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The r-Bessel and restricted r-Bell numbers

Ji-Hwan Jung

Department of Mathematics
Sungkyunkwan University
Suwon 440-746
Republic of Korea
jh56k@skku.edu

István Mező*

Department of Mathematics
Nanjing University of Information Science and Technology
Nanjing 210044
P. R. China
istvanmezo81@gmail.com

José L. Ramírez†

Departamento de Matemáticas
Universidad Nacional de Colombia
Bogotá
Colombia
jlramirezr@unal.edu.co

Abstract

In the present article we establish some combinatorial properties involving r-Bessel numbers of the second kind. These identities are deduced from the combinatorial interpretation by using restricted set partitions. Additionally, we introduce the restricted r-Bell numbers in analogy to the well-known Bell numbers. We derive several recurrence relations, combinatorial sums, arithmetical properties (2-adic valuation) and asymptotic results for these sequences.

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1 Introduction

The Bessel numbers of the second kind, $B(n,k)$, count the number of set partitions of $[n] := \{1, 2, \ldots, n\}$ into k blocks of size one or two. These numbers can be defined as the connecting coefficients between some special polynomials. Namely,

$$\frac{d^n}{dt^n} f(t)\bigg|_{t=0} := f^{(n)}(0) = \sum_{k=1}^{n} B(n,k)(x)_k,$$

where $f(t) = \left(1 + t + \frac{t^2}{2!}\right)^x$ and $(x)_n$ is the falling factorial, i.e., $(x)_n := x(x-1)\cdots(x-n+1)$ if $n \geq 1$ and $(x)_0 = 1$.
The Bessel numbers of the second kind satisfy the following recurrence (cf. [15])

$$B(n,k) = B(n-1,k-1) + (n-1)B(n-2,k-1).$$

Moreover, $B(n,k) = S(n,k)$ for $n-k < 2$, where $S(n,k)$ are the Stirling numbers of the second kind. Note that the Bessel numbers are complementary to the associated Stirling numbers of the second kind defined as the number of set partitions of $[n]$ into k blocks of size at least 2 (cf. [16]). See [15] for the unimodality of Bessel numbers, [18] for a combinatorial proof of the log-convave property. See [17] for their connections to Bessel polynomials, [19, 20] for their relations to special polynomials as Cauchy polynomials and poly-Bernoulli polynomials. For some generalizations see [13, 14]. See [4, 9, 28, 26] for some recent relations of the associated Stirling numbers.

Recently, Cheon et al. [12] introduced a generalization of this sequence called *r-Bessel numbers* $B_r(n,k)$. This new sequence counts the number of set partitions of $[n+r] := \{1, 2, \ldots, n+r\}$ into $k+r$ blocks of size one or two such that the first r elements are in distinct blocks. It is clear that if $r = 0$ then $B_0(n,k) = B(n,k)$.
For example, $B_2(2,1) = 5$ with the partitions being

$$\left\{\{\overline{1}\},\{\overline{2}\},\{3,4\}\right\}, \quad \left\{\{\overline{1},3\},\{\overline{2}\},\{4\}\right\}, \quad \left\{\{\overline{1},4\},\{\overline{2}\},\{3\}\right\},$$
$$\left\{\{\overline{1}\},\{\overline{2},3\},\{4\}\right\}, \quad \left\{\{\overline{1}\},\{\overline{2},4\},\{3\}\right\}.$$

There is a combinatorial formula to evaluate the r-Bessel numbers (cf. [12]):

$$B_r(n,k) = \frac{n!}{k!} \sum_{j=0}^{n-k} \binom{r}{j} \binom{k}{n-k-j} \frac{1}{2^{n-k-j}}.$$

Moreover, its exponential generating function is given by

$$\sum_{n=k}^{\infty} B_r(n,k) \frac{z^n}{n!} = \frac{1}{k!}(1+z)^r \left(z + \frac{z^2}{2}\right)^k.$$

The r-Bessel numbers can be defined by means of Riordan arrays [12] as follows

$$[B_r(n,k)] = \left((1+z)^r, z + \frac{z^2}{2}\right),$$

where (,) denoted an element of an exponential Riordan group. For example, if $r = 2$ we have the following array:

$$[B_2(n,k)] = \left((1+z)^2, z + \frac{z^2}{2}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 12 & 9 & 1 & 0 & 0 & 0 \\ 0 & 12 & 39 & 14 & 1 & 0 & 0 \\ 0 & 0 & 90 & 95 & 20 & 1 & 0 \\ 0 & 0 & 90 & 375 & 195 & 27 & 1 \\ \vdots & & & \vdots & & & \vdots \end{bmatrix}$$

An infinite lower triangular matrix $L = [l_{n,k}]_{n,k\in\mathbb{N}}$ is called an exponential Riordan array [3] if its column k has generating function $g(z)\,(f(z))^k/k!$, $k = 0, 1, 2, \ldots$, where $g(z)$ and $f(z)$ are formal power series with $g(0) \neq 0$, $f(0) = 0$ and $f'(0) \neq 0$. The matrix corresponding to the pair $f(z), g(z)$ is denoted by $(g(z), f(z))$. If we multiply (g, f) by a column vector $(c_0, c_1, \ldots)^T$ with the exponential generating function $h(z)$ then the resulting column vector has exponential generating function $gh(f)$. This property is known as the fundamental theorem of exponential Riordan arrays or summation property. The product of two exponential Riordan arrays $(g(z), f(z))$ and $(h(z), l(z))$ is defined by:

$$(g(z), f(z)) * (h(z), l(z)) = (g(z)h(f(z)), l(f(z))).$$

The set of all exponential Riordan matrices is a group under the operator $*$ (cf. [3, 27]).

