Some new large sets of t–designs

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Abstract

By constructing a new large set of 4–(13,5,3) designs, and using a recursive construction described recently by Qiu-rong Wu, we produce an infinite family of large sets and as a byproduct an infinite family of new 4-designs. Similarly, we construct a new large set of 3–(13,4,2) designs, and obtain an infinite family of large sets of 3–designs. We also include a large set of 2-(14,4,6) designs, which implies a new family of large sets of 2-designs.

1 Introduction and Wu's result

A *t-design*, denoted by t–(v,k,λ), is a collection $\mathcal{B}$ of $k-$subsets, called blocks of a set X of v points such that every t–set of X is a subset of exactly λ blocks of $\mathcal{B}$. A t–design is called *simple* if no block of $\mathcal{B}$ is repeated; a t–$(v,k,1)$ design is called a *t–Steiner system.*

The appearance of Teirlinck's remarkable papers [6], [7] made clear that a major route to the recursive construction of t–designs with large t is by constructing *large sets* of t–designs. Khosrovshahi and Ajoodani-Namini [3] gave an independent proof of a similar recursive construction for large sets of t–designs when $k = t+1$. Recently, Qiu-rong Wu [9] generalized the Khosrovshahi, Ajoodani-Namini construction.

By elementary counting, if $s \leq t$, a t–(v,k,λ) design is also an s–(v,k,μ) design, where $\mu = \lambda\binom{v-s}{t-s}/\binom{k-s}{t-s}$. Since μ must be an integer for each s, one gets necessary conditions for the existence of a t-design. Given t, k and v, there is a smallest positive integer $\lambda^*(t,k,v)$ such that these conditions are satisfied for all $0 \leq s \leq t$. Since complementing each block of a t–(v,k,λ) design, with respect to the point set X, yields a t–$(v, v-k, \lambda')$ design, where $\lambda' = \lambda\binom{v-k}{t}/\binom{k}{t}$, we shall assume that $k \leq v/2$.

Let $\binom{X}{k}$ denote the set of all k–subsets of a v–set X. A *large set* of t–(v,k,λ) designs is a partition of $\binom{X}{k}$ into n t–(v,k,λ) designs, where $n = \binom{v-t}{k-t}/\lambda$, and we will denote this large set by $LS[n](t,k,v)$. Note that λ can be computed from the parameters n, t, k, and v. The designs in a large set are simple but we do not insist that $\lambda = \lambda^*(t,k,v)$.

By taking all blocks of a t–(v,k,λ) design through a point x, and deleting x, one gets a $(t-1)$–$(v-1,k-1,\lambda)$ design, called the *derived* design. Note that the n derived designs of an $LS[n](t,k,v)$ through x will also form an $LS[n](t-1,k-1,v-1)$, called the *derived large set* through x.

A table of t–(v,k,λ) designs, for $v \leq 30$, appears in [2] along with information about the existence of large sets. In [4] Kramer, Magliveras and Stinson construct large sets for several parameter cases and give an updated table for $v \leq 15$. In addition, a survey of large sets of disjoint designs has been written by Teirlinck [8].

Our notation now simplifies the statements of two result of Wu [9], namely:

Theorem 1.1 *If there exist large sets $LS[n](t,k,v_1)$, $LS[n](t,k,v_2)$, $LS[n](k-2,k-1,v_1-1)$, $LS[n](k-2,k-1,v_2-1)$, then there exists a large set $LS[n](t,k,v_1+v_2-k+1)$.*

As an immediate corollary we get:

Corollary 1.2 *If there exist large sets $LS[n](t,k,v)$ and $LS[n](k-2,k-1,v-1)$ then there exist large sets $LS[n](t,k,v+m(v-k+1))$ for all $m \geq 0$.*

2 An infinite family of large sets of 4-designs

We now present a new large set. Let the group G of order and degree 13 be generated by $(1,2,3,4,5,6,7,8,9,a,b,c,d)$. When G acts on each of the 3-sets of starting blocks listed below three disjoint 4–(13,5,3) designs are produced.

