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The product of the total restrained domination numbers of a graph and its complement

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Abstract

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a total restrained dominating set if every vertex is adjacent to a vertex in S, and every vertex in $V - S$ is adjacent to a vertex in $V - S$. The total restrained domination number of G, denoted by $\gamma_{tr}(G)$, is the smallest cardinality of a total restrained dominating set of G. In this paper we show that if G is a graph of order $n \geq 4$, then $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 4n$. We also characterize the graphs achieving the upper bound.

1 Introduction

For notation and graph theory terminology, we generally follow [5]. Specifically, let $G = (V, E)$ be a graph of order n with vertex set V and edge set E. For a set $S \subseteq V$, the *subgraph induced by* S in G is denoted by $\langle S \rangle$. If H is a subgraph of G, then $G-H$ will denote the induced graph $\langle V(G) - V(H) \rangle$. The *minimum degree* (respectively, *maximum degree*) among the vertices of G is denoted by $\delta(G)$ (respectively, $\Delta(G)$).

If $v \in V$, then the *open neighborhood of v in G* is defined as $N_G(v) = \{x \in V - \{v\} \mid x$ is adjacent to v in $G\}$, while the *closed neighborhood of v in G* is given by $N_G[v] = N_G(v) \cup \{v\}$. A degree one vertex of a graph G will be referred to as a *leaf*, while a degree zero vertex of G will be referred to as an *isolate*.

A set $S \subseteq V$ is a *dominating set* of G, denoted by **DS**, if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a **DS**. The concept of domination in graphs, with its many

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variations, is now well studied in graph theory. A thorough study of domination appears in [5, 6].

A set $S \subseteq V$ is a *restrained set* if every vertex in $V - S$ is adjacent to a vertex in $V - S$. A **DS** $S \subseteq V$ is a *restrained dominating set*, denoted by **RDS**, if S is also a restrained set. Every graph has a restrained dominating set, since $S = V$ is such a set. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a **RDS** of G.

A **DS** $S \subseteq V$ is a *total dominating set*, denoted by **TDS**, if every vertex in S is adjacent to a vertex in S. Every graph without isolated vertices has a total dominating set, since $S = V$ is such a set. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a **TDS** of G.

A **RDS** $S \subseteq V$ is a *total restrained dominating set*, denoted by **TRDS**, if S is a **TDS**. Every graph without isolated vertices has a total restrained dominating set, since $S = V$ is such a set. The *total restrained domination number* of G, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a **TRDS** of G. Total restrained domination was introduced by Telle and Proskurowski [11], although indirectly, as a vertex partitioning problem and further studied, for example, in [3, 2, 7, 12].

Nordhaus and Gaddum presented best possible bounds on the sum and product of the chromatic number of a graph and its complement in [10]. Bounding the sum and product of the domination number of a graph and its complement were investigated by Jaeger and Payan, in [8]: If G is a graph of order $n \geq 2$, then $\gamma(G) + \gamma(\overline{G}) \leq n + 1$ and $\gamma(G)\gamma(\overline{G}) \leq n$. Furthermore, these problems were also examined for the restrained domination number, and these results appear in [1, 3, 4].

Define K as the graph obtained by joining an isolated vertex to the vertices of degree two of a P_4. It is shown in [3] that if G is a graph of order $n \geq 2$ such that neither G nor $\overline{G}$ contains isolated vertices or is isomorphic to K, then $4 \leq \gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 4$. Extremal graphs G of order n achieving these two bounds are also characterized.

The aim of this paper is to bound the product of the total restrained domination numbers of a graph and its complement. We show that if $n \geq 4$, and neither G nor $\overline{G}$ contains isolated vertices or is isomorphic to K, then $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 4n$. We also characterize the graphs achieving the upper bound.

The following two results will prove to be useful in the proof of our main result.

Theorem 1.1 *[7] Let G be a connected graph with $\delta \geq 2$ and order $n \geq 4$. Then $\gamma_{tr}(G) \leq n - \frac{\Delta}{2} - 1$.*

Theorem 1.2 *[2] Let G be a connected graph with $3 \leq \delta \leq n - 2$. Then $\gamma_{tr}(G) \leq n - \delta$.*

2 Preliminary Results

Let $\mathcal{L}$ be the class of all graphs constructed in the following way: Let u and v be two distinct isolates and consider the complete graph K_n, where $n = 2$ or $n \geq 4$. Let u' and v' be two distinct vertices of K_n. Join u to u', and join v to v'. Recall the definition of the graph K. In order to prove our main result, we will first prove a sequence of necessary lemmas.

Lemma 2.1 *If $G \in \mathcal{L}$, then $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) = 4n$.*

Proof. Let $G \in \mathcal{L}$ and let S be any **TRDS** of G of cardinality $\gamma_{tr}(G)$. Then, as u and v are leaves of G, adjacent to u' and v', respectively, $\{u, v, u', v'\} \subseteq S$, whence $\gamma_{tr}(G) = |S| \geq 4$. Moreover, $\{u, v, u', v'\}$ is a **TRDS** of G, whence $\gamma_{tr}(G) = 4$.

