AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **69(3)** (2017), Pages 368–374

On quotient digraphs and voltage digraphs*

C. Dalfó

Departament de Matemàtiques
Universitat Politècnica de Catalunya
Barcelona
Catalonia
cristina.dalfo@upc.edu

M. A. Fiol

Departament de Matemàtiques
Universitat Politècnica de Catalunya
Barcelona Graduate School of Mathematics, Barcelona
Catalonia
miguel.angel.fiol@upc.edu

M. Miller†

School of Mathematical and Physical Sciences
The University of Newcastle
Newcastle, NSW 2308
Australia

J. Ryan

School of Electrical Engineering and Computing
The University of Newcastle
Newcastle, NSW 2308
Australia
joe.ryan@newcastle.edu.au

In memory of Mirka Miller

* Research of the first two authors is supported by MINECO under project MTM2014-60127-P, and by AGAUR under project 2014SGR1147.
The first author has also received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 734922.

† Was also at: Department of Mathematics, University of West Bohemia, Plzeň, Czech Republic.

Abstract

We study the relationship between two key concepts in the theory of digraphs, those of quotient digraphs and voltage digraphs. These techniques contract or expand a given digraph in order to study its characteristics, or to obtain more involved structures. As an application, we relate the spectrum of a digraph Γ, called a voltage digraph or base, with the spectrum of its lifted digraph Γ^α. We prove that all the eigenvalues of Γ (including multiplicities) are, in addition, eigenvalues of Γ^α. This study is carried out by introducing several reduced matrix representations of Γ^α. As an example of our techniques, we study some basic properties of the Alegre digraph and its base.

1 Introduction

In the study of interconnection and communication networks, the theory of digraphs plays a key role as, in many cases, the links between nodes are unidirectional. Within this theory, there are two concepts that have shown to be very fruitful to construct good and efficient networks. Namely, those of quotient digraphs and voltage digraphs. Roughly speaking, quotient digraphs allow us to give a simplified or "condensed" version of a larger digraph, while the voltage digraph technique do the converse by "expanding" a smaller digraph. From this point of view, it is natural that both techniques have close relationships. In this paper we explore some of such interrelations.

The paper is organized as follows. In the rest of this section, we give some basic background information. In Section 2 we present the basic definition and results on regular partitions and their corresponding quotient digraphs. In Section 3, we recall the definitions of voltage and lifted digraphs. Finally, Section 4 is devoted studying a representation of a lifted digraph with a matrix whose size equals the order of the (much smaller) base digraph.

1.1 Background

Here, we recall some basic terminology and simple results concerning digraphs and their spectra. For the concepts and/or results not presented here, we refer the reader to some of the basic textbooks on the subject; for instance Chartrand and Lesniak [3] or Diestel [5].

Throughout this paper, $\Gamma = (V, E)$ denotes a digraph, with vertex set V and arc set E, that is *strongly connected*, namely, each vertex is connected to all other vertices by traversing the arcs in their corresponding direction. An arc from vertex u to vertex v is denoted by either (u, v), uv, or $u \to v$. We allow *loops* (that is, arcs from a vertex to itself), and *multiple arcs*. The set of vertices adjacent to and from $v \in V$ is denoted by $\Gamma^-(v)$ and $\Gamma^+(v)$, respectively. Such vertices are referred to as *in-neighbors* and *out-neighbors* of v, respectively. Moreover, $\delta^-(v) = |\Gamma^-(v)|$

and $\delta^+(v) = |\Gamma^+(v)|$ are the *in-degree* and *out-degree* of vertex v, and Γ is *d-regular* when $\delta^+(v) = \delta^-(v) = d$ for all $v \in V$. Similarly, given $U \subset V$, $\Gamma^-(U)$ and $\Gamma^+(U)$ represent the sets of vertices adjacent to and from (the vertices) of U, respectively. Given two vertex subsets $X, Y \subset V$, the subset of arcs from X to Y is denoted by $e(X, Y)$.

The spectrum of Γ, denoted by $\mathrm{sp}\Gamma = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, is constituted by the distinct eigenvalues λ_i (with the corresponding algebraic multiplicities m_i), $i = 0, 1, \ldots, d$, of its adjacency matrix $\boldsymbol{A}$.

2 Regular partitions and quotient digraphs

Let $\Gamma = (V, E)$ be a digraph with n vertices and adjacency matrix $\boldsymbol{A}$. A partition π of its vertex set $V = U_1 \cup U_2 \cup \cdots \cup U_m$, for $m \leq n$, is called *regular* if the number of arcs from a vertex $u \in U_i$ to vertices in U_j only depends on i and j. Let c_{ij} be the number of arcs which join a fixed vertex in U_i to vertices in U_j. A matrix characterization of this property is the following: Let $\boldsymbol{S}$ be the $n \times m$ matrix whose i-th column is the normalized characteristic vector of U_i, so that $\boldsymbol{S}^\top \boldsymbol{S} = \boldsymbol{I}$ (the identity matrix), and consider the so-called *quotient matrix*