Note that the r-Bessel numbers are related to r-Stirling numbers of the second kind [10]. The r-Stirling numbers of the second kind, $S_r(n,k)$, are defined as the number of set partitions of $[n+r]$ into $k+r$ blocks such that the first r elements are in distinct blocks.

In the present article, we study the r-Bessel numbers of the second kind from their combinatorial interpretation, and moreover we introduce the restricted r-Bell numbers in analogy to the well-known Bell numbers. In particular, in Sections 2 and 3 we obtain several recurrence relations and combinatorial sums of the r-Bessel numbers of the second kind and the restricted r-Bell numbers. In some cases, we use the fundamental theorem of exponential Riordan arrays to establish the identities. In Section 4 we introduce the restricted r-Bell polynomials. Then we show their relations with the Hermite polynomials. In particular, for the case $r = 0$ we prove that these polynomials have only real zeros. In Section 5, we use the Riordan arrays to derive several connections between the r-Bessel numbers and the r-Whitney numbers. In Section 6, we show that the restricted Bell numbers form a log-convex sequence. Moreover, a conjecture on the log-concavity of the r-Bessel numbers is proposed here. In Section 7, we study the 2-adic valuation of the restricted r-Bell numbers. Finally, in Section 8 we determine the asymptotic behavior of the restricted r-Bell numbers.

2 Some Combinatorial Identitites

Recall that a *partition* of a set A is a class of disjoint subsets of A such that the union of them covers A. The subsets are often called *blocks*. Any fixed partition can be written uniquely: we order the elements in the blocks in increasing order and we put the blocks into increasing order with respect to their first elements. This representation is called the partition's *standard form*.
Let $n, r \geq 0$ be integers. Let $\Pi_{n,r}$ denote the set of partitions of the set $[n+r]$, such that the first r elements are in distinct blocks. The elements $\{1, 2, \ldots, r\}$ will be called *special elements*. A block of a partition of the above set is called *special* if it contains a special element.

Theorem 2.1. *The r-Bessel numbers satisfy the following recursive relation:*

$$B_r(n,k) = B_r(n-1,k-1) + rB_{r-1}(n-1,k) + (n-1)B_r(n-2,k-1),$$

with the initial conditions $B_r(n,k) = 0$ if $n, k \leq 0$ and $B_r(n,0) = (r)_n$ if $n \geq 0$.

Proof. For any set partition of $\Pi_{n,r}$ into k blocks, there are three options: either $n+r$ forms a single block or $n+r$ is in a special block or $n+r$ is in a non-special block. In the first case, it is clear that there are $B_r(n-1,k-1)$ possibilities. In the second case, the element $n+r$ can be placed into one of the r special blocks. The remaining elements can be chosen in $B_{r-1}(n-1,k)$ ways. Altogether, we have $rB_{r-1}(n-1,k)$ possibilities. In the third case, we can use a similar argument as was used in the previous case, and then we obtain $(n-1)B_r(n-2,k-1)$ possibilities. $\square$

Theorem 2.2. *The r-Bessel numbers satisfy the following recursive relation:*

$$B_r(n,k) = B_r(n-1,k-1) + (k+r)B_r(n-1,k) -$$
$$\binom{n-1}{2}B_r(n-3,k-1) - r(n-1)B_{r-1}(n-2,k).$$

Proof. For any set partition of $\Pi_{n,r}$ into k blocks we can do the following construction. The element $n+r$ forms a single block; then the other elements can be chosen in $B_r(n-1,k-1)$ ways. Or the element $n+r$ is in one of the $k+r$ existing blocks after constructing a partition of $\Pi_{n,r}$ into k blocks. Then we have $(k+r)B_r(n-1,k)$ ways. However, we have to subtract the possibilities where the existing block is size 2. In this case we have two options. If the block is special then there are $r(n-1)$ ways to construct this block, while if the block is non-special then there are $\binom{n-1}{2}$ ways. Therefore we have to subtract the following possibilities $\binom{n-1}{2}B_r(n-3,k-1) + r(n-1)B_{r-1}(n-2,k)$. $\square$

Theorem 2.3. *For any positive integer r we have*

$$B_r(n,k) = B_{r-1}(n,k) + nB_{r-1}(n-1,k). \tag{1}$$

Proof. For any set partition of $\Pi_{n,r}$ into k blocks we can do the following construction: either 1 (the first element) forms a single special block or 1 is in a non-special block. In the first case, it is clear that there are $B_{r-1}(n,k)$ possibilities, while in the second case, the element 1 can be placed into one of the n non-special blocks. The other elements can be chosen in $B_{r-1}(n-1,k)$ ways. Altogether, we have $nB_{r-1}(n-1,k)$ possibilities. □

Theorem 2.4. *For any integers $r, s \geq 0$ we have*

$$B_{r+s}(n,k) = \sum_{i=k}^{n} \binom{n}{i} (r)_{n-i} B_s(i,k). \tag{2}$$

Proof. By means of the Riordan multiplication and taking in count that $(1+z)^r = \sum_{i=0}^{\infty} (r)_i \frac{z^i}{i!}$, we obtain

$$[B_{r+s}(n,k)] = ((1+z)^r, z) \left((1+z)^s, z + \frac{z^2}{2} \right) = \left[\binom{n}{k} (r)_{n-k} \right] [B_s(n,k)] .$$

By equating the (n,k)th element of the above equation, we obtain the desired result. □