Let S be any **TRDS** of $\overline{G}$ of cardinality $\gamma_{tr}(\overline{G})$. Then, as u' and v' are leaves of $\overline{G}$, adjacent to v' and u', respectively, we have $\{u, v, u', v'\} \subseteq S$. Moreover, $\langle V(G) - \{u, v, u', v'\}\rangle_{\overline{G}}$ only contains isolated vertices, whence $V(G) - \{u, v, u', v'\} \subseteq S$, and so $|S| \geq n$. It now follows that $\gamma_{tr}(\overline{G}) = n$, whence $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) = 4n$. □

Lemma 2.2 *Suppose $n \geq 4$ and neither G nor $\overline{G}$ contains isolated vertices. If* $\text{diam}(G) = \text{diam}(\overline{G}) = 2$, *then $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) < 4n$.*

Proof. Suppose $n \geq 4$, neither G nor $\overline{G}$ contains isolated vertices and $\text{diam}(G) = \text{diam}(\overline{G}) = 2$. Let $\delta^* = \min\{\delta(G), \delta(\overline{G})\}$ and $\delta^{**} = \max\{\delta(G), \delta(\overline{G})\}$. As $\text{diam}(G) = \text{diam}(\overline{G}) = 2$, both G and $\overline{G}$ are connected, and $n \geq 5$.

Let u (u', respectively) be a vertex of G ($\overline{G}$, respectively) such that $\deg_G(u) = \delta(G)$ ($\deg_{\overline{G}}(u') = \delta(\overline{G})$, respectively). Suppose $\delta^* = 1$. Without loss of generality, assume $\delta(G) = 1$, and let v be adjacent to u in G. As $\text{diam}(G) = 2$, we have $N_G(v) = V - \{v\}$, and so v is isolated in $\overline{G}$, which is a contradiction. Thus, $\delta^* \geq 2$.

Let $X = V - N_G[u]$, $X' = V - N_{\overline{G}}[u']$, $T_0 = N_G(u)$ and $T_0' = N_{\overline{G}}(u')$. If $X = \emptyset$, then u is isolated in $\overline{G}$, which is a contradiction. We conclude that $X \neq \emptyset$. Similarly, $X' \neq \emptyset$. As $\text{diam}(G) = \text{diam}(\overline{G}) = 2$, sets T_0 and T_0' dominate X and X' respectively.

To complete the proof of Lemma 2.2, we will prove a sequence of claims. We will eventually show that there exists an integer $k \geq 1$, such that $n \geq k^2 + 2k + 3$, whence $4n \geq 4k^2 + 8k + 12$. Then we will show that $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq 2(2k+1) = 4k+2$. Since the sum is then bounded, we can, using calculus, bound the product $\gamma_{tr}(G)\gamma_{tr}(\overline{G})$, and deduce that $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 4k^2 + 4k + 1 < 4n$.

Claim 2.1 *If $\delta^* \leq 3$, then $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) < 4n$.*

Proof. Assume that $\delta^* \leq 3$. Without loss of generality, assume that $\delta^* = \delta(G)$. Hence, $\Delta(\overline{G}) = n - \delta(G) - 1 \geq n - 4$. So, by Theorem 1.1, $\gamma_{tr}(\overline{G}) \leq n - \frac{\Delta(\overline{G})}{2} - 1 \leq \frac{n}{2} + 1$.

Let $U = \{x \in X \mid N_G(x) = N_G(u)\}$. If $U = \emptyset$, then $N_G[u]$ is a **TRDS** of G and so $\gamma_{tr}(G) \leq 4$. As $\Delta(\overline{G}) \geq \delta(\overline{G}) \geq 2$, we have, by Theorem 1.1, $\gamma_{tr}(\overline{G}) \leq n-2$, whence $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) < 4n$. Hence, $U \neq \emptyset$. ◻

Claim A. If $\gamma_{tr}(G) \leq 6$, then $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) < 4n$.

Proof. Suppose $\gamma_{tr}(G) \leq 6$. Then $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 6(\frac{n}{2}+1) = 3n+6$. If $n \geq 7$, then $3n+6 < 4n$, and so $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) < 4n$. We therefore assume that $n = 5$ or $n = 6$.

First consider the case when $\delta(G) = 3$. It follows easily that $1 \leq |U| \leq 2$, whence $\gamma_{tr}(G) = 2$, and so $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 2(\frac{n}{2}+1) = n+2 < 4n$.

Next consider the case when $\delta(G) = 2$. If $|U| \geq 2$, then it is clear that $\gamma_{tr}(G) = 2$, whence $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 2(\frac{n}{2}+1) = n+2 < 4n$. It follows that $|U| = 1$. As $\delta(G) \geq 2$, we have that $|X - U| \geq 2$, and so $n = 6$. Let $X - U = \{x, y\}$. Note that x and y cannot be adjacent to a common neighbor in G that lies in $N_G(u)$. Let $x' \in N_G(u)$ be adjacent to x. The set $\{y, x', x\}$ is a **TRDS** of G, hence $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 3n < 4n$. ◻

If $|U| \leq 2$, then $U \cup N_G[u]$ is a **TRDS** of G of cardinality at most 6, and the result follows from Claim A. Hence $|U| \geq 3$.