$$\boldsymbol{B} = \boldsymbol{S}^\top \boldsymbol{A} \boldsymbol{S}. \tag{1}$$

Then, π is regular if and only if

$$\boldsymbol{S}\boldsymbol{B} = \boldsymbol{A}\boldsymbol{S}. \tag{2}$$

The digraph $\pi(\Gamma)$ whose (weighted) adjacency matrix is the quotient matrix is called *quotient digraph*, and their arcs can have weight different from 1. More precisely, the vertices of the quotient digraph are the subsets U_i, for $i = 1, 2, \ldots, m$, and the arc from vertex U_i to vertex U_j has weight c_{ij}. For the case of quotient digraphs obtained from non-directed graphs, see Godsil [7] (Lemma 2.1). From this, we have the following basic result, where the regular partition of V is called a *regular* partition of $\boldsymbol{A}$.

Lemma 2.1. *Every eigenvalue of the quotient matrix $\boldsymbol{B}$ of a regular partition of $\boldsymbol{A}$ is also an eigenvalue of $\boldsymbol{A}$, that is,* $\mathrm{sp}\boldsymbol{B} \subset \mathrm{sp}\boldsymbol{A}$.

3 Voltage and lifted digraphs

Voltage (di)graphs are, in fact, a type of compounding that consists in connecting together several copies of a (di)graph by setting some (directed) edges between any two copies. Usually, the symmetry of the obtained constructions yields digraphs with large automorphism groups. As far as we know, one of the first papers where voltage graphs were used for construction of dense graphs is Alegre, Fiol and Yebra [1], but without using the name of 'voltage graphs'. This name was coined previously by Gross [8]. For more information, see Gross and Tucker [9], Baskoro, Branković, Miller, Plesník, Ryan and Siráň [2], and Miller and Siráň [10].

Let Γ be a digraph with vertex set $V = V(\Gamma)$ and arc set $E = E(\Gamma)$. Then, given a group G with generating set Δ, a voltage assignment of Γ is a mapping $\alpha : E \to \Delta$. The lift Γ^α is the digraph with vertex set $V(\Gamma^\alpha) = V \times G$ and arc set $E(\Gamma^\alpha) = E \times G$, where there is an arc from vertex (u, g) to vertex (v, h) if and only if $uv \in E$ and $h = g\alpha(uv)$. Such an arc is denoted by (uv, g).

3.1 The adjacency matrix of the lifted digraph

It is clear that the base digraph with the voltage assignment univocally determines the adjacency matrix of its lift. To define it we need to consider the following concepts. Given a (multiplicative) group G together with a given order of its elements $g_1(= 1), g_2, \dots, g_n$, a *$G$-circulant matrix* is defined as a square matrix $\boldsymbol{A}$ of order n indexed by elements of G, with first row $a_{1,1} = a_{g_1}$, $a_{1,2} = a_{g_2}$, …, $a_{1,n} = a_{g_n}$, and elements

$$(\boldsymbol{A})_{g,h} = a_{hg}, \qquad g, h \in G.$$

Thus, the elements of row g are identical to those of the first row, but they are permuted for the action of g on G. In particular, a *circulant matrix* (see Davis [4]) corresponds to a G-circulant matrix with the cyclic group $G = \mathbb{Z}_n$ and natural order $0, 1, \dots, n-1$. Another example is the adjacency matrix $\boldsymbol{A}$ of the Cayley digraph $\mathrm{Cay}(G, \Delta)$ of the group G with generating set Δ, which is a G-circulant matrix whose first row has elements $a_{1,j} = 1$ if $g_j \in \Delta$, and $a_{1,j} = 0$, otherwise.

The concept of block G-circulant matrix is similar, but now the elements a_{g_1}, $a_{g_2}, \dots, a_{g_n}$ of the first row (and, consequently, the other rows) are replaced by the $m \times m$ matrices (or blocks) $\boldsymbol{A}_1 = \boldsymbol{A}_{g_1}, \dots, \boldsymbol{A}_n = \boldsymbol{A}_{g_n}$.

From the above definitions, the following result is straightforward.

Lemma 3.1. *Let Γ be a base graph with voltage assignment α on the group $G = \{g_1(= 1), \dots, g_m\}$. Let Γ_i be the spanning subgraph of Γ with arc set $\alpha^{-1}(g_i)$, and adjacency matrix $\boldsymbol{A}_i$, for $i = 1, \dots, m$. Then the adjacency matrix $\boldsymbol{A}$ of the lifted digraph Γ^α is the block G-circulant matrix with first block-row $\boldsymbol{A}_1, \boldsymbol{A}_2, \dots, \boldsymbol{A}_m$.* □

3.2 The spectrum of the lifted digraph

Apart from the obvious approach of computing the characteristic polynomial of the adjacency matrix, it seems to be difficult, in general, to get a general result about the whole spectrum of the lifted digraph Γ^α. However, we have the following proposition.

