AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **70(2)** (2018), Pages 288–296

Singly even self-dual codes constructed from Hadamard matrices of order 28

Masaaki Harada   Ken Saito

Research Center for Pure and Applied Mathematics
Graduate School of Information Sciences
Tohoku University, Sendai 980–8579
Japan
mharada@m.tohoku.ac.jp   kensaito@ims.is.tohoku.ac.jp

Abstract

We give a classification of singly even self-dual $[56, 28, d]$ codes with $d \in \{10, 12\}$ constructed from Hadamard matrices of order 28.

1 Introduction

A (binary) $[n, k]$ *code* C is a k-dimensional vector subspace of $\mathbb{F}_2^n$, where $\mathbb{F}_2$ is the field of 2 elements. All codes in this note are binary. The parameter n is called the *length* of C. The elements of C are called *codewords* and the *weight* $\mathrm{wt}(x)$ of a codeword x is the number of non-zero coordinates. An $[n, k, d]$ code is an $[n, k]$ code with minimum (non-zero) weight d. The *dual* code $C^\perp$ of C is defined as $C^\perp = \{x \in \mathbb{F}_2^n \mid x \cdot y = 0 \text{ for all } y \in C\}$ under the standard inner product $x \cdot y$. A code C is *self-dual* if $C = C^\perp$. A code is *doubly even* if all codewords have weight divisible by four, and *singly even* if all weights are even and there is at least one codeword x with $\mathrm{wt}(x) \equiv 2 \pmod 4$. A doubly even self-dual code of length n exists if and only if $n \equiv 0 \pmod 8$, while a singly even self-dual code of length n exists if and only if n is even. The minimum weight $d(C)$ of a self-dual code C of length n is bounded by $d(C) \leq 4\lfloor n/24 \rfloor + 4$ unless $n \equiv 22 \pmod{24}$ when $d(C) \leq 4\lfloor n/24 \rfloor + 6$ [8]. A self-dual code meeting the upper bound is called *extremal*.

A *Hadamard* matrix H of order n is an $n \times n$ $(1, -1)$-matrix such that $HH^T = nI_n$, where H^T is the transpose of H and I_n is the identity matrix of order n. It is known that the order n is necessarily $1, 2$, or a multiple of four. Two Hadamard matrices H and H' are said to be *equivalent* if there are $(0, \pm 1)$-monomial matrices P, Q with $H' = PHQ$. All Hadamard matrices of orders up to 32 were classified (see [6] for order 28 and [5] for order 32).

Two codes C and C' are *equivalent* if one can be obtained from the other by permuting the coordinates. It is a fundamental problem to classify self-dual codes of modest length and determine the largest minimum weight among self-dual codes of that length. For example, the largest minimum weight among singly even self-dual codes of length 56 is 10 or 12 [3]. In this note, we give a partial classification of singly even self-dual $[56, 28, d]$ codes with $d \in \{10, 12\}$. Extremal doubly even self-dual codes of lengths 40 and 56 constructed from Hadamard matrices of orders 20 and 28 were classified in [2] and [7], respectively (see Proposition 2 for the construction). Extremal singly even self-dual codes of length 40 constructed from Hadamard matrices of order 20 were classified in [4] (see Proposition 3 for the construction). The aim of this note is to demonstrate the following theorem.

Theorem 1. *There is no extremal singly even self-dual code of length* 56 *constructed from Hadamard matrices of order* 28*. There are* 944 *inequivalent singly even self-dual* $[56, 28, 10]$ *codes constructed from Hadamard matrices of order* 28*.*

All computer calculations in this note were done with the help of MAGMA [1].

2 Weight enumerators of singly even self-dual $[56, 28, 10]$ codes

In this section, we give the possible weight enumerators of singly even self-dual $[56, 28, 10]$ codes.

