\textbf{Abstract}

A simple graph $G = (V, E)$ admits an $H$-covering if every edge in $E$ is contained in a subgraph $H' = (V', E')$ of $G$ which is isomorphic to $H$. In this case we say that $G$ is $H$-supermagic if there is a bijection $f : V \cup E \to \{1, \ldots, |V| + |E|\}$ such that $f(V) = \{1, \ldots, |V|\}$ and
\[\sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)\] is constant over all subgraphs \(H'\) of \(G\) which are isomorphic to \(H\). Extending results from [M. Roswitha and E.T. Baskoro, Amer. Inst. Physics Conf. Proc. 1450 (2012), 135-138], we show that the firecracker \(F_{k,n}\) is \(F_{2,n}\)-supermagic, the banana tree \(B_{k,n}\) is \(B_{k-1,n}\)-supermagic and the flower \(F_n\) is \(C_3\)-supermagic.

1 Introduction

The graphs considered in this paper are finite, undirected and simple. For a positive integer \(n\) we denote the set \(\{1, \ldots, n\}\) by \([n]\), and for integers \(a \leq b\), the set \([a, b]\) is denoted by \([a, b]\). Let \(V(G)\) and \(E(G)\) be the set of vertices and edges of a graph \(G\). A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labeling was first introduced by Rosa \([8]\) in 1967. Since then there are various types of labeling that have been studied and developed (see [1]).

For a graph \(H\), a graph \(G\) admits an \(H\)-covering if every edge of \(G\) belongs to at least one subgraph of \(G\) which is isomorphic to \(H\). A graph \(G = (V, E)\) which admits an \(H\)-covering is called \(H\)-magic if there exists a bijection \(f : V \cup E \rightarrow |V| + |E|\) and a constant \(f(H)\), which we call the \(H\)-magic sum of \(f\), such that \(\sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) = f(H)\) for every subgraph \(H' \subseteq G\) with \(H' \cong H\). Additionally, if \(f(V) = |V|\) then we say that \(G\) is \(H\)-supermagic.

The concept of \(H\)-supermagic labeling was introduced by Gutiérrez and Lladó \([2]\) in 2005, for \(H\) being a star or a path. In \([4]\), Lladó and Moragas constructed cycle-supermagic labelings for some graphs. Furthermore, Maryati et al. \([5]\) studied path-supermagic labelings while Ngurah et al. \([7]\), Roswitha et al. \([10]\) and Kojima \([3]\) proved that some graphs have cycle-supermagic labelings. Some results for certain shackles and amalgamations of a connected graph have been proved by Maryati et al. \([6]\). Recently, Roswitha and Baskoro \([9]\) established \(H\)-supermagic coverings for some trees.

Roswitha and Baskoro \([9]\) show that for any integer \(k\) and even \(n\), the firecracker graph \(F_{k,n}\) is \(F_{2,n}\)-supermagic and the banana tree graph \(B_{k,n}\) is \(B_{k-1,n}\)-supermagic and left the remaining cases as open problems. In this paper, we solve these two problems. The result for banana trees is an immediate consequence of a theorem about amalgamations of graphs from \([6]\) which we recall in Section 2. The result for firecrackers in Section 3 is obtained by a similar method. In addition, we prove in Section 4 that for odd \(n\), the flower graph \(F_n\) is \(C_3\)-supermagic.

2 Amalgamations and banana trees

Let \(H\) be a graph with \(n\) vertices, say \(V(H) = \{v_1, \ldots, v_n\}\) and \(m\) edges, say \(E(H) = \{e_1, \ldots, e_m\}\). Take \(k\) copies of \(H\) denoted by \(H^1, \ldots, H^k\) and let the vertex and edge sets be \(V(H') = \{v_1^i, \ldots, v_n^i\}\) and \(E(H') = \{e_1^i, \ldots, e_m^i\}\). Fix a vertex \(v \in V(H)\), without loss of generality \(v = v_n\), and form a graph, \(G = A_k(H, v)\) by identifying
all the vertices $v_n^1, \ldots, v_n^k$ (and denoting the identified vertex by $v_n$). The following theorem was proved in [6].

