1, p_t q_t+1; r) > (m-1)(n-1),$$

resulting in the statement of the theorem. □

Using the observation that

$$R(k_1, k_2, \ldots, k_i; r) = R(k_1, k_2, \ldots, k_i, \underbrace{r, r, \ldots, r}_{t-i \text{ copies}}; r)$$

and

$$R(k_{i+1}, k_{i+2}, \ldots, k_t; r) = R(\underbrace{r, r, \ldots, r}_{i \text{ copies}}, k_{i+1}, k_{i+2}, \ldots, k_t; r),$$

one obtains the following corollary.

Corollary 6. *If $r \geq 3$ and each $k_j \geq r$, then $R(k_1, k_2, \ldots, (k_t - 1)(r-1) + 1; r) \geq (R(k_1, k_2, \ldots, k_i; r) - 1)(R(k_{i+1}, k_{i+2}, \ldots, k_t; r) - 1) + 1$.*

This corollary can be viewed as a generalization of the multicolor Ramsey inequality for graphs proved by Xiaodong, Zheng, Exoo, and Radziszowski in Theorem 2 of [10]. Note that this generalization is different than the one given in Theorem 3.7 of [4] and it also differs from the conjectured generalization described in the conclusion of [4].

In the special case where $t = 2$, $p = p_1 + 1 = q_1 + 1$, and $r = p_2 + 1 = q_2 + 1$, and using the observation that $R(p, r; r) = p$, Theorem 5 implies the following corollary to Theorem 5.

Corollary 7. *If $r \geq 3$ and $p \geq r$, then*

$$R(p, (r-1)^2 + 1; r) \geq (p-1)^2 + 1.$$

As another application of Theorem 5, we prove the following lower bounds for diagonal t-colored hypergraph Ramsey numbers:

$$R^t(n; r) := R(\underbrace{n, n, \ldots, n}_{t \text{ colors}}; r).$$

Theorem 8. *Let $r \geq 3$, $t \geq 2$, and $n = (r-1)^2 + 1$. If $R(n, n; r) \geq m$, then*

$$R^t(n; r) \geq (m-1)^{2^{t-2}} + 1.$$

Proof. When $t = 2$, the theorem is trivial. Now we proceed by induction on $t \geq 2$. Assume the theorem is true for $t = k$:

$$R(n, n; r) \geq m \quad \Longrightarrow \quad R^k(n; r) \geq (m-1)^{2^{k-2}} + 1.$$

Then the assumption that $n = (r-1)^2 + 1$ along with Theorem 5 implies that

$$\begin{aligned} R^{k+1}(n; r) &\geq (R(\underbrace{n, n, \ldots, n}_{k \text{ copies}}, r; r) - 1)^2 + 1 \\ &\geq (R^k(n; r) - 1)^2 + 1 \\ &\geq ((m-1)^{2^{k-2}} + 1 - 1)^2 + 1 \\ &\geq (m-1)^{2^{(k+1)-2}} + 1, \end{aligned}$$

completing the proof of the theorem. □

For example, it is known that $R(5, 5; 3) \geq 82$ (Exoo [5]), from which Corollary 8 implies that

$$R^3(5; 3) \geq 81^2 + 1 = 6,562.$$

This improves the previous best known lower bound for this Ramsey number from 163, as was proven in [4]. In fact, this one lower bound offered by Exoo results in all of the following bounds:

$$\begin{aligned}
R^4(5;3) &\geq 43,046,722 \\
R^5(5;3) &\geq 1,853,020,188,851,842 \\
R^6(5;3) &\geq 3,433,683,820,292,512,484,657,849,089,282 \\
&\vdots \\
R^t(5;3) &\geq 81^{2^{t-2}} + 1.
\end{aligned}$$

These bounds offer a significant improvement over the bounds given in Proposition 3.8 of [4].

References

[1] H. Abbott, *Some Problems in Combinatorial Analysis*, Ph.D. thesis, Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, 1965.

[2] C. Berge, *Hypergraphs*, North-Holland Mathematical Library, Elsevier Science Publishers, 1989.

[3] M. Budden, J. Hiller, J. Lambert and C. Sanford, The Lifting of Graphs to 3-Uniform Hypergraphs and Some Applications to Hypergraph Ramsey Theory, *Involve* 10 (2017), 65–76.

[4] M. Budden, J. Hiller and A. Rapp, Generalized Ramsey Theorems for r-Uniform Hypergraphs, *Australas. J. Combin.* 63 (2015), 142–152.

[5] G. Exoo, Indiana State University, Cited by Radziszowski [9], Construction available at `http://ginger.indstate.edu/ge/RAMSEY`.

[6] G. Geller and S. Stahl, The Chromatic Number and Other Functions of the Lexicographic Product, *J. Combin. Theory Ser. B* 19 (1975), 87–95.

[7] R. Graham, B. Rothschild and J. Spencer, *Ramsey Theory,* second ed., John Wiley & Sons, 1990.

[8] M. Hellmuth, L. Ostermeier and P. Stadler, A Survey on Hypergraph Products, *Math. Comput. Sci.* 6 (2012), 1–32.

[9] S. Radziszowski, Small Ramsey Numbers — Revision 15 *Electron. J. Combin.* Vol. DS1.15 (2017), 1–104.

[10] X. Xiaodong, X. Zheng, G. Exoo and S. Radziszowski, Constructive Lower Bounds on Classical Multicolor Ramsey Numbers, *Electron. J. Combin.* 11 (1) (2004), #R35.

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