Let C be a singly even self-dual code of length n. Let C_0 denote the doubly even subcode of C. The *shadow* S of C is defined to be $C_0^{\perp} \setminus C$. Let A_i and B_i be the numbers of vectors of weight i in C and S, respectively. The weight enumerators W_C and W_S of C and S are given by $\sum_{i=0}^{n} A_i y^i$ and $\sum_{i=d(S)}^{n-d(S)} B_i y^i$, respectively, where $d(S)$ denotes the minimum weight of S. If we write

$$W_C = \sum_{j=0}^{\lfloor n/8 \rfloor} a_j (1 + y^2)^{n/2-4j} (y^2(1 - y^2)^2)^j,$$

for suitable integers a_j, then

$$W_S = \sum_{j=0}^{\lfloor n/8 \rfloor} (-1)^j a_j 2^{n/2-6j} y^{n/2-4j} (1 - y^4)^{2j},$$

[3, (10), (11)].

Suppose that C is a singly even self-dual $[56, 28, 10]$ code. Since the minimum weight is 10, we have

$$a_0 = 1, a_1 = -28, a_2 = 238, a_3 = -672, a_4 = 525.$$

Then the weight enumerator of the shadow S is written as:

$$\begin{aligned}&\frac{-a_7}{16384}+\left(\frac{7a_7}{8192}+\frac{a_6}{256}\right)y^4+\left(\frac{-91a_7}{16384}-\frac{3a_6}{64}-\frac{a_5}{4}\right)y^8\\&+\left(8400+\frac{91a_7}{4096}+\frac{33a_6}{128}+\frac{5a_5}{2}\right)y^{12}\\&+\left(620928-\frac{1001a_7}{16384}-\frac{55a_6}{64}-\frac{45a_5}{4}\right)y^{16}+\cdots.\end{aligned}$$

Since the shadow contains no zero-vector, $a_7 = 0$.

- Suppose that $d(S) = 4$.
  This gives that $a_6 = 256$ and a_5 is divisible by four, say $a_5 = 4\alpha$. Therefore, the possible weight enumerators of C and S are as follows:

  $$\begin{aligned}W^C_{56,1} =&1+(308+4\alpha)y^{10}+(4246-8\alpha)y^{12}+(40852-28\alpha)y^{14}+\cdots,\\W^S_{56,1} =&y^4+(-12-\alpha)y^8+(8466+10\alpha)y^{12}+(620708-45\alpha)y^{16}+\cdots,\end{aligned}$$

  respectively, where α is an integer.

- Suppose that $d(S) \geq 8$.
  This gives that $a_6 = 0$ and a_5 is divisible by four, say $a_5 = 4\alpha$. Therefore, the possible weight enumerators of C and S are as follows:

  $$\begin{aligned}W^C_{56,2} =&1+(308+4\alpha)y^{10}+(3990-8\alpha)y^{12}+(42900-28\alpha)y^{14}+\cdots,\\W^S_{56,2} =&-\alpha y^8+(8400+10\alpha)y^{12}+(620928-45\alpha)y^{16}+\cdots,\end{aligned}$$

  respectively, where α is an integer.

3 Self-dual codes from Hadamard matrices

In this section, we review methods for constructing self-dual codes from Hadamard matrices. Then we give a condition that singly even self-dual codes of length 56 constructed from Hadamard matrices of order 28 have minimum weight at least 10.

A *generator matrix* of a code C is a matrix whose rows generate C. Let J_n denote the $n \times n$ all-one matrix and let $\mathbf{1}_n$ denote the all-one vector of length n. Let M be an $n \times n$ $(1,-1)$-matrix and let x be a $(1,-1)$-vector of length n. Throughout this note, we regard $\overline{M} = (M + J_n)/2$ as a binary matrix and we regard $\overline{x} = (x + \mathbf{1}_n)/2$ as a binary vector.