**Theorem 1** ([6]). Let $H$ be any graph, and let $v \in V(H)$. If $G = A_k(H, v)$ contains exactly $k$ subgraphs isomorphic to $H$ then $G$ is $H$-supermagic with $H$-supermagic sums

$$f(H) = \begin{cases} \frac{3(n+m)-1}{2} + \frac{k(n+m-1)^2}{2} & \text{if } (m+n-1)(k-1) \text{ is even}, \\ \frac{3(n+m)-2}{2} + \frac{k[n(m-1)^2+1]}{2} & \text{if } (m+n-1)(k-1) \text{ is odd}. \end{cases}$$

For the convenience of the reader we provide an explicit description of the labeling.

**Proof.** The graph $A_k(H, v)$ has $k(n-1) + 1$ vertices and $km$ edges. We define the labeling

$$f : V(A_k(H, v)) \cup E(A_k(H, v)) \to [k(n+m-1)+1]$$

as follows.

**Case 1** If $n+m$ is odd, we start with $f(v_n) = 1$. Then we use the labels $2, \ldots, k(n-1) + 1$ for the remaining vertices:

$$f(v_i^j) = \begin{cases} 1 + (i-1)k + j & \text{if } i \text{ is odd}, \\ ik + 2 - j & \text{if } i \text{ is even}, \end{cases} \text{ for } i \in [n-1], j \in [k]. \quad (1)$$

Finally we use the labels $k(n-1) + 2, \ldots, k(n+m-1)+1$ for the edges:

$$f(e_i^j) = \begin{cases} 1 + (i+n-2)k + j & \text{if } i + n - 1 \text{ is odd}, \\ (i+n-1)k + 2 - j & \text{if } i + n - 1 \text{ is even}, \end{cases} \text{ for } i \in [n-1], j \in [k]. \quad (2)$$

The sum of the labels used for $H^j$ is independent of $j$:

$$f(v_n) + \sum_{i=1}^{n-1} f(v_i^j) + \sum_{i=1}^{m} f(e_i^j) = 1 + \sum_{i=1, i \text{ odd}}^{n+m-1} [1 + (i-1)k + j]$$

$$+ \sum_{i=1, i \text{ even}}^{n+m-1} [ik + 2 - j]$$

$$= \frac{3(n+m)-1}{2} + \frac{k(n+m-1)^2}{2}.$$

**Case 2** If $n+m$ is even and $k$ is odd, we start with $f(v_n) = 1$. Next we use the labels $2, \ldots, 3k+1$ to label the vertices $v_i^j$ for $i \in [3], j \in [k]$ (assuming that $n \geq 4$, otherwise use the first edges in the obvious way):

$$f(v_1^1) = 1 + j$$

$$f(v_1^j) = \begin{cases} 3(k+1)/2 + j & \text{for } j \in [(k-1)/2], \\ (k+3)/2 + j & \text{for } j \in [(k+1)/2, k], \end{cases}$$

$$f(v_3^j) = \begin{cases} 3k + 2 - 2j & \text{for } j \in [(k-1)/2], \\ 4k + 2 - 2j & \text{for } j \in [(k+1)/2, k]. \end{cases}$$
Then we use the labels $3k + 2, \ldots, k(n - 1) + 1$ for the remaining vertices, applying (1) for $i \in [4, n - 1]$. Finally, we use the labels $k(n - 1) + 2, \ldots, k(n + m - 1) + 1$ for the edges, applying (2). The sum of the labels used for $H^j$ is independent of $j$:

\[
\begin{align*}
 f(v_n) + \sum_{i=1}^{3} f(v_i^1) + \sum_{i=4}^{n-1} f(v_i^1) + \sum_{i=1}^{m} f(e_i^j) & = 1 + \frac{9(k+1)}{2} + \sum_{i=4, i \text{ odd}}^{n+m-1} [1 + (i - 1)k + j] + \sum_{i=4, i \text{ even}}^{n+m-1} [ik + 2 - j] \\
 & = \frac{3(n + m) - 1}{2} + \frac{k(n + m - 1)^2}{2}.
\end{align*}
\]