Combinatorial Proof: We can construct any partition of $\Pi_{n,k}$ into k blocks as follows: we put $n-i$ non-special elements in the r-special blocks. Since that none of the blocks contain more than 2 elements then we have $r(r-1)\cdots(r-(n-i)) = (r)_{n-i}$ possibilities. Moreover, the $n-i$ elements can be chosen in $\binom{n}{n-i} = \binom{n}{i}$ ways. Altogether, we have $\binom{n}{i}(r)_{n-i}$ ways. The remaining $s+i$ elements can be chosen in $B_s(i,k)$ ways. Summing over i gives (2). □

In particular if $s = 0$ we obtain the following relation between r-Bessel and Bessel numbers

$$B_r(n,k) = \sum_{i=k}^{n} \binom{n}{i} (r)_{n-i} B(i,k). \tag{3}$$

Corollary 2.5. *For any integers $r, s \geq 0$ with $r \geq s$ we have*

$$B_r(n,k) = \sum_{i=k}^{n} \binom{n}{i} (r-s)_{n-i} B_s(i,k). \tag{4}$$

Note that if $s = r-1$ we obtain Equation (1).

3 The Restricted r-Bell numbers

In analogy to the usual definition of Bell numbers we define the *restricted r-Bell numbers*, $T_r(n)$, as the number of set partitions of $[n+r] = \{1, 2, \ldots, n+r\}$ into blocks of size one or two such that the first r elements are in distinct blocks. Then it is clear that for any integer $n \geq 0$

$$T_r(n) = \sum_{k=0}^{n} B_r(n,k),$$

with $T_r(0) = 1$. If $r = 0$ we recover the restricted Bell numbers (cf. [22, 25]), which count also the number of involutions in the set of permutations (cf. [1]). From Theorems 2.1 and 2.3 we obtain the following recursion:

$$T_r(n) = rT_{r-1}(n-1) + T_r(n-1) + (n-1)T_r(n-2), \tag{5}$$

$$T_r(n) = T_{r-1}(n) + nT_{r-1}(n-1). \tag{6}$$

Moreover, from the summation property for the exponential Riordan arrays we get the following theorem:

Theorem 3.1. *The exponential generating function of the restricted r-Bell numbers is*

$$\sum_{n=0}^{\infty} T_r(n)\frac{z^n}{n!} = (1+z)^r e^{z+\frac{z^2}{2}}.$$

From above theorem we can obtain the following explicit expression for the restricted r-Bell numbers. However, we give a combinatorial proof.

Theorem 3.2. *For any integer $n \geq 1$ we have*

$$T_r(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i}\binom{2i}{i}\frac{i!}{2^i}\sum_{j=0}^{n-2i}\binom{n-2i}{j}(r)_j.$$

Proof. Count the set of all partitions of $\Pi_{n,r}$ into block of size at most 2, with exactly i non-special blocks of size 2. First we chose a subset of $[n]$ of size $2i$ for the i non-special blocks of size 2. It can be chosen in $\binom{n}{2i}$ ways. Moreover, the number of set partitions of $[2i]$ such that each block has two elements is given by $\frac{(2i)!}{(2!)^i\, i!} = \frac{i!}{2^i}\binom{2i}{i}$. Now suppose there are j special blocks of size two, then it can be chosen in $\binom{n-2i}{j}(r)_j$ ways. To complete the argument sum over i and j. □

In particular if $r = 0$ we obtain the following expression to the restricted Bell numbers:

$$T(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i}\binom{2i}{i}\frac{i!}{2^i}.$$

Theorem 3.3. *For any integers $n, m \geq 0$ we have*

$$T_r(n+m) = \sum_{j\geq 0}\sum_{l=0}^{j} j!\binom{n}{j}\binom{m}{l}\binom{r}{j-l}T(n-j)T_{r-(j-l)}(m-l).$$

Proof. Split up the set of $n+m+r$ elements into two disjoint sets A_1 and A_2 such that $|A_1| = n$ and $|A_2| = m + r$. Count the set of all partitions of $\Pi_{n+m,r}$ into block of size at most 2, with exactly j blocks of size 2 with one element from each set A_1 and A_2. The first element can be chosen in $\binom{n}{j}$ ways and the second one can be chosen in $\sum_{l=0}^{j}\binom{m}{l}\binom{r}{j-l}$ ways. Therefore, there are $j!\binom{n}{j}\sum_{l=0}^{j}\binom{m}{l}\binom{r}{j-l}$ ways to chose these blocks. The remaining elements in A_1 and A_2 can be chosen in $T(n-j)$ and $T_{r-(j-l)}(m-l)$ ways, respectively. Summing over j gives the desired identity. □

In particular, if $m = 0$ we obtain the following corollary:

Corollary 3.4. *The restricted r-Bell numbers satisfy the following recursive relations:*

$$T_r(n) = \sum_{j\geq 0} j!\binom{n}{j}\binom{r}{j}T(n-j) = \sum_{j\geq 0}\binom{n}{j}(r)_j T(n-j).$$

4 The Restricted r-Bell polynomials

Let us begin with definition of the restricted r-Bell polynomials.