Proposition 2 (Tonchev [9]). *Let H be a Hadamard matrix of order $n \equiv 4 \pmod 8$ such that the number of coordinates equal to $+1$ in each row and column is congruent to $3 \pmod 4$. Then the code with generator matrix $\begin{pmatrix} I_n & \overline{H} \end{pmatrix}$ is a doubly even self-dual code of length $2n$.*

Proposition 3 (Harada and Tonchev [4]). *Let H be a Hadamard matrix of order $n \equiv 4 \pmod 8$ such that the number of coordinates equal to $+1$ in each row and column is congruent to $1 \pmod 4$. Then the code with generator matrix $\begin{pmatrix} I_n & \overline{H} \end{pmatrix}$ is a singly even self-dual code of length $2n$.*

Starting from a particular Hadamard matrix, one can transform it into many different Hadamard matrices by negating rows and columns such that the number of coordinates equal to $+1$ in each row and column is congruent to $3 \pmod 4$ (resp. $1 \pmod 4$). Hence, doubly even (resp. singly even) self-dual codes are constructed from Hadamard matrices by Proposition 2 (resp. Proposition 3). Extremal doubly even self-dual codes of lengths 40 and 56 constructed by Proposition 2 from Hadamard matrices of orders 20 and 28 were classified in [2] and [7], respectively. Extremal singly even self-dual codes of length 40 constructed by Proposition 3 from Hadamard matrices of order 20 were classified in [4]. In this note, we give a classification of singly even self-dual $[56, 28, d]$ codes with $d \in \{10, 12\}$ constructed by Proposition 3 from Hadamard matrices of order 28.

Without loss of generality, we may assume that the particular Hadamard matrix of order n has the following form

$$H = \begin{pmatrix} -1 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & D & \\ 1 & & & \end{pmatrix}. \tag{1}$$

Let $H^3(\Gamma, \Lambda)$ and $H^1(\Gamma, \Lambda)$ be the Hadamard matrices obtained from H in form (1) by negating columns $i \in \Gamma \subset \{1, 2, \dots, n\}$ and negating rows $j \in \Lambda \subset \{1, 2, \dots, n\}$ such that the number of coordinates equal to $+1$ in each row and column is congruent to $3 \pmod 4$ and $1 \pmod 4$, respectively. It is easy to see that $|\Gamma|$ and $|\Lambda|$ are even. We remark that Λ is uniquely determined for a given set Γ. We often denote the matrices $H^3(\Gamma, \Lambda)$ and $H^1(\Gamma, \Lambda)$ by $H^3(\Gamma)$ and $H^1(\Gamma)$, respectively, without listing Λ. Let $C^3(H, \Gamma, \Lambda)$ and $C^1(H, \Gamma, \Lambda)$ denote the doubly even self-dual code and the singly even self-dual code constructed by Propositions 2 and 3 from $H^3(\Gamma, \Lambda)$ and $H^1(\Gamma, \Lambda)$, respectively. Similar to the matrices $H^3(\Gamma, \Lambda)$ and $H^1(\Gamma, \Lambda)$, we often denote the codes $C^3(H, \Gamma, \Lambda)$ and $C^1(H, \Gamma, \Lambda)$ by $C^3(H, \Gamma)$ and $C^1(H, \Gamma)$, respectively, without listing Λ.

From now on, we assume that $n = 28$. Throughout this note, X denotes the set $\{1, 2, \dots, 28\}$. We say that the set of four rows of a Hadamard matrix of order 28 is a *Hall set* if the four rows can be converted to the following form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \mathbf{1}_6 & \mathbf{1}_6 & \mathbf{1}_6 & \mathbf{1}_6 \\ 1 & 1 & -1 & -1 & \mathbf{1}_6 & \mathbf{1}_6 & -\mathbf{1}_6 & -\mathbf{1}_6 \\ 1 & -1 & 1 & -1 & \mathbf{1}_6 & -\mathbf{1}_6 & \mathbf{1}_6 & -\mathbf{1}_6 \\ 1 & -1 & -1 & 1 & -\mathbf{1}_6 & \mathbf{1}_6 & \mathbf{1}_6 & -\mathbf{1}_6 \end{pmatrix},$$

by permuting and negating rows and columns.