**Case 3** If $n + m$ is even and $k$ is even, we start with $f(v_n) = k/2 + 1$. Next we use the labels $1, \ldots, k/2, k/2 + 2, \ldots, 3k + 1$ to label the vertices $v_i^j$ for $i \in [3], j \in [k]$ (assuming that $n \geq 4$, otherwise use the first edges in the obvious way):

\[
\begin{align*}
 f(v_i^1) & = \begin{cases} 
 j & \text{for } j \in [k/2], \\
 j + 1 & \text{for } j \in [k/2 + 1, k], 
\end{cases} \\
 f(v_i^2) & = \begin{cases} 
 3k/2 + 1 + j & \text{for } j \in [k/2], \\
 k/2 + 1 + j & \text{for } j \in [k/2 + 1, k], 
\end{cases} \\
 f(v_i^3) & = \begin{cases} 
 3(k + 1) - 2j & \text{for } j \in [k/2], \\
 4k + 2 - 2j & \text{for } j \in [k/2 + 1, k]. 
\end{cases}
\end{align*}
\]

Then we use the labels $3k + 2, \ldots, k(n - 1) + 1$ for the remaining vertices, applying (1) for $i = 4, \ldots, n-1$. Finally, we use the labels $k(n-1)+2, \ldots, k(n + m - 1) + 1$ for the edges, applying (2). The sum of the labels used for $H^j$ is independent of $j$:

\[
\begin{align*}
 f(v_n) + \sum_{i=1}^{3} f(v_i^1) + \sum_{i=4}^{n-1} f(v_i^1) + \sum_{i=1}^{m} f(e_i^j) & = (k/2 + 1) + \frac{9k + 8}{2} + \sum_{i=4, i \text{ odd}}^{n+m-1} [1 + (i - 1)k + j] + \sum_{i=4, i \text{ even}}^{n+m-1} [ik + 2 - j] \\
 & = \frac{3(n + m) - 2}{2} + \frac{k[(n + m - 1)^2 + 1]}{2}. \quad \Box
\end{align*}
\]

Let $H$ be the graph obtained by taking a star with $n$ vertices and connecting an additional vertex $v$ to exactly one leaf of the star. The **banana tree** $B_{k,n}$ is the graph $A_k(H, v)$.

**Corollary 1.** For any integers $k$ and $n \geq k + 2$, the banana tree $B_{k,n}$ is $B_{1,n}$-supermagic.
The condition \( n \geq k + 2 \) is needed because otherwise \( B_{k,n} \) contains more than \( k \) subgraphs isomorphic to \( B_{1,n} \). We do not have this problems for \( H = B_{t,n} \) with \( \ell \geq 2 \), and therefore we get the following result.

**Corollary 2.** For any integers \( n \), \( k \) and \( \ell \in [2, k - 1] \), the banana tree \( B_{k,n} \) is \( B_{\ell,n} \)-supermagic. In particular, for \( \ell = k - 1 \), this solves the open problem in \([9]\).

**Remark 1.** Note that the labeling strategy in the first case of the proof of Theorem 11 immediately gives the following result. Fix an induced subgraph \( H' \) of \( H \), say induced by the last \( \ell \) vertices, and form a graph, \( G = A_k(H, H') \) by identifying the vertices \( v_1^k, \ldots, v_k^k \) for \( i = n - \ell + 1, \ldots, n \). If \( n - \ell + |E(H) \setminus E(H')| \) is even and \( G \) contains exactly \( k \) subgraphs isomorphic to \( H \), then \( G \) is \( H \)-supermagic.