Definition 4.1. The restricted r-Bell polynomials are defined by

$$T_{r,n}(x) = \sum_{k=0}^{n} B_r(n,k)x^k.$$

From Theorems 2.1 and 2.3, we easily establish the following recursions:

$$\begin{aligned} T_{r,n}(x) &= xT_{r,n-1}(x) + rT_{r-1,n-1}(x) + (n-1)xT_{r,n-2}(x), \\ T_{r,n}(x) &= T_{r-1,n}(x) + nT_{r-1,n-1}(x). \end{aligned} \tag{7}$$

The exponential generating function of the restricted r-Bell polynomials is given, by Theorem 3.1, as

$$\sum_{n\geq 0} T_{r,n}(x)\frac{z^n}{n!} = (1+z)^r e^{x\left(z+\frac{z^2}{2}\right)} \tag{8}$$

The first few restricted r-Bell polynomials are:

$$\begin{aligned} &T_{r,0}(x) = 1, \quad T_{r,1}(x) = x + r, \quad T_{r,2}(x) = x^2 + (2r+1)x + (r-1)r, \\ &T_{r,3}(x) = x^3 + 3(r+1)x^2 + 3r^2x + (r-2)(r-1)r, \\ &T_{r,4}(x) = x^4 + (6+4r)x^3 + (3+6r+6r^2)x^2 + (2r-6r^2+4r^3)x \\ &\qquad\qquad + (r-3)(r-2)(r-1)r, \ldots \end{aligned}$$

Suppose that $P(x)$ and $Q(x)$ have only real zero. Let $\{a_k\}$ and $\{b_k\}$ be the sequences of all zeros of $P(x)$ and $Q(x)$ in nondecreasing order respectively. We say that $Q(x)$ *interlaces* $P(x)$, denoted by $Q(x) \preceq P(x)$, if deg$P(x) = n+1$, deg$Q(x) = n$ and

$$a_1 \leq b_1 \leq a_2 \cdots \leq b_n \leq a_{n+1}.$$

In this section, we show that the restricted Bell polynomials $T_{n,0}(x) := T_n(x)$ with degree $n \geq 1$ have the only non-positive real zeros and $T_n(x) \preceq T_{n+1}(x)$ for $n \geq 1$. The Hermite polynomials $H_n(x)$ are defined by the exponential generating function:

$$\sum_{n\geq 0} H_n(x)\frac{z^n}{n!} = e^{xz-\frac{z^2}{2}}. \tag{9}$$

It is known that the Hermite polynomials with degree $n \geq 1$ have only simple real zeros and $H_n(x) \preceq H_{n+1}(x)$. Furthermore, they are symmetric.

Theorem 4.2. *For $n \geq 1$, the polynomials $T_n(x)$ have the only non-positive real zeros and $T_n(x) \preceq T_{n+1}(x)$.*

Proof. Since the polynomial $H_n(x)$ is symmetric, the polynomial can be expressed as

$$H_n(x) = \begin{cases} x(x^2 - x_1^2)(x^2 - x_2^2)\cdots(x^2 - x_{m-1}^2), & \text{if } n = 2m-1; \\ (x^2 - x_1^2)(x^2 - x_2^2)\cdots(x^2 - x_m^2), & \text{if } n = 2m; \end{cases}$$

where m is a positive integer. It follows from (8) and (9) that $T_n(-x^2) = (-x)^n H_n(x)$. Thus the polynomials $T_n(x)$ can be expressed as

$$T_n(x) = \begin{cases} x^m(x + x_1^2)(x + x_2^2)\cdots(x + x_{m-1}^2), & \text{if } n = 2m-1; \\ x^m(x + x_1^2)(x + x_2^2)\cdots(x + x_m^2), & \text{if } n = 2m; \end{cases}$$

i.e., $T_n(x)$ have the only non-positive real zeros. Since $H_n(x) \preceq H_{n+1}(x)$, one may see that $T_n(x) \preceq T_{n+1}(x)$. Hence the proof follows. □

The authors have tried, without success, to establish the next statement:

Conjecture 4.3. *For $n, r \geq 1$, $T_{n,r}(x)$ have only non-positive real zeros and $T_{r,n}(x) \preceq T_{r,n+1}(x)$.*

Theorem 4.4. *For any integers $r, s, n \geq 0$ we have*

$$T_{r+s,n}(x) = \sum_{i=0}^{r} \binom{n}{i} (r)_{n-i} T_{s,i}(x).$$

In particular, if $s = 0$,

$$T_{r,n}(x) = \sum_{i=0}^{n} \binom{n}{i} (r)_{n-i} T_i(x).$$

Proof. This statement is a consequence of Theorem 2.4. □

Combinatorial proof. By Definition 4.1, $T_{r+s,n}(x)$, when x is a positive integer, is the number of partitions of $n+r+s$ elements into blocks of size one or two such that the blocks containing none of the first $r+s$ elements are colored independently with one of the x colors. Such a partition can be formed if we choose some, say i, elements from that of n and put them into blocks of size one or two where the first s elements are in different blocks, and then the remaining $n-i$ elements go one-by-one to the blocks of the first r elements (each into separate blocks in $r(r-1)\cdots(r-(n-i)+1) = (r)_{n-i}$ ways). Since we must permit some (or all) of the first r elements to be alone in a block, we must restrict $i \le r$. Note that the coloring is appropriately counted in the factor $T_{s,i}(x)$. Summing over i we are done. $\square$

Remark 4.5. Considering the generating functions of the sequence $T_r(n)$ and of the Hermite polynomials, it follows easily that

$$T_r(n) = \sum_{m=0}^{n} H_m\left(\frac{\sqrt{2}}{2i}\right)\left(\frac{i}{\sqrt{2}}\right)^m \frac{1}{m!}\binom{r}{n-m}.$$