Let x be an arbitrary vertex in $N_G(u)$. If the set $S = N_G[u] - \{x\}$ is a **DS** of G, then S is a **TRDS** of G, whence $\gamma_{tr}(G) \leq 3$, a contradiction. It follows that there exists a vertex $y \in X - U$, such that y and x are adjacent in G and $N_G(u) \cap N_G(y) = \{x\}$.

If x dominates X in G, then $\{x, u\}$ is a **TRDS** of G, and so $\gamma_{tr}(G) = 2$, a contradiction. Thus x does not dominate X in G, and so there exists a vertex $z \in X - U - \{y\}$, such that x is not adjacent to z in G.

Let $y' \in U$. The set $S = \{y, z, y'\}$ is a **TDS** of $\overline{G}$. If S is a **TRDS** of $\overline{G}$, then $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 3n < 4n$. Note that if $t \in U - \{y'\}$, then every vertex of $X \cup \{u\} - S - \{t\}$ is adjacent to t in $\overline{G}$. Thus, there exists a vertex $x' \in N_G(u)$ such that x' is adjacent in G to every vertex of $V - S$. The set $S' = S \cup \{x'\}$ is a **TDS** of $\overline{G}$. If S' is a **TRDS** of $\overline{G}$, then, by Theorem 1.1, $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 4(n-2) < 4n$. Hence, there exists a vertex $x'' \in N_G(u) - \{x'\}$ such that x'' is adjacent in G to every vertex of $V - S - \{x'\}$.

Suppose first that $x = x''$. Since $\deg(u) = \delta(G)$, vertex z is adjacent to a vertex $z' \in X - U - \{z\}$ in G. As x' is adjacent to every vertex of $V - \{x', x, y, z, y'\}$, the set $\{x, y, z, z'\}$ is a **TRDS** of G, and the result follows from Claim A. Hence $x \neq x''$, and by a similar argument, $x \neq x'$. It immediately now follows that $\delta(G) = 3$.

If y is adjacent to z in G, then $\{x, x', y\}$ is a **TRDS** of G, and the result follows from Claim A. As $\delta(G) \geq 3$, the vertex z is adjacent to a vertex $z' \in X - U - \{z, y\}$. If z is adjacent to x' in G, then $\{x, x'\}$ is a **TRDS** of G and the result follows as before. If z is not adjacent to x' in G, then $\{x, x', z'\}$ is a **TRDS** of G, and the result follows as before. This completes the proof of our claim. ◻

By Claim 2.1, $\delta^* \geq 4$.

As in [9], for an arbitrary graph G, let S_0 be the largest subset of T_0 that does not dominate X. Let $T_1 = T_0 - S_0$. By the maximality of S_0, every vertex of T_1 dominates $X - N(S_0)$, but T_1 may or may not dominate X. Note that if $S_0 = \emptyset$, then $\gamma_{tr}(G) \leq 2$ and so $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 2n < 4n$. We continue, constructing sets $T_0, T_1, \ldots, T_k$ with $T_0 \supset T_1 \supset \ldots \supset T_k$ (where $k \geq 1$) and sets $S_0, \ldots, S_{k-1}$ such that

1. for $i < k$, the set T_i dominates X.
2. for $i < k$, the set S_i is the largest subset of T_i that does not dominate X, and $T_{i+1} = T_i - S_i$.
3. T_k does not dominate X.

Since T_i dominates X but S_i does not (when $i < k$), all of $T_0, \ldots, T_k$ (and $S_0, \ldots, S_{k-1}$) are nonempty.

Analogously, for the graph $\overline{G}$, construct sets $T'_0, T'_1, \ldots, T'_\ell$ with $T'_0 \supset T'_1 \supset \ldots \supset T'_\ell$ (where $\ell \geq 1$) and sets $S'_0, \ldots, S'_{\ell-1}$ such that

1. for $i < \ell$, the set T'_i dominates X'.
2. for $i < \ell$, the set S'_i is the largest subset of T'_i that does not dominate X', and $T'_{i+1} = T'_i - S'_i$.
3. T'_ℓ does not dominate X'.

Again, $T'_i \neq \emptyset$ for $i = 0, \ldots, \ell$, while $S'_i \neq \emptyset$ for $i = 0, \ldots, \ell - 1$.