### 3 Attaching copies of a fixed graph to a path

Let \( G \) be a graph with \( n \) vertices, say \( V(G) = \{v_1, \ldots, v_n\} \) and \( m \) edges, say \( E(G) = \{e_1, \ldots, e_m\} \). Let \( P_k, k \geq 2 \), be a path with vertex set \( V(P_k) = \{w_1, w_2, \ldots, w_k\} \) and edge set \( E(P_k) = \{w_1w_2, \ldots, w_{k-1}w_k\} \). Take \( k \) copies of \( G \) denoted by \( G^1, G^2, \ldots, G^k \) and let the vertex and edge sets be \( V(G_i) = \{v_i^1, \ldots, v_n^i\} \) and \( E(G_i) = \{e_i^1, \ldots, e_m^i\} \). Fix a vertex \( v \in V(G) \), without loss of generality \( v = v_n \), and attach the copies of \( G \) to the path such that the vertex \( v_i^k \in V(G^k) \) is identified with the vertex \( w_i \) in \( P_k \), \( i = 1, 2, \ldots, k \). The resulting graph is denoted by \( P_k(G, v) \).

**Theorem 2.** Let \( G \) be a graph with \( n \) vertices and \( m \) edges and let \( k \geq 2 \) be an integer. If \( (n + m - 1)(k - 1) \) is even and \( P_k(G, v) \) contains exactly \( k - 1 \) subgraphs isomorphic to \( P_2(G, v) \), then \( P_k(G, v) \) is \( P_2(G, v) \)-supermagic with supermagic sum \( (n + m)[(n + m + 1)k + 1] + [k/2] \).

**Proof.** The graph \( P_k(G, v) \) has \( kn \) vertices and \( (m + 1)k - 1 \) edges. We define the labeling
\[
f : V(P_k(G, v)) \cup E(P_k(G, v)) \to [(m + n + 1)(k - 1)]
\]
as follows. For the path vertices, we use the first \( k \) labels \( 1, \ldots, k \):
\[
f(w_i) = \begin{cases} 
    (i + 1)/2 & \text{if } i \text{ is odd}, \\
    [k/2] + i/2 & \text{if } i \text{ is even}, 
\end{cases}
\]
for \( i \in [k] \).

For the path edges, we use the the labels in \( [(m + n)k + 1, (m + n)k + (k - 1)] \):
\[
f(w_iw_{i+1}) = (n + m + 1)k - i \quad \text{for } i \in [k - 1].
\]

For labeling the remaining elements we distinguish two cases.

**Case 1** If \( n + m - 1 \) is even, we set
\[
f(v_j^i) = \begin{cases} 
    jk + i & \text{if } j \text{ is odd}, \\
    (j + 1)k + 1 - i & \text{if } j \text{ is even}, 
\end{cases}
\]
for \( j \in [n - 1], i \in [k] \).

**Case 2** If \( n + m - 1 \) is odd, we set
\[
f(v_j^i) = \begin{cases} 
    (n - 1 + j)k + i & \text{if } n - 1 + j \text{ is odd}, \\
    (n + j)k + 1 - i & \text{if } n - 1 + j \text{ is even}, 
\end{cases}
\]
for \( j \in [m], i \in [k] \). (4)
Case 2 If $n + m − 1$ is odd and $k$ is odd, we set
\[
\begin{align*}
f(v_1^i) &= k + i \\
f(v_2^i) &= \begin{cases} (5k + 1)/2 + i & \text{for } i \in \lfloor (k - 1)/2 \rfloor, \\ (3k + 1)/2 + i & \text{for } i \in \lceil (k + 1)/2, k \rceil, \end{cases} \\
f(v_3^i) &= \begin{cases} 4k + 1 - 2i & \text{for } i \in \lfloor (k - 1)/2 \rfloor, \\ 5k + 1 - 2i & \text{for } i \in \lceil (k + 1)/2, k \rceil. \end{cases}
\end{align*}
\]
As in the proof of Theorem 1 (3) and (4) are used for labeling the remaining vertices and edges.

Denoting the sum of the labels used for $G^i$ by $A_i$, we obtain
\[
A_i = \sum_{j=1}^{n-1} f(v_j^i) + \sum_{j=1}^{m} f(e_j^i) + f(w_i)
= \frac{(n + m - 1)(n + m + 1)k + (n + m - 1)}{2} + \begin{cases} (i + 1)/2 & \text{if } i \text{ is odd}, \\ \lceil k/2 \rceil + i/2 & \text{if } i \text{ is even}. \end{cases}
\]
Finally, the sum of the labels of the subgraph isomorphic to $P_2(G, v)$ which is formed by $G^i$, $G^{i+1}$ and the edge $w_iw_{i+1}$ is independent of $i$:
\[
A_i + A_{i+1} + f(w_iw_{i+1}) = (n + m)[(n + m + 1)k + 1] + \lceil k/2 \rceil. \tag*{□}
\]

Remark 2. We think that it might be possible that the parity assumption in Theorem 2 is not necessary, and we leave the case that both $n + m$ and $k$ are even for future work.