Proposition 4. *Let C be a singly even self-dual code of length 56 constructed by Proposition 3 from a Hadamard matrix H of order 28. If the minimum weight of C is at least 10, then H is equivalent to H_1 or H_2 given in [7, p. 160].*

Proof. Let H be a Hadamard matrix of order 28 in form (1). Let Γ be a subset of X such that $|\Gamma|$ is even. Let r_i be the i-th row of $\overline{H}$. Let r_i^3 and r_i^1 be the i-th rows of $\overline{H^3(\Gamma)}$ and $\overline{H^1(\Gamma)}$, respectively. Let x_Γ be the vector of $\mathbb{F}_2^{28}$ whose support is Γ. Then we have

$$\begin{aligned} r_i^3 =& r_i + x_\Gamma + (x_\Gamma \cdot r_i + \gamma)\mathbf{1}_{28}, \\ r_i^1 =& r_i + x_\Gamma + (x_\Gamma \cdot r_i + \gamma + 1)\mathbf{1}_{28}, \end{aligned}$$

where $\gamma = 0$ if $|\Gamma| \equiv 0 \pmod 4$ and $\gamma = 1$ otherwise. Hence, for $A \subset X$ such that $|A|$ is even, we obtain

$$\sum_{i \in A} r_i^3 = \sum_{i \in A} r_i^1. \tag{2}$$

Let $\{r_{i_1}, r_{i_2}, r_{i_3}, r_{i_4}\}$ be a Hall set of H. Then $\overline{r_{i_1}} + \overline{r_{i_2}} + \overline{r_{i_3}} + \overline{r_{i_4}}$ has weight 4 or 24 [7]. If $C^3(H, \Gamma, \Lambda)$ is extremal, that is, it has minimum weight 12, then we obtain [7]

$$|\Lambda \cap \{i_1, i_2, i_3, i_4\}| = \begin{cases} 1 \text{ or } 3 & \text{if } \mathrm{wt}(\overline{r_{i_1}} + \overline{r_{i_2}} + \overline{r_{i_3}} + \overline{r_{i_4}}) = 4, \\ 0, 2 \text{ or } 4 & \text{if } \mathrm{wt}(\overline{r_{i_1}} + \overline{r_{i_2}} + \overline{r_{i_3}} + \overline{r_{i_4}}) = 24. \end{cases} \tag{3}$$

In addition, Kimura [7] showed that there is no (Γ, Λ) satisfying (3) for all Hadamard matrices H such that H is not equivalent to H_1 and H_2 given in [7, p. 160]. In other words, if H is not equivalent to H_1 and H_2 in [7], then $C^3(H, \Gamma)$ has a codeword of weight 8 for all Γ and H. Hence, by (2), if H is not equivalent to H_1 and H_2 in [7], then $C^1(H, \Gamma)$ has a codeword of weight 8 for all Γ and H. This completes the proof. □

4 Singly even self-dual $[56, 28, d]$ codes with $d \in \{10, 12\}$ from Hadamard matrices

In this section, we give a classification of singly even self-dual $[56, 28, d]$ codes with $d \in \{10, 12\}$ constructed by Proposition 3 from Hadamard matrices of order 28. By Proposition 4, it is sufficient to consider only Hadamard matrices H_1 and H_2 given in [7, p. 160].