Claim 2.2 *$\gamma_{tr}(G)\gamma_{tr}(\overline{G}) < 4n$ or for $i < k$ ($i < \ell$, respectively) we have $\gamma_{tr}(G) \leq |S_i| + 2$ ($\gamma_{tr}(\overline{G}) \leq |S'_i| + 2$, respectively).*

Proof. Without loss of generality, consider the graph G and the set S_i and recall that every vertex in T_{i+1} dominates $U = X - N_G(S_i)$. Let $W = N_G(u) - S_i$, let $y \in T_{i+1}$, and let $S = \{u, y\} \cup S_i$. Obviously S is a **TDS** of G and has cardinality $|S_i| + 2$. Observe that if y dominates X, then, since $\delta^* \geq 4$, we have that $\{y, u\}$ is a **TRDS** of G, and so $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 2n < 4n$. We may therefore assume that y does not dominate X.

Case 1. $|W| \geq 3$.

Then $3 \leq |W| = \delta(G) - |S_i|$, and so $|S| = |S_i| + 2 \leq \delta(G) - 1$. Every vertex in X has at most $\delta(G) - 2$ neighbors in S, while every vertex in W has at most $\delta(G) - 1$ neighbors in S. It follows that S is a **TRDS** of G and so $\gamma_{tr}(G) \leq |S_i| + 2$.

Case 2. $|W| = 2$.

We show that S is a restrained set of G, since then the conclusion will follow. Suppose, to the contrary, that S is not restrained. As $|S_i| = \delta(G) - 2$, each vertex in X

has at most $\delta(G) - 1$ neighbors in S, and so there exists a vertex in $x \in W - \{y\}$ such that $N_G(x) = \{y, u\} \cup S_i$.

Consider the set $S' = S_i \cup \{x\}$. Note that S' is a restrained set. If S' is a **DS** of G, then S' is a **TRDS** of G, and so $\gamma_{tr}(G) \leq |S_i| + 1$. So, suppose S' is not a **DS** of G, and let $z \in X$ be a vertex which is not adjacent to any of the vertices in $S_i \cup \{x\}$. As $\text{diam}(G) = 2$, z is adjacent to y in G. Since y does not dominate X, there is a vertex $z' \in X - \{z\}$ that is not adjacent to y in G. As $\delta(\overline{G}) \geq 4$, the set $\{z', z, u\}$ is a **TRDS** of $\overline{G}$, whence $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 3n < 4n$.

Case 3. $W = \{y\}$.

We show that S is a restrained set of G, since then the conclusion will follow. Suppose, to the contrary, that S is not restrained. Thus there exists a vertex x in X such that $N_G(x) = N_G(u)$. Let $z \in X - N_G(S_i)$. Then y is adjacent to z in G, and z dominates S_i in $\overline{G}$. Let $z' \in X - \{z, x\}$, such that z' is not adjacent to y in G. As $\delta(\overline{G}) \geq \delta^* \geq 4$, the set $\{x, z', z\}$ is a **TRDS** of $\overline{G}$, whence $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 3n < 4n$. ▫

For $i = 0, 1, \ldots, k-1$, let x_i be a vertex of X that is not dominated by S_i, and let x_k be a vertex that is not dominated by T_k.

Let $0 < i \leq k-1$ and let $0 \leq j < i$. We show that $x_i \neq x_j$. Note that x_j is adjacent to every vertex of T_{j+1}. As $S_i \subseteq T_{j+1}$, vertex x_j is adjacent to every vertex of S_i. But x_i is non-adjacent to every vertex of S_i, and so $x_i \neq x_j$. A similar argument shows that $x_i \neq x_k$ for $i = 0, \ldots, k-1$. Let $U = \cup_{i=0}^{k}\{x_i\}$. Then $|U| = k + 1$.

Similarly, for $i = 0, 1, \ldots, \ell - 1$, let x'_i be a vertex of X that is not dominated by S'_i, let x'_ℓ be a vertex that is not dominated by T'_ℓ, and let $U' = \cup_{i=0}^{\ell}\{x'_i\}$. Then $|U'| = \ell + 1$.

We say that a vertex $v \in N_G(u) \cup (X - U)$ has **Property P** if either $v \in N_G(u)$ and $N_G(v) \supseteq N_G[u] \cup (X - U) - \{v\}$ or $v \in X - U$ and $N_G(v) \supseteq N_G(u) \cup (X - U) - \{v\}$. A similar property is described for a vertex $v \in N_{\overline{G}}(u') \cup (X' - U')$.

Claim 2.3 *If G ($\overline{G}$, respectively) has no vertices with* ***Property P****, then $\gamma_{tr}(\overline{G}) \leq k + 2$ ($\gamma_{tr}(G) \leq \ell + 2$, respectively).*

Proof. Suppose G has no vertices with **Property P**. Given the non-existence of vertices with **Property P**, the set $S = U \cup \{u\}$ is a **TRDS** of $\overline{G}$, whence $\gamma_{tr}(\overline{G}) \leq k + 2$ ▫

Claim 2.4 *If G or $\overline{G}$ has no vertices with* ***Property P****, then $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq \delta^{**} - (\gamma_{tr}(G) - 3)(\gamma_{tr}(\overline{G}) - 3) + 4$.*