Example 1. We illustrate the construction in Theorem 2 for $k = 5$, $G = K_4^-$ (the graph obtained from a complete graph on 4 vertices by deleting one edge) and $v$ being a vertex of degree 3 in $G$. We obtain the $P(K_4^-, v_4)$-supermagic labeling shown in Figure 1.

Corollary 3. Let $G = K_{1,n-1}$ be a star with $n \geq 4$ vertices, and let $v$ be a pendant vertex of $G$. The firecracker graph is $F_{k,n} = P_k(G, v)$. Since $|V(G)| + |E(G)| = 2n − 1$ is odd, and there are exactly $k - 1$ subgraphs isomorphic to $F_{2,n}$, the firecracker $F_{k,n}$ is $F_{2,n}$-supermagic with supermagic sum $(2n - 1)(2nk + 1) + \lceil k/2 \rceil$.

4 C3-Supermagic Labeling of the Flower Graph $\mathcal{F}_n$

A flower graph $\mathcal{F}_n$ is constructed from a wheel $W_n$ by adding $n$ vertices, each new vertex adjacent to one vertex on the cycle and the center of the wheel with vertex set $V = \{x_0\} \cup \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\}$ and edge set $E = \{x_0x_i : 1 \leq i \leq n\} \cup \{x_0y_i : 1 \leq i \leq n\} \cup \{x_iy_i : 1 \leq i \leq n\} \cup \{x_ix_{i+1} : 1 \leq i \leq n\}$, where indices are interpreted modulo $n$ in the obvious way.
We consider four permutations \(\pi_1, \ldots, \pi_4\) of the set \([n]\), and define a total labeling of the flower graph \(F_n\) as follows.

\[
\begin{align*}
 f(x_0) &= n + 1, \\
 f(x_i) &= \pi_1(i) & \text{for } i \in [n], \\
 f(y_i) &= \pi_2(i) + n + 1 & \text{for } i \in [n], \\
 f(x_0x_i) &= \pi_2(i) + 5n + 1 & \text{for } i \in [n], \\
 f(x_0y_i) &= \pi_2(i) + 4n + 1 & \text{for } i \in [n], \\
 f(x_iy_i) &= \begin{cases} 
 \pi_3(i) + 2n + 1 & \text{odd } i \\
 \pi_3(i) + 3n + 1 & \text{even } i
\end{cases} & \text{for } i \in [n], \\
 f(x_ix_{i+1}) &= \pi_4(i) + 2n + 1 + (n + 1)/2 & \text{for } i \in [n - 1].
\end{align*}
\]

**Lemma 1.** Define

\[
\varphi_k^i(\pi_1, \ldots, \pi_4) = \pi_1(i) + \pi_1(i + 1) + \pi_2(i) + \pi_2(i + 1) + \pi_4(i) + (n + 1)/2 - 1,
\]

\[
\varphi_k^j(\pi_1, \ldots, \pi_4) = \pi_1(i) + 3\pi_2(i) + \pi_3(i) + \begin{cases} 
 n & \text{if } i \text{ is even}, \\
 0 & \text{if } i \text{ is odd}.
\end{cases}
\]

If \(\varphi_k^i(\pi_1, \ldots, \pi_4)\) is equal to a constant \(\varphi\) for all \(i \in [n]\) and \(k \in \{1, 2\}\), then the labeling given above is \(C_3\)-supermagic with supermagic sum \(f(C_3) = 13n + 5 + \varphi\).

**Proof.** The flower graph \(F_n\) contains \(2n\) subgraphs \(H_1, \ldots, H_{2n}\) isomorphic to \(C_3\). We distinguish two types of 3-cycles: (1) cycles induced by vertex sets \(\{x_0, x_i, x_{i+1}\}\), and (2) cycles induced by vertex sets \(\{x_0, x_i, y_i\}\).