4.1 Determinantal Identity

Let $A_r := [B_r(n,k)] = \left((1+z)^r, z+\frac{z^2}{2}\right)$ be the exponential Riordan array of the r-Bessel numbers. Then the inverse exponential Riordan array of A_r is given by ([12]):

$$B_r := [b_r(n,k)] = A_r^{-1} = \left((1+2z)^{-r/2}, -1+\sqrt{1+2z}\right).$$

The numbers $\widehat{b}_r(n,k) := (-1)^{n-k} b_r(n,k)$ are called *the unsigned r-Bessel numbers of the first kind*. Cheon et al. [12] gave an interesting combinatorial interpretation of this sequence by using the new concept of G-partitions. Its exponential generating function is given by

$$\sum_{n=k}^{2k+r} \widehat{b}_r(n,k)\frac{t^n}{n!} = \frac{1}{k!}\frac{(1-\sqrt{1-2z})^k}{(1-2z)^{r/2}}.$$

Moreover, it is clear that the r-Bessel numbers satisfy the following orthogonality relation:

$$\sum_{i=s}^{n}(-1)^{i-s}B_r(n,i)\widehat{b}_r(i,s) = \sum_{i=s}^{n}(-1)^{n-i}\widehat{b}_r(n,i)B_r(i,s) = \delta_{s,n},$$

where $\delta_{s,n} = 1$ if $s = n$ and 0 otherwise. From above relations we obtain the inverse relation:

$$f_n = \sum_{s=0}^{n}(-1)^{n-s}\widehat{b}_r(n,s)g_s \iff g_n = \sum_{s=0}^{n} B_r(n,s)f_s.$$

From definition of the restricted r-Bell polynomials we obtain the following equality

$$X = A_r^{-1}T_r,$$

where $X = [1, x, x^2, \dots]^T$ and $T_r = [T_{r,0}(x), T_{r,1}(x), T_{r,2}(x), \dots]^T$. Then $X = B_r T_r$ and

$$x^n = \sum_{k=0}^{n} b_r(n,k) T_{r,k}(x).$$

Therefore

$$T_{r,n}(x) = x^n - \sum_{k=0}^{n-1} b_r(n,k) T_{r,k}(x), \quad n \geq 0. \tag{10}$$

From the above identity we obtain the following determinantal identity.

Theorem 4.6. *The restricted r-Bell polynomials satisfy*

$$T_{r,n}(x) = (-1)^n \begin{vmatrix} 1 & x & \cdots & x^{n-1} & x^n \\ 1 & b_r(1,0) & \cdots & b_r(n-1,0) & b_r(n,0) \\ 0 & 1 & \cdots & b_r(n-1,1) & b_r(n,1) \\ \vdots & & \cdots & & \vdots \\ 0 & 0 & \cdots & 1 & b_r(n,n-1) \end{vmatrix}$$

Proof. This identity follows from Equation (10) and by expanding the determinant by the last column. □

5 Some Additional Identities

From the product of Riordan arrays we obtain the following identities involving r-Stirling numbers and r-Whitney numbers of both kinds. The r-Whitney numbers of the first kind $w_{m,r}(n,k)$ and the second kind $W_{m,r}(n,k)$ were defined by Mező [21] as the connecting coefficients between some particular polynomials.
For non-negative integers n, k and r with $n \geq k \geq 0$ and for any integer $m > 0$

$$(mx + r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k)(x)_k, \tag{11}$$

and

$$m^n (x)_n = \sum_{k=0}^{n} w_{m,r}(n,k)(mx + r)^k. \tag{12}$$

The r-Whitney numbers of both kinds have the following exponential generating functions [21]:

$$\sum_{n=k}^{\infty} W_{m,r}(n,k) \frac{z^n}{n!} = \frac{e^{rz}}{k!} \left(\frac{e^{mz} - 1}{m} \right)^k, \tag{13}$$

$$\sum_{n=k}^{\infty} w_{m,r}(n,k) \frac{z^n}{n!} = (1 + mz)^{\frac{-r}{m}} \frac{\ln^k(1 + mz)}{m^k k!}. \tag{14}$$

Moreover, these sequences satisfy the orthogonality relation:

$$\sum_{i=s}^{n}(-1)^{i-s}W_r(n,i)w_r(i,s) = \sum_{i=s}^{n}(-1)^{n-i}w_r(n,i)W_r(i,s) = \delta_{s,n}.$$

Then we have the inverse relation:

$$f_n = \sum_{s=0}^{n}(-1)^{n-s}w_r(n,s)g_s \iff g_n = \sum_{s=0}^{n} W_r(n,s)f_s.$$

Note that if $(m,r)=(1,0)$ we obtain the Stirling numbers, if $(m,r)=(1,r)$ we have the r-Stirling (or noncentral Stirling) numbers [10], and if $(m,r)=(m,1)$ we have the Whitney numbers [6, 7]. The Stirling numbers of the first and the second kind are denoted by $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$, respectively. Moreover, the r-Stirling numbers of the first and the second kind are denoted by $\left[{n \atop k}\right]_r$ and $\left\{{n \atop k}\right\}_r$, respectively.

From Equations (13) and (14) we obtain that the matrices $\mathcal{W}_2 := [W_{m,r}(n,k)]_{n,k\geq 0}$ and $\mathcal{W}_1 := [w_{m,r}(n,k)]_{n,k\geq 0}$ are exponential Riordan arrays given by ([11])

$$\mathcal{W}_2 = \left(e^{rz}, \frac{e^{mz}-1}{m}\right), \qquad \mathcal{W}_1 = \left((1-mz)^{\frac{-r}{m}}, -\frac{1}{m}\ln(1-mz)\right).$$

In [11, 23, 24], authors found additional identities for these sequences by using a matrix approach.