12345	12347	12358	1235b	12367	12368	1237b	1238a	1239a	1239b	1239c
123ac	12457	12459	12469	1246a	1246b	1248a	1248b	1248c	1249c	12569
1256b	1257a	12589	1258c	125ac	1267c	12689	126ac	1279c	13579	1358b
12346	12349	1234c	1235a	1235c	12369	1236a	12378	12379	1237c	1238b
123ab	12458	1245a	1245b	12467	12468	1247a	1247b	12489	124ac	1256a
1256c	12578	12579	1259b	1268c	1269b	1269c	1279b	1357b	1358a	1368b
12348	1234a	1234b	12356	12357	12359	1236b	1236c	1237a	12389	1238c
123bc	1245c	1246c	12478	12479	1247c	1249a	1249b	124ab	1257b	1257c
1258a	1258b	1259a	1259c	1267b	1268a	1268b	127ac	128ac	1357a	1359b

The full automorphism group on each design is G and the three designs are nonisomorphic. It should be added that initial attempts to find this particular large set were quite unsuccessful. Eventually the first author found this large set by a non-deterministic approach in which several initial 4–(13,5,3) designs were found and then the remaining orbits were searched for a second (and hence a complementary third) 4–(13,5,3) design.

Corollary 2.1 *If* $m \geq 1$ *then there exists a family of large sets* $LS[3](4,6,9m+5)$.

Proof. By using the $LS[3](4,6,14)$ (see [1]) and the $LS[3](4,5,13)$, Corollary 1.2 produces the family of large sets of $LS[3](4,6,9m+5)$ for $m \geq 1$.

In this family of $t-$designs the 4-(23,6,57) is known (see [5]) but the other 4-designs for $m > 2$ are apparently new.

3 An infinite family of large sets of 3-designs

The following new $LS[5](3,4,13)$ was constructed by methods akin to those in [4]. Let the group G of order and degree 13 be generated by $\alpha = (1,2,3,4,5,6,7,8,9,a,b,c,d)$. When G acts on each of the five sets of starting blocks listed below, five disjoint 3–$(13,4,2)$ designs are produced.

1234	124a	1257	1258	126a	126c	1279	128b	129b	1357	136b
1235	123c	124c	125a	1267	126b	1278	1368	1379	137a	147a
1236	1239	1248	124b	1256	127a	127c	129c	1358	135a	137b
1237	123b	1247	1249	1259	125b	1268	128c	12ac	1359	138b
1238	123a	1245	1246	125c	1269	127b	128a	135b	1369	136a

The first three designs in this large set are pairwise nonisomorphic, although the automorphism group of each of the first two is dihedral of order 26 and is generated by α and $\beta = (1,12)(2,11)(3,10)(4,9)(5,8)(6,7)$. The third, fourth, and fifth designs are isomorphic; each has G as full automorphism group; and the permutation $(1,9,3)(2,5,6)(4,10,12)(7,11,8)$ interchanges these three designs.

Theorem 3.1 *If $m \geq 1$ then there exists a family of large sets $LS[5](3,4,10m+3)$.*

Proof. The derivation of the large set $LS[5](3,4,13)$ yields an $LS[5](2,3,12)$ and Corollary 1.2 produces our result.

4 A family of $LS[11](2,4,11m+14)$

An $LS[11](2,4,14)$ is produced in the following way:
Apply the permutation $(1,4,5,9,3)(2,8,a,7,6)$ $(b)(c)(d)(e)$ to the first 18 blocks in the following list (the 19th block is fixed by this permutation). These $91 = 5 \times 18 + 1$ blocks form a $2-(14,4,6)$ design which is also a transversal of the 91 orbits of 4-sets under the group generated by $(1,2,3,4,5,6,7,8,9,a,b)$ $(c)(d)(e)$. This clearly produces an $LS[11](2,4,14)$. Application of Corollary 1.2 now yields a family of $LS[11](2,4,11m+14)$, for all $m \geq 1$.

$19ab$	5679	$39ab$	1237	$17ab$	$345c$	$345d$	$567e$	$679a$	$124c$
$568d$	$346e$	$67bc$	$56ad$	$178e$	$78cd$	$89ce$	$1bde$	$bcde$	

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