Proof. Assume, without loss of generality, that G has no vertices with **Property P**. Observe that $|S_0| = \delta(G) - |T_k| - \sum_{i=1}^{k-1} |S_i|$. By Claims 2.2 and 2.3, we have

$$\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq (|S_0| + 2) + (k + 2)$$

$$
\begin{aligned}
&= \delta(G) - |T_k| - \sum_{i=1}^{k-1} |S_i| + k + 4 \\
&\leq \delta(G) - 1 - \sum_{i=1}^{k-1} (\gamma_{tr}(G) - 2) + k + 4 \\
&= \delta(G) + (k-1) - (k-1)(\gamma_{tr}(G) - 2) + 4 \\
&= \delta(G) - (k-1)(\gamma_{tr}(G) - 3) + 4 \\
&\leq \delta^{**} - (\gamma_{tr}(G) - 3)(\gamma_{tr}(\overline{G}) - 3) + 4. \qquad \square
\end{aligned}
$$

Assume G does not have any vertices with **Property P**. By Claim 2.4, we have that $\gamma_{tr}(G)+\gamma_{tr}(\overline{G}) \leq \delta^{**}+4-(\gamma_{tr}(G)-3)(\gamma_{tr}(\overline{G})-3)$ and it follows that $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq \delta^{**} + 2(\gamma_{tr}(G) + \gamma_{tr}(\overline{G})) - 5$. Hence, as $\delta^* \geq 4$, we have (cf. Theorem 1.2) that

$$\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq \delta^{**} + 4n - 2(\delta(G) + \delta(\overline{G})) - 5 < 4n.$$

We may assume, henceforth, that both G and $\overline{G}$ must have vertices with **Property P**.

Claim 2.5 *If $\gamma_{tr}(\overline{G}) \leq \ell + 2$ ($\gamma_{tr}(G) \leq k + 2$, respectively), then $2\gamma_{tr}(\overline{G}) \leq \delta(\overline{G}) + 4 - (\gamma_{tr}(\overline{G}) - 3)^2$ ($2\gamma_{tr}(G) \leq \delta(G) + 4 - (\gamma_{tr}(G) - 3)^2$, respectively).*

Proof. Consider the graph G and assume $\gamma_{tr}(G) \leq k + 2$. By Claim 2.2 we have that $\gamma_{tr}(G) \leq |S_0| + 2$. Observe that $|S_0| = \delta(G) - |T_k| - \sum_{i=1}^{k-1} |S_i|$. Following the proof of Claim 2.4, we obtain $2\gamma_{tr}(G) \leq \delta(G) + 4 - (\gamma_{tr}(G) - 3)^2$. Similarly, one establishes that $2\gamma_{tr}(\overline{G}) \leq \delta(\overline{G}) + 4 - (\gamma_{tr}(\overline{G}) - 3)^2$. $\square$

Without loss of generality, we may assume that $\gamma_{tr}(\overline{G}) \leq \gamma_{tr}(G)$.

Claim 2.6 *For all $i < k$, $|S_i| \geq k + 1$.*

Proof. Assume that for some $i < k$, we have that $|S_i| \leq k$. By Claim 2.2, $\gamma_{tr}(G) \leq |S_i| + 2 \leq k + 2$. Applying Theorem 1.2 and Claim 2.5, we obtain

$$
\begin{aligned}
\gamma_{tr}(G)\gamma_{tr}(\overline{G}) &\leq \gamma_{tr}(G)^2 \\
&\leq \delta(G) - 5 + 4\gamma_{tr}(G) \\
&\leq \delta(G) + 4(n - \delta(G)) - 5 < 4n. \qquad \square
\end{aligned}
$$

By Claim 2.6, we have that $\delta(G) = |T_k| + \sum_{i=0}^{k-1} |S_i| \geq k(k+1) + 1$, whence $n = |N_G[u]| + |U| + |X - U| \geq k(k+1) + 2 + k + 1$.

Claim 2.7 *The graph G has a **TRDS** of cardinality at most $2k + 1$.*

Proof. Let x be the vertex of G that has **Property P** and observe that either $x \in N_G(u)$ or $x \in X - U$. Consider the vertices x_0 and x_1. For each vertex x_i, where $2 \leq i \leq k$, consider a neighbor of x_i in G, say y_i. Let $W = \cup_{i=2}^{k}\{x_i, y_i\}$.

Assume first that x_0 and x_1 are adjacent. As $\text{diam}(G) = 2$, x_0 and u have a common neighbor in G, say y. Since the vertex x dominates $N_G(u) \cup X - U$, we have that if $y \neq x$, then x is adjacent to y. Let $S = W \cup \{x, x_0, y\}$, and observe that $|S| \leq |W| + 3 = 2k + 1$. It is clear that S is a **TDS** of G. If $k = 1$ then $|S| = 3$, and so, since $\delta^* \geq 4$, S is a **TRDS** of G. If $k \geq 2$ then $\delta(G) \geq k(k+1) + 1 > 2k + 1$, whence S is a **TRDS** of G.