Theorem 5.1. *For any integers $n,k,r \geq 0$, we have*

1. $\displaystyle\sum_{j=0}^{n}\left\{{n \atop j}\right\}B_r(j,k) = W_{2,r}(n,k).$

2. $\displaystyle\sum_{j=0}^{n}\left\{{n \atop j}\right\}_r B_r(j,k) = W_{2,2r}(n,k).$

Proof. By means of the Riordan multiplication we get

$$\begin{aligned}\left[\left\{{n \atop k}\right\}\right][B_r(n,k)] &= (1, e^z-1)\left((1+z)^r, z+\frac{z^2}{2}\right)\\ &= \left(e^{rz}, (e^z-1)+\frac{(e^z-1)^2}{2}\right) = \left(e^{rz}, \frac{e^{2z}-1}{2}\right) = [W_{2,r}(n,k)].\end{aligned}$$

By equating the (n,k)th element of the above equation, we obtain the first identity. The last identity is obtained in a similar manner. ∎

The r-Whitney numbers of the second kind satisfy the following identity ([11])

$$W_{m,r}(n,k) = \sum_{j=k}^{n}\binom{n}{j}m^{j-k}r^{n-j}\left\{{j \atop k}\right\}. \tag{15}$$

Therefore from the first identity in Theorem 5.1 we have

$$\sum_{j=0}^{n} \left\{ {n \atop j} \right\} B_r(j,k) = \sum_{j=0}^{n} \binom{n}{j} 2^{j-k} r^{n-j} \left\{ {j \atop k} \right\}.$$

In particular, if $r = 0$ we obtain the following interesting relation for the Bessel numbers of the second kind ([30]):

$$\sum_{j=0}^{n} \left\{ {n \atop j} \right\} B(j,k) = 2^{n-k} \left\{ {n \atop k} \right\}.$$

Corollary 5.2. *For any integers $n, k, r \geq 0$, we have*

1. $\displaystyle\sum_{j=0}^{n} (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix} W_{2,r}(j,k) = B_r(n,k).$

2. $\displaystyle\sum_{j=0}^{n} (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_r W_{2,2r}(j,k) = B_r(n,k).$

Proof. From (1) in Theorem 5.1 and the inverse relation for the Stirling numbers with

$$f_n = B_r(n,k), \quad \text{and} \quad g_n = W_{2,r}(n,k),$$

we obtain the identity (1). The second relation can be verified in a similar way. □

From the first identity in the above corollary and Equation (15) we obtain the following interesting identity

$$B_r(n,k) = \sum_{l=0}^{n} \sum_{j=0}^{n} (-1)^{n-j} \binom{j}{l} \left\{ {l \atop k} \right\} \begin{bmatrix} n \\ j \end{bmatrix} 2^{l-k} r^{j-l}.$$

Theorem 5.3. *For any integers $n, k, r \geq 0$, we have*

1. $\displaystyle\sum_{j=0}^{n} \widehat{b}_r(n,j) \begin{bmatrix} j \\ k \end{bmatrix} = w_{2,r}(n,k).$

2. $\displaystyle\sum_{j=0}^{n} \widehat{b}_r(n,j) \begin{bmatrix} j \\ k \end{bmatrix}_r = w_{2,2r}(n,k).$

Proof. By means of the Riordan multiplication we get

$$\begin{aligned} \left[\widehat{b}_r(n,k)\right] \left[\begin{bmatrix} n \\ k \end{bmatrix} \right] &= \left((1-2z)^{-r/2}, 1 - \sqrt{1-2z}\right) (1, -\ln(1-z)) \\ &= \left((1-2z)^{(-r/2)}, \frac{1}{2}\ln(1-2z)\right) = \left[w_{2,r}(n,k)\right]. \end{aligned}$$

By equating the (n,k)th element of the above equation, we obtain the first identity. The last identity is obtained in a similar manner. □

The (unsigned) r-Whitney numbers of the first kind satisfy the following identity ([11])

$$w_{m,r}(n,k) = \sum_{j=k}^{n} \binom{j}{k} m^{n-j} r^{j-k} \begin{bmatrix} n \\ j \end{bmatrix}. \tag{16}$$

Therefore from the first identity in Theorem 5.3 we have

$$\sum_{j=0}^{n} \widehat{b}_r(n,j) \begin{bmatrix} j \\ k \end{bmatrix} = \sum_{j=k}^{n} \binom{j}{k} 2^{n-j} r^{j-k} \begin{bmatrix} n \\ j \end{bmatrix}.$$

In particular, if $r = 0$ we obtain the following relation for the (unsigned) Bessel numbers of the first kind ([30]):

$$\sum_{j=0}^{n} \widehat{b}(n,j) \begin{bmatrix} j \\ k \end{bmatrix} = 2^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}.$$

From the inverse relation for the r-Bessel numbers we obtain next corollary.

Corollary 5.4. *For any integers $n, k, r \geq 0$, we have*

1. $\displaystyle\sum_{j=0}^{n} B_r(n,j) w_{2,r}(j,k) = (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}.$

2. $\displaystyle\sum_{j=0}^{n} B_r(n,j) w_{2,2r}(j,k) = (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_r.$

6 Log-Convex and Log-Concavity Properties

Given a positive real sequence $\mathcal{A} = \{a_k\}_{0 \leq k}$, we say that $\mathcal{A}$ is *unimodal* if there exists an integer $0 \leq j$ such that $a_0 \leq \cdots \leq a_{j-1} \leq a_j \geq a_{j+1} \geq \cdots$. The integer j is called the *mode* of the sequence $\mathcal{A}$. We say that the sequence $\mathcal{A}$ is log-concave if $a_n a_{n+2} \leq a_{n+1}^2$, for all $n \geq 0$. It is called log-convex if $a_n a_{n+2} \geq a_{n+1}^2$ for all $n \geq 0$. For more information, the reader is referred to the general overview on unimodality and log-concavity written by P. Brändén in [8, Chapter 7].