4.1 Case H_2 in [7]

Suppose that H is a Hadamard matrix in form (1) obtained from H_2 given in [7, p. 160] by negating the first row and the i-th columns ($i = 2, 3, \ldots, 28$). Let Γ be a subset of X. Let $\{r_{i_1}, r_{i_2}, r_{i_3}, r_{i_4}\}$ be a set of four distinct rows of $H^1(\Gamma)$. Our exhaustive search shows that the only sets

$$\begin{aligned}\Gamma_1 =&\{1,2,3,4,5,6,7,8,9,10\},\\ \Gamma_2 =&\{1,11,12,13,14,15,16,17,18,19\},\\ \Gamma_3 =&X \setminus \Gamma_1,\\ \Gamma_4 =&X \setminus \Gamma_2\end{aligned}$$

are the subsets Γ satisfying $\mathrm{wt}(\overline{r_{i_1}} + \overline{r_{i_2}} + \overline{r_{i_3}} + \overline{r_{i_4}}) \neq 4$ for all i_1, i_2, i_3, i_4 with $1 \leq i_1 < i_2 < i_3 < i_4 \leq 28$. We verified that the four singly even self-dual codes $C^1(H, \Gamma_k)$ $(k = 1, 2, 3, 4)$ have the following weight enumerator

$$1 + 3y^6 + 197y^{10} + 4446y^{12} + 44361y^{14} + 309738y^{16} + 1577375y^{18} + \cdots .$$

Therefore, there is no singly even self-dual $[56, 28, d]$ code with $d \in \{10, 12\}$ constructed by Proposition 3. In addition, we verified that these codes are equivalent.

4.2 Case H_1 in [7]

Suppose that H is a Hadamard matrix in form (1) obtained from H_1 given in [7, p. 160] by negating the first row and the i-th columns $(i = 2, 3, \ldots, 28)$.

Lemma 5. *The codes $C^1(H, \Gamma, \Lambda)$ and $C^1(H, X \setminus \Gamma, X \setminus \Lambda)$ are the same.*

Proof. It follows from the fact that $H^1(\Gamma, \Lambda) = H^1(X \setminus \Gamma, X \setminus \Lambda)$. □

Hence, without loss of generality, we may assume that Γ satisfies the condition

$$|\Gamma| \leq 14. \tag{4}$$

Although the following proposition is somewhat trivial, we give a proof for the sake of completeness.

Proposition 6. *Suppose that $C^1(H, \Gamma, \Lambda)$ has minimum weight at least 10 satisfying (4).*

(i) *If $1 \in \Gamma$ and $1 \in \Lambda$, then*

$$(|\Gamma|, |\Lambda|) \in \{(a, b) \mid a \in \{10, 14\}, b \in \{10, 14, 18, 22, 26\}\}.$$

(ii) *If $1 \in \Gamma$ and $1 \notin \Lambda$, then*

$$(|\Gamma|, |\Lambda|) \in \{(a, b) \mid a \in \{4, 8, 12\}, b \in \{8, 12, 16, 20, 24\}\}.$$

(iii) *If $1 \notin \Gamma$ and $1 \in \Lambda$, then*

$$(|\Gamma|, |\Lambda|) \in \{(a, b) \mid a \in \{8, 12\}, b \in \{4, 8, 12, 16, 20\}\}.$$

(iv) *If* $1 \notin \Gamma$ *and* $1 \notin \Lambda$, *then*

$$(|\Gamma|, |\Lambda|) \in \{(a, b) \mid a \in \{2, 6, 10, 14\}, b \in \{2, 6, 10, 14, 18\}\}.$$

Proof. Since $C^1(H, \Gamma, \Lambda)$ are self-dual,

$$\begin{pmatrix} I_{28} & \overline{H(\Gamma, \Lambda)} \end{pmatrix} \text{ and } \begin{pmatrix} \overline{H(\Gamma, \Lambda)}^T & I_{28} \end{pmatrix}$$

are generator matrices of $C^1(H, \Gamma, \Lambda)$. Since $C^1(H, \Gamma, \Lambda)$ has minimum weight at least 10, the weight of every row of $\overline{H^1(\Gamma, \Lambda)}$ and $\overline{H^1(\Gamma, \Lambda)}^T$ is at least 9.