We may assume, without loss of generality, that the set U is independent in G. As $\text{diam}(G) = 2$, x_0 and x_1 have a common neighbor in G, say y, and $y \in N_G(u) \cup X - U$. Since x dominates $N_G(u) \cup X - U$, recall that if $y \neq x$, then x is adjacent to y. Let $x' \in N_G(u) - \{x\}$ be a vertex adjacent to x in G, and let $S = W \cup \{y, x, x'\}$. Note that $|S| \leq |W| + 3 = 2k + 1$. It is clear that S is a **TDS** of G. If $k = 1$, then, since $\delta^* \geq 4$, S is a **TRDS** of G. If $k \geq 2$, then $\delta(G) \geq k(k+1) + 1 > 2k + 1$, and so S is a **TRDS** of G. □

Claim 2.8 *If $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq 4k + 2$, then $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 4k^2 + 4k + 1$.*

Proof. This follows from the fact that $ab \leq \frac{(a+b)^2}{4}$, where a and b are non-negative real numbers. □

To complete the proof of Lemma 2.2, recall that $n \geq k^2 + 2k + 3$, whence $4n \geq 4k^2 + 8k + 12$. By Claim 2.7, we have $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq \gamma_{tr}(G) + \gamma_{tr}(G) \leq 2(2k+1) = 4k+2$, and so (cf. Claim 2.8) $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 4k^2 + 4k + 1 < 4n$.□

Lemma 2.3 *Let G be a graph of order $n \geq 4$ and assume that neither G nor $\overline{G}$ has any isolates. If G or $\overline{G}$ is disconnected, then $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) < 4n$.*

Proof. Assume G is disconnected, $n \geq 4$ and neither G nor $\overline{G}$ has any isolates. Let $G_1, G_2, \ldots, G_k$ be the components of G, and observe that each component has order at least two. Let $x \in V(G_1)$ and $y \in V(G_2)$. The set $\{x, y\}$ is a **TRDS** of $\overline{G}$, whence $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 2n < 4n$. □

Lemma 2.4 *Let G be a graph of order $n \geq 4$ such that G and $\overline{G}$ are both connected, and neither G nor $\overline{G}$ has isolates or is isomorphic to K. If $\delta(G) = 1$ or $\delta(\overline{G}) = 1$, then $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) < 4n$, or, either $G \in \mathcal{L}$ or $\overline{G} \in \mathcal{L}$.*

Proof. Let G be a graph of order $n \geq 4$ such that G and $\overline{G}$ are both connected, and neither G nor $\overline{G}$ has isolates or is isomorphic to K.

Case A. G has at least two leaves.

Let u and v be leaves of G and let u' (v', respectively) be the neighbor of u (v, respectively) in G. Let us assume first that $u' = v'$. Since $\overline{G}$ has no isolates, there exists a vertex $x \in V(G) - N_G[u']$. The set $\{x, u, u'\}$ is a **TRDS** of $\overline{G}$, and so $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 3n < 4n$.

We therefore assume that $u' \neq v'$. If $V(G) - \{u, u', v, v'\} = \emptyset$, then $G \cong P_4 \in \mathcal{L}$ for which $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) = 4n$. We henceforth assume that $V(G) - \{u, u', v, v'\} \neq \emptyset$, and so $n \geq 5$.

If u' and v' are not adjacent, then the set $\{u, v'\}$ is a **TRDS** of $\overline{G}$, whence $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 2n < 4n$. We may assume that u' and v' are adjacent.

Assume that there exists a vertex $x \in V(G) - \{u, v, u', v'\}$ such that x and u' are not adjacent in G. The set $\{u, v', x\}$ is a **TRDS** of $\overline{G}$ and so $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 3n < 4n$. Hence, in G, u' is adjacent to every vertex in $V(G) - \{u, v, u', v'\}$, and, by symmetry, in G, v' is adjacent to every vertex in $V(G) - \{u, v, u', v'\}$.

Observe that $\deg_{\overline{G}}(u') = \deg_{\overline{G}}(v') = 1$, so every **TRDS** of $\overline{G}$ must contain u, u', v and v'. Also note that u is adjacent to v in $\overline{G}$.

Consider the set $S = \{u, u', v', v\}$. As $G \ncong K$, $|V - S| \geq 2$. Suppose S is a **TRDS** of G. Then $\gamma_{tr}(G) = 4$ and so $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 4n$. If $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) < 4n$, we are done. So suppose $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) = 4n$, whence $\gamma_{tr}(\overline{G}) = n$. If two distinct vertices x and y in $V - S$ are non-adjacent in G, then $V - \{x, y\}$ is a **TRDS** of $\overline{G}$, and so $\gamma_{tr}(\overline{G}) \leq n - 2$, which is a contradiction. Thus, $V - S$ forms a clique in G, and so $G \in \mathcal{L}$.

Thus, we may assume that S is not a **TRDS** of G, and, similarly, that S is not a **TRDS** of $\overline{G}$. As S is a **TDS** of both G and $\overline{G}$, there exist vertices x and y in $V - S$ such that in G, x is not adjacent to every vertex of $V - S - \{x\}$ and y is adjacent to every vertex of $V - S - \{y\}$. We conclude that $x \neq y$ and that x and y are both adjacent and non-adjacent in G, which is a contradiction.