**Case 1** Cycle \((x_0, x_i, x_{i+1})\). The sum of the vertex labels is

\[
 f(x_0) + f(x_i) + f(x_{i+1}) = n + 1 + \pi_1(i) + \pi_1(i + 1),
\]
and the sum of the edge labels is
\[ f(x_0x_i) + f(x_ix_{i+1}) + f(x_0x_{i+1}) = \pi_2(i) + \pi_2(i+1) + \pi_4(i) + 12n + 3 + (n+1)/2. \tag{6} \]

Taking the sum of (5) and (6), we have the supermagic sum
\[ f(C_3) = n + 1 + \pi_1(i) + \pi_1(i+1) + \pi_2(i) + \pi_2(i+1) + \pi_4(i) + 12n + 3 + (n+1)/2 \]
\[ = 13n + 5 + [\pi_1(i) + \pi_1(i+1) + \pi_2(i) + \pi_2(i+1) + \pi_4(i) - 1 + (n+1)/2] \]
\[ = 13n + 5 + \varphi_1^i(\pi_1, \ldots, \pi_4) \]
\[ = 13n + 5 + \varphi. \]

**Case 2** Cycle \((x_0, x_i, y_i)\). The sum of the vertex labels is
\[ f(x_0) + f(x_i) + f(y_i) = 2n + 2 + \pi_1(i) + \pi_2(i), \tag{7} \]
and the sum of the edge labels is
\[ f(x_0x_i) + f(x_iy_i) + f(x_0y_i) = \pi_2(i) + \pi_2(i) + \pi_3(i) + \begin{cases} 11n + 3 & \text{if } i \text{ is odd}, \\ 12n + 3 & \text{if } i \text{ is even}. \end{cases} \tag{8} \]

Taking the sum of (7) and (8) gives the supermagic sum
\[ f(C_3) = 13n + 5 + \varphi_2^i(\pi_1, \ldots, \pi_4) = 13n + 5 + \varphi. \qedhere \]

In the following lemma we provide permutations \(\pi_1, \ldots, \pi_4\) which satisfy the condition in Lemma [1].

**Lemma 2.** Define the permutations by
\[
\begin{align*}
\pi_1(i) &= i, \\
\pi_2(i) &= n + 1 - \begin{cases} (i + 1)/2 & \text{for odd } i, \\ i/2 + (n + 1)/2 & \text{for even } i, \end{cases} \\
\pi_3(i) &= \begin{cases} (i + 1)/2 & \text{for odd } i, \\ i/2 + (n + 1)/2 & \text{for even } i, \end{cases} \\
\pi_4(i) &= n + 1 - i.
\end{align*}
\]

Then for every \(i \in [n]\) we have \(\varphi_1^i(\pi_1, \ldots, \pi_4) = \varphi_2^i(\pi_1, \ldots, \pi_4) = 3n + 2\).

**Theorem 3.** For any odd integer \(n\), the flower graph \(F_n\) is \(C_3\)-supermagic.

**Proof.** The total labeling of \(F_n\) can be obtained by applying the permutations in Lemma [2] to the labeling construction. Using the value of \(\varphi\) in Lemma [2] and the supermagic the sum of the permutations in Lemma [2] we have the constant supermagic sum on flower graph \(F_n\)
\[ f(C_3) = 13n + 5 + \varphi = 13n + 5 + 3n + 2 = 16n + 7. \]
Hence, flower graph \(F_n\) is \(C_3\)-supermagic, for odd \(n\). \qedhere

**Example 2.** Using Lemma [2] for \(n = 7\) we get the permutations \(\pi_1 = (1, 2, 3, 4, 5, 6, 7), \pi_2 = (7, 3, 6, 2, 5, 1, 4), \pi_3 = (1, 5, 2, 6, 3, 7, 4)\) and \(\pi_4 = (7, 6, 5, 4, 3, 2, 1)\). These permutations give the labeling for \(F_7\) shown in Figure [2].
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