(i) Suppose that $1 \in \Gamma$ and $1 \in \Lambda$. Then the weights of the first rows of $\overline{H^1(\Gamma, \Lambda)}$ and $\overline{H^1(\Gamma, \Lambda)}^T$ are $|\Gamma| - 1$ and $|\Lambda| - 1$, respectively. Hence, we have that $|\Gamma| \geq 10$, $|\Lambda| \geq 10$ and $|\Gamma| \equiv |\Lambda| \equiv 2 \pmod 4$. The result follows.

(ii) Suppose that $1 \in \Gamma$ and $1 \notin \Lambda$. Then the weight of the first row of $\overline{H^1(\Gamma, \Lambda)}$ is $29 - |\Gamma|$. Hence, we have that $|\Gamma| \leq 20$ and $|\Gamma| \equiv 0 \pmod 4$. The weight of the first row of $H^1(\Gamma, \Lambda)^T$ is $1 + |\Lambda|$. Hence, we have that $|\Lambda| \geq 8$ and $|\Lambda| \equiv 0 \pmod 4$. The result follows.

(iii) Suppose that $1 \notin \Gamma$ and $1 \in \Lambda$. Then the weight of the first row of $\overline{H^1(\Gamma, \Lambda)}$ is $|\Gamma| + 1$. Hence, we have that $|\Gamma| \geq 8$ and $|\Gamma| \equiv 0 \pmod 4$. The weight of the first row of $\overline{H^1(\Gamma, \Lambda)}^T$ is $29 - |\Lambda|$. Hence, we have that $|\Lambda| \leq 20$ and $|\Lambda| \equiv 0 \pmod 4$. The result follows.

(iv) Suppose that $1 \notin \Gamma$ and $1 \notin \Lambda$. Then the weights of the first rows of $\overline{H^1(\Gamma, \Lambda)}$ and $\overline{H^1(\Gamma, \Lambda)}^T$ are $27 - |\Gamma|$ and $27 - |\Lambda|$, respectively. Hence, we have that $|\Gamma| \leq 18$, $|\Lambda| \leq 18$ and $|\Gamma| \equiv |\Lambda| \equiv 2 \pmod 4$. The result follows.

This completes the proof. □

By considering all subsets Γ satisfying the conditions given in Proposition 6, we found all distinct singly even self-dual $[56, 28, d]$ codes with $d \in \{10, 12\}$ constructed from H_1. Our computer search shows that no extremal singly even self-dual code is constructed from H_1. The singly even self-dual $[56, 28, 10]$ codes have 16 distinct weight enumerators. The weight enumerators were determined by calculating the numbers of codewords of weights 10 and 12. For each weight enumerator, we checked whether codes are equivalent or not. Then 944 inequivalent singly even self-dual $[56, 28, 10]$ codes are constructed from H_1.

4.3 Results

By Proposition 4, there are no other singly even self-dual $[56, 28, d]$ codes with $d \in \{10, 12\}$ constructed from Hadamard matrices of order 28. Hence, there is no extremal singly even self-dual code and there are 944 inequivalent singly even self-dual $[56, 28, 10]$ codes constructed from Hadamard matrices of order 28. This completes the proof of Theorem 1.

We denote the 944 codes by $C_{56,i}$ $(i = 1, 2, \dots, 944)$. For the 16 distinct weight enumerators, the values (i, α) in $W^{C}_{56,i}$ are listed in Table 1. For each (i, α), we list in Table 1 the number $N(i, \alpha)$ of the inequivalent codes and we list in Table 2 the set Γ of one code as an example. All the sets Γ for the 944 codes can be obtained electronically from "`http://www.math.is.tohoku.ac.jp/~mharada/Paper/F2-56-d10.txt`".