Case B. G has exactly one leaf.

We may therefore assume that G has exactly one leaf, say u. Consider a path $u, u_1, \ldots, u_{\mathrm{ecc}(u)}$ of G. Define V_i, $i = 0, \ldots, \mathrm{ecc}(u)$, as the set of vertices at distance i from u. If $\mathrm{ecc}(u) = 2$, then $\overline{G}$ is disconnected, which is a contradiction. So assume that $\mathrm{ecc}(u) \geq 3$. If $\mathrm{ecc}(u) \geq 5$, then $\{u, u_{\mathrm{ecc}(u)}\}$ is a **TRDS** of $\overline{G}$, whence $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 2n < 4n$. For convenience, set $x = u_1$ and $z = u_3$. Observe that the set $V(G) - \{x, z\}$ is a **TRDS** of $\overline{G}$, and so $\gamma_{tr}(\overline{G}) \leq n - 2$.

First consider the case when $\mathrm{ecc}(u) = 4$. If $S = \{u, x, z\}$ is a **TRDS** of $\overline{G}$, then $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 3n < 4n$. We therefore assume that S is not a **TRDS** of $\overline{G}$. As S is a **TDS** of $\overline{G}$ and every vertex of V_2 is non-adjacent to every vertex of V_4 in $\overline{G}$, there exists a vertex $y \in V_3 - \{z\}$ which is adjacent to every vertex of $V_2 \cup V_3 - \{z\} \cup V_4$ in G. If $\{x, u, y\}$ is a **TRDS** of $\overline{G}$, then $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 3n < 4n$. So assume $\{x, u, y\}$ is not a **TRDS** of $\overline{G}$, and, as before, there exists a vertex $y' \in V_3 - \{y\}$ which is adjacent to every vertex of $V_2 \cup V_3 - \{y\} \cup V_4$ in G. If $y' \neq z$, let $S' = \{u, x, y', z\}$, while if $y' = z$, let $S' = \{u, x, u_2, z\}$. Then S' is a **TRDS** of G and so $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 4(n-2) < 4n$.

Next consider the case when $\mathrm{ecc}(u) = 3$. Suppose $V_3 = \{z\}$. Then, as before, $S = \{u, x, z\}$ is not a **TRDS** of $\overline{G}$, and there exists a vertex $y \in V_2$ which is adjacent to every vertex of $V_2 - \{y\}$ in G. Since u is the only leaf of G, we can pick a vertex $y' \in N_G(z) - \{y\}$. The set $\{u, y', x\}$ is a **TRDS** of G and so $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 3n < 4n$. It follows that $|V_3| \geq 2$.

As S is not a **TRDS** of $\overline{G}$, there exists a vertex $y \in V_2 \cup V_3$ which is adjacent to every vertex of $V_2 \cup V_3 - \{y, z\}$ in G.

Subcase i. $y \in V_3$.

If $\{x, u, y\}$ is a **TRDS** of $\overline{G}$, then $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 3n < 4n$. Thus, $\{x, u, y\}$ is not a **TRDS** of $\overline{G}$, and so there exists a vertex $y' \in V_2 \cup V_3$ which is adjacent to every vertex of $V_2 \cup V_3 - \{y, y'\}$ in G. If $y' \in V_3 - \{z, y\}$, then $\{u, x, y', z\}$ is a **TRDS** of G and we have that $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 4(n-2) < 4n$. If $y' \in V_2$, then $\{u, x, y'\}$ is a **TRDS** of G, whence $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 3n < 4n$. We may assume that $y' = z$, and let $y'' \in V_2$. Let $S' = \{u, x, y'', y'\}$. If $V_3 - \{z, y\} \neq \emptyset$, then S' is a **TRDS** of G. If $V_3 - \{z, y\} = \emptyset$, then $\{u, x, y''\}$ is a **TRDS** of G. Hence, in both cases, $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) < 4n$.

Subcase ii. $y \in V_2$.

Assume first that y is adjacent to z. The set $\{u, x, y\}$ is a **TDS** of G and if it is a **TRDS** of G, then $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) < 4n$. Hence, as u is the only leaf of G, we have that every vertex in V_3 must have degree at least two in G, and so there is a vertex $y' \in V_2$ such that $N_G(y') = \{x, y\}$. The set $\{z, y, u\}$ is a **TRDS** of $\overline{G}$, whence $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) < 4n$.

We may assume that y is not adjacent to z. Let $z' \in V_3 - \{z\}$. By considering the set $\{u, x, z'\}$, there is a vertex $y' \in V_2 \cup V_3$ such that $N_G(y') \supseteq V_2 \cup V_3 - \{y', z'\}$. Similar to what have been shown for the set $\{u, x, z\}$ above, we have $y' \in V_2$ with y' not adjacent to z' in G. The facts that $z' \neq z$ and y' is adjacent to z imply that $y \neq y'$. The set $S = \{u, x, y, y'\}$ is a **TDS** of G, and if it is a **TRDS** of G, then $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 4(n-2) < 4n$.