The unimodality of the restricted Stirling numbers $(B(n,k))_{k \geq 0}$ was proved in [15]. Moreover, in [18] authors give a combinatorial proof of the log-concavity of this sequence.

A sequence $\{a_0, a_1, \ldots, a_n\}$ of the coefficients of a polynomial $f(x) = \sum_{k=0}^{n} a_k x^k$ of degree n with only real zeros is called the Pólya frequency sequence (PF). It is well know that if a sequence is PF then it is log-concave (See [29, Theorem 4.5.2] for a proof). Therefore 4.3 implies the following conjecture.

Conjecture 6.1. *The sequence of r-Bessel numbers $(B_r(n,k))_{k \geq 0}$ is log-convave. Hence, it is unimodal.*

It is well-known that the Bell sequence is log-convex [2]. It is not difficult to show that the restricted r-Bell numbers are also log-convex.
A sequence $(a_n)_{n\in\mathbb{N}}$ has no internal zeros if there do not exist integers $0 \le i < j < k$ such that $a_i \neq 0, a_j = 0, a_k \neq 0$.

Theorem 6.2 (Bender-Canfield Theorem, [5]). *Let $1, w_1, w_2, \ldots$ be a log-concave sequence of nonnegative real numbers with no internal zeros and define the sequence $(a_n)_{n\geq 0}$ by*

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = \exp\left(\sum_{j=1}^{\infty} \frac{w_i}{i} x^j\right).$$

Then the sequence $(a_n)_{n\geq 0}$ is log-convex and the sequence $(a_n/n!)_{n\geq 0}$ is log-convave.

Theorem 6.3. *The restricted Bell sequence $(T(n))_{n\geq 0}$ is log-convex and the sequence $(T(n)/n!)_{n\geq 0}$ is log-concave.*

Proof. From Theorems 3.1 (with $r = 0$) and 6.2, it suffices to prove that the sequence

$$w_i = \begin{cases} \frac{1}{(i-1)!}, & \text{if } 1 \le i \le m; \\ 0, & i > m. \end{cases}$$

is a log-concave sequence, which is clear. □

Remark 6.4. Note that the above statement cannot be generalized to the restricted r-Bell sequence $(T_r(n))_{n\geq 0}$. For example,

$$8 = T_2(0)T_2(2) < T_2(1)^2 = 9,$$

while

$$528 = T_2(2)T_2(4) > T_2(3)^2 = 484.$$

7 2-adic Valuation of the Restricted r-Bell Numbers

In this section, we analyze the 2-adic valuation of the restricted r-Bell numbers. Let p be a prime number, the p-adic valuation of $n \in \mathbb{N}$, denoted by $\nu_p(n)$, is the largest nonnegative integer m such that p^m divides n. The valuation of $n = 0$ is defined by $+\infty$.

Amdeberhan and Moll [1] studied the 2-adic valuation of the restricted Bell numbers. In particular they found the following expression:

$$\nu_2(T(n)) = \begin{cases} k, & \text{if } n = 4k; \\ k, & \text{if } n = 4k+1; \\ k+1, & \text{if } n = 4k+2; \\ k+2, & \text{if } n = 4k+3. \end{cases}$$

The 2-adic valuation of the restricted r-Bell numbers follows a similar pattern. In Figure 1 we show the first few values of $\nu_2(T_5(n))$.

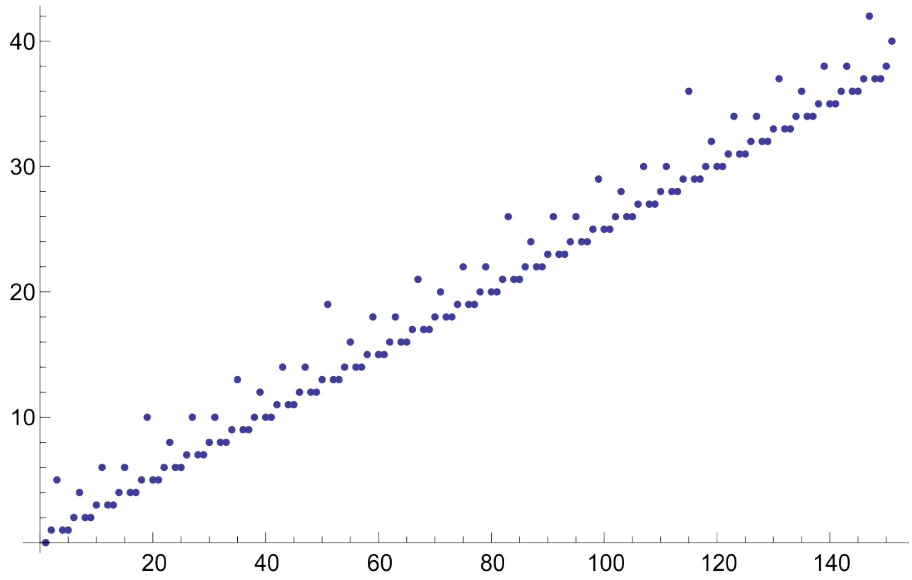

Figure 1: The 2-adic valuation of $T_5(n)$.