Table 1: Weight enumerators

(i, α)	$N(i, \alpha)$	(i, α)	$N(i, \alpha)$	(i, α)	$N(i, \alpha)$	(i, α)	$N(i, \alpha)$
$(1, -18)$	1	$(2, -2)$	6	$(2, -10)$	230	$(2, -18)$	29
$(1, -24)$	4	$(2, -4)$	10	$(2, -12)$	180	$(2, -20)$	11
$(1, -22)$	1	$(2, -6)$	94	$(2, -14)$	138	$(2, -22)$	1
$(2, 0)$	1	$(2, -8)$	160	$(2, -16)$	77	$(2, -24)$	1

Table 2: Sets Γ for some codes $C_{56,i}$

Codes	Sets Γ	(i, α)
$C_{56,1}$	$\{1, 4, 5, 8\}$	$(1, -18)$
$C_{56,2}$	$\{1, 2, 3, 4\}$	$(1, -24)$
$C_{56,6}$	$\{1, 2, 4, 6, 7, 9, 10, 12, 14, 15, 16, 20\}$	$(1, -22)$
$C_{56,7}$	$\{1, 3, 4, 5, 7, 8, 9, 10, 11, 13, 16, 20\}$	$(2, 0)$
$C_{56,8}$	$\{1, 2, 3, 5, 6, 7, 8, 10, 12, 13, 14, 17\}$	$(2, -2)$
$C_{56,14}$	$\{1, 2, 3, 4, 6, 11, 14, 15\}$	$(2, -4)$
$C_{56,24}$	$\{1, 3, 4, 5, 6, 8, 9, 11\}$	$(2, -6)$
$C_{56,118}$	$\{1, 2, 3, 5, 6, 7, 9, 11\}$	$(2, -8)$
$C_{56,278}$	$\{1, 2, 3, 4, 5, 6, 7, 11\}$	$(2, -10)$
$C_{56,508}$	$\{1, 2, 3, 4, 5, 6, 7, 8\}$	$(2, -12)$
$C_{56,688}$	$\{1, 2, 3, 4, 5, 6, 8, 9\}$	$(2, -14)$
$C_{56,826}$	$\{1, 2, 3, 4, 5, 7, 9, 11\}$	$(2, -16)$
$C_{56,903}$	$\{1, 2, 4, 5, 6, 7, 8, 11\}$	$(2, -18)$
$C_{56,932}$	$\{1, 3, 5, 6, 7, 9, 11, 14\}$	$(2, -20)$
$C_{56,943}$	$\{1, 2, 3, 6, 7, 11, 14, 17\}$	$(2, -22)$
$C_{56,944}$	$\{1, 2, 4, 5, 6, 8, 10, 11, 12, 13, 14, 15\}$	$(2, -24)$

Acknowledgments

This work was supported by JSPS KAKENHI Grant Number 15H03633.

References

[1] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system I: The user language, *J. Symbolic Comput.* **24** (1997), 235–265.

[2] F. C. Bussemaker and V. D. Tonchev, Extremal doubly-even codes of length 40 derived from Hadamard matrices of order 20, *Discrete Math.* **82** (1990), 317–321.

[3] J. H. Conway and N. J. A. Sloane, A new upper bound on the minimal distance of self-dual codes, *IEEE Trans. Inform. Theory* **36** (1990), 1319–1333.

[4] M. Harada and V. D. Tonchev, Singly-even self-dual codes and Hadamard matrices, *Lec. Notes in Comp. Sci.* 948, Springer, Berlin, 1995, pp. 279–284.

[5] H. Kharaghani and B. Tayfeh-Rezaie, On the classification of Hadamard matrices of order 32, *J. Combin. Des.* **18** (2010), 328–336.

[6] H. Kimura, Classification of Hadamard matrices of order 28, *Discrete Math.* **133** (1994), 171–180.

[7] H. Kimura, Extremal doubly even $(56, 28, 12)$ codes and Hadamard matrices of order 28, *Australas. J. Combin.* **10** (1994), 153–161.

[8] E. M. Rains, Shadow bounds for self-dual codes, *IEEE Trans. Inform. Theory* **44** (1998), 134–139.

[9] V. D. Tonchev, Self-orthogonal designs and extremal doubly-even codes, *J. Combin. Theory Ser. A* **52** (1989), 197–205.

(Received 22 Aug 2017)