We may assume that S is not a **TRDS** of G. Hence, there exists a vertex $v \notin S$ such that either $v \in V_2$ and $N_G(v) \subseteq \{x, y, y'\}$ or $v \in V_3$ and $N_G(v) \subseteq \{y, y'\}$. If $v \in V_3$, then as u is the only leaf of G, we have $N_G(v) = \{y, y'\}$, whence $v \notin \{z, z', y, y'\}$. Thus, in both cases, $v \notin \{z, z', y, y'\}$. The set $\{u, y, z\}$ is a **TRDS** of $\overline{G}$, whence $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 3n < 4n$. □

Lemma 2.5 *Let G be a connected graph of order $n \geq 4$ with $\min\{\delta(G), \delta(\overline{G})\} \geq 2$. Moreover, suppose $\overline{G}$ is connected. Then $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) < 4n$.*

Proof. Let G be a connected graph of order $n \geq 4$ with $\min\{\delta(G), \delta(\overline{G})\} \geq 2$. Moreover, suppose $\overline{G}$ is connected. We may assume, by Lemma 2.2, that $\mathrm{diam}(G) \geq 3$ or $\mathrm{diam}(\overline{G}) \geq 3$. Without loss of generality, suppose $\mathrm{diam}(G) \geq 3$, and consider a diametrical path $u = u_0, u_1, \ldots, u_{\mathrm{diam}(G)} = v$ of G. Define V_i, $i = 0, \ldots, \mathrm{diam}(G)$, as the set of vertices at distance i from u. It is easy to see that if $\mathrm{diam}(G) \geq 4$, then the set $\{u, x\}$ for $x \in V_3$ is a **TRDS** of $\overline{G}$ and so $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 2n$. Thus, we may assume that $\mathrm{diam}(G) = 3$.

Consider the case where $V_3 = \{v\}$. Obviously, $|V_1| \geq 2$ ($|V_2| \geq 2$, respectively), since otherwise u (v, respectively) is a leaf of G. The set $\{u, v\}$ is a **TDS** of $\overline{G}$. If it is a **TRDS** of $\overline{G}$, then $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 2n$. So suppose $\{u, v\}$ is not a **RDS** of $\overline{G}$. Then there exists a vertex $y \in V_1 \cup V_2$ such that $N_G(y) \supseteq V_1 \cup V_2 - \{y\}$.

Suppose there is a vertex $y' \in V_1 \cup V_2 - \{y\}$ such that $N_G(y') \supseteq V_1 \cup V_2 - \{y, y'\}$. Let $z \in N_G(v)$ with $z \in V_2 - \{y'\}$, and let $z' \in N_G(u)$ with $z' \in V_1 - \{y'\}$. If $y \in V_1$, then $\{y, u, z\}$ is a **TRDS** of G. If $y \in V_2$, then $\{z, y, z'\}$ is a **TRDS** of G. Hence $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 3n$.

Assume, therefore, that each vertex $y' \in V_1 \cup V_2 - \{y\}$ is adjacent to a vertex of $V_1 \cup V_2 - \{y, y'\}$ in $\overline{G}$. But then $\{y, u, v\}$ is a **TRDS** of $\overline{G}$ and so $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 3n$.

Now suppose $V_3 \supset \{v\}$. We may assume that the **TDS** $\{u, v\}$ of $\overline{G}$ is not a **RDS** of $\overline{G}$. Hence, there exists a vertex $y \in V_2$ such that $N_G(y) \supseteq V_1 \cup V_2 \cup V_3 - \{v, y\}$. Now suppose that there is a vertex $y' \in V_2 - \{y\}$ such that $N_G(y') \supseteq V_1 \cup V_2 \cup V_3 - \{v, y'\}$. Let $z \in N_G(v) - \{y'\}$. The set $\{y, u_1, z\}$ is a **TRDS** of G and so $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 3n$. Hence, $\{y, u, v\}$ is a **TRDS** of $\overline{G}$ and so $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 3n$. □

3 Main Result

We are now ready to state and prove our main result:

Theorem 3.1 *Let G be a graph of order $n \geq 4$, and suppose neither G nor $\overline{G}$ contains isolated vertices or is isomorphic to K. Then $\gamma_{tr}(G)\gamma_{tr}(\overline{G}) \leq 4n$ with equality holding if and only if either $G \in \mathcal{L}$ or $\overline{G} \in \mathcal{L}$.*

Proof. Let G be a graph of order $n \geq 4$, such that G has no isolates and is not isomorphic to K. By Lemmas 2.3 and 2.4, both G and $\overline{G}$ must be connected and $\min\{\delta(G), \delta(\overline{G})\} \geq 2$. Our result now follows from Lemma 2.5. By Lemmas 2.1, 2.2, 2.3, 2.4 and 2.5 equality holds if and only if $G \in \mathcal{L}$ or $\overline{G} \in \mathcal{L}$. □

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