Theorem 7.1. *The 2-adic valuation of the restricted r-Bell numbers is: if $r \equiv 0 \pmod 4$ then*

$$\nu_2(T_r(n)) = \begin{cases} k, & \text{if } n = 4k; \\ k, & \text{if } n = 4k+1; \\ k+1, & \text{if } n = 4k+2; \\ \alpha, & \text{if } n = 4k+3. \end{cases}$$

where $\alpha \geq k+2$. The remaining cases are:

$$\nu_2(T_{4l+1}(n)) = \begin{cases} k, & \text{if } n = 4k; \\ k+1, & \text{if } n = 4k+1; \\ \alpha, & \text{if } n = 4k+2; \\ k+1, & \text{if } n = 4k+3. \end{cases} \quad \nu_2(T_{4l+2}(n)) = \begin{cases} k, & \text{if } n = 4k; \\ k, & \text{if } n = 4k+1; \\ \alpha, & \text{if } n = 4k+2; \\ k+1, & \text{if } n = 4k+3. \end{cases}$$

$$\nu_2(T_{4l+3}(n)) = \begin{cases} k, & \text{if } n = 4k; \\ \alpha, & \text{if } n = 4k+1; \\ k+1, & \text{if } n = 4k+2; \\ k+1, & \text{if } n = 4k+3. \end{cases}$$

with $\alpha \geq k+2$.

Proof. We proceed by induction on n. The proof is divided into four cases according to the residue of n modulo 4. The symbols O_i (resp. E_i) denote an odd number (resp. even number). If $n = 4k$ then from Equation 6 and by induction hypothesis we have

$$T_{4l}(4k) = T_{4l-1}(4k) + 4kT_{4l-1}(4k-1) = 2^k O_1 + 4k \cdot 2^k O_2 = 2^k(O_1 + 4kO_2) = 2^k O_3.$$

Therefore $\nu_2(T_{4l}(4k)) = k$.
If $n = 4k+1$ then from Equation 6 and by induction hypothesis we have that

$$\begin{aligned} T_{4l}(4k+1) &= T_{4l-1}(4k+1) + (4k+1)T_{4l-1}(4k) = 2^\alpha O_1 + (4k+1)2^k O_2 \\ &= 2^k(2^{\alpha-k}O_1 + (4k+1)O_2) = 2^k O_3. \end{aligned}$$

Therefore $\nu_2(T_{4k}(4k+1)) = k$.
If $n = 4k+2$ then from Equation 6 and by induction hypothesis we have that

$$\begin{aligned} T_{4l}(4k+2) &= T_{4l-1}(4k+2) + (4k+2)T_{4l-1}(4k+1) = 2^{k+1}O_1 + (4k+2)2^\alpha O_2 \\ &= 2^{k+1}(O_1 + 2^{\alpha+1}(2k+1)O_2) = 2^{k+1}O_3. \end{aligned}$$

Therefore $\nu_2(T_{4k}(4k+2)) = k+1$.
If $n = 4k+3$ then from Equation 6 and by induction hypothesis we get

$$\begin{aligned} T_{4l}(4k+3) &= T_{4l-1}(4k+3) + (4k+3)T_{4l-1}(4k+2) = 2^{k+1}O_1 + (4k+3)2^{k+1}O_2 \\ &= 2^{k+1}(O_1 + (4k+3)O_2) = 2^{k+1}E_1. \end{aligned}$$

Therefore $\nu_2(T_{4k}(4k+3)) = \alpha \geq k+2$.
The remaining cases are analyzed in a similar manner. □

8 Asymptotics of $T_r(n)$

In this section we are going to determine the asymptotic behavior of the $T_r(n)$ sequence as $n \to \infty$ and r is fixed. This will be done via Hayman's method [29]. This method says that if we have an entire function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

then the asymptotics of a_n as $n \to \infty$ is given by

$$a_n \sim \frac{f(r_n)}{r_n^n \sqrt{2\pi b(r_n)}}, \tag{17}$$

where

$$b(r) = r\left(r\frac{f'(r)}{f(r)}\right)', \tag{18}$$

and r_n is the unique solution of

$$r\frac{f'(r)}{f(r)} = n. \tag{19}$$

To assure that this equation really has a unique root, some assumptions on the generating function f must be made. In our case Theorem 3.1 is suitable, because the left hand side of (19) is strictly increasing on $[0, +\infty[$ and the equation has a unique solution. However, if we substitute the function under 3.1 into (19), the resulting transcendental equation is too complicated to solve. But one substantial simplification can be made: since we need only asymptotic results, it is enough to solve an equation "close to" the original. It turns out that as $r \to \infty$

$$r\frac{f'(r)}{f(r)} = x^2 + x + r + O(x^{-1}).$$

Hence

$$r_n \sim \frac{1}{2}\left(-1 + \sqrt{4n + 1 - 4r}\right).$$

as $n \to \infty$ and r is fixed.
Substituting this r_n into (17), we can determine the asymptotic expansion of the resulting function at $n = \infty$. This results in

$$T_r(n) \sim \frac{1}{\sqrt{2}e^{1/4}} n^{n/2+r/2} \exp\left(\sqrt{n} + \frac{1}{24\sqrt{n}} - \frac{n}{2}\right)\left(1 + \frac{1+2r}{4\sqrt{n}}\right).$$

This result, in a somewhat weaker form, can be found in [29, p. 187].
This can be rewritten in a somewhat weaker form:

$$\frac{T_r(n)}{n^{n/2+r/2}\exp\left(\sqrt{n} - \frac{n}{2}\right)} \to \frac{1}{\sqrt{2}e^{1/4}}.$$

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