Proposition 3.2. *Let Γ be a base graph with vertices $u_1, \dots, u_n$, and a given voltage assignment α on the group $G = \{g_1, \dots, g_m\}$. Let $\boldsymbol{B} = \sum_{i=1}^m \boldsymbol{A}_i$, where $\boldsymbol{A}_i$ is the adjacency matrix of the subgraph of Γ with arc set $\alpha^{-1}(g_i)$. Then*

$$\mathrm{sp}\boldsymbol{B} \subset \mathrm{sp}\Gamma^\alpha.$$

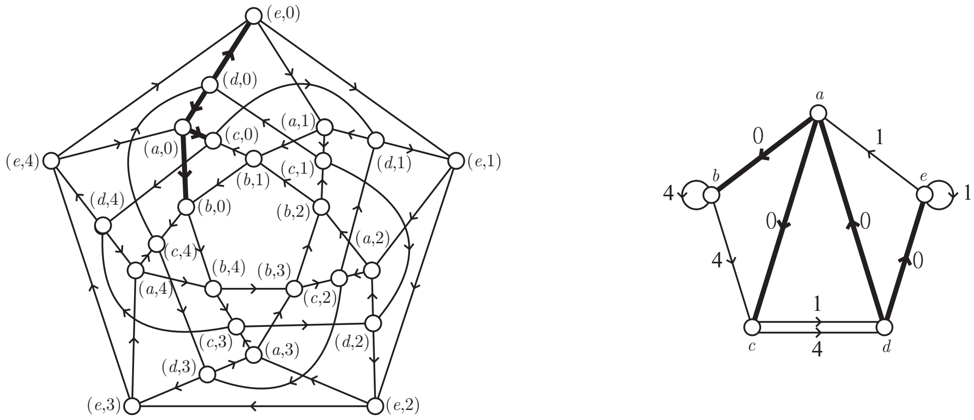

Figure 1: The Alegre digraph (left), and its quotient digraph (right). The adjacencies in the copy 0 are represented with a thick line.

Proof. Just note that $\boldsymbol{B}$ is the quotient matrix of a regular partition, so Lemma 2.1 applies. □

An example: The Alegre digraph

The Alegre digraph is the 2-regular digraph with $n = 25$ vertices and diameter $k = 4$ represented in Figure 3.2 (left). This digraph was found by Fiol, Yebra, and Alegre in [6]. The Alegre digraph can be seen as the lifted digraph Γ^α of the base digraph Γ with the voltage assignments shown in Figure 3.2 (right).

Then, the nonzero blocks of the first row constituting the adjacency matrix of Γ^α are

$$\boldsymbol{A}_0 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{A}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, Γ^α has a regular partition with five sets of five vertices each. As expected, the corresponding quotient digraph is the base graph Γ, with quotient matrix

$$\boldsymbol{B} = \sum_{i=0}^{4} \boldsymbol{A}_i = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and spectrum $\mathrm{sp}\boldsymbol{B} = \{2, 0^{(2)}, \mathrm{i}, -\mathrm{i}\}$.

In fact, the spectrum of the Alegre digraph is

$$\mathrm{sp}\Gamma^\alpha = \left\{2, 0^{(10)}, \mathrm{i}^{(5)}, -\mathrm{i}^{(5)}, \frac{1}{2}(-1+\sqrt{5})^{(2)}, \frac{1}{2}(-1-\sqrt{5})^{(2)}\right\},$$

where, in agreement with Proposition 3.2, we observe that $\mathrm{sp}\boldsymbol{B} \subset \mathrm{sp}\Gamma^{\alpha}$. Notice also that, in this case, the other eigenvalues of Γ are those ($\neq 2$) of the undirected cycle C_5, whose spectrum is

$$\mathrm{spC}_5 = \left\{2, \frac{1}{2}(-1+\sqrt{5})^{(2)}, \frac{1}{2}(-1-\sqrt{5})^{(2)}\right\}.$$

4 Matrix representations

In this section, we study how to fully represent a lifted digraph with a matrix whose size equals the order of the base graph.

We deal with the case when the group G of the voltage assignments is cyclic. Thus, let $\Gamma = (V, E)$ be a digraph with voltage assignment α on the group $G = \mathbb{Z}_m = \{0, 1, \ldots, m-1\}$. Its *polynomial matrix* $\boldsymbol{B}(x)$ is a square matrix indexed by the vertices of Γ, and whose elements are polynomials in the quotient ring $\mathbb{R}^c_{m-1}[x] = \mathbb{R}[x]/(x^m)$, where (x^m) is the ideal generated by the polynomial x^m. More precisely, each entry of $\boldsymbol{B}(x)$ is fully represented by a polynomial of degree at most $m-1$, say $(\boldsymbol{B}(x))_{uv} = p_{uv}(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{m-1}x^{m-1}$, where

$$\alpha_i = \begin{cases} 1, & \text{if } uv \in E \text{ and } \alpha(uv) = g_i, \\ 0, & \text{otherwise.} \end{cases} \qquad i = 0, \ldots, m-1.$$

For example, in the case of the Alegre digraph in Fig. 3.2, the polynomial matrix is

$$\boldsymbol{B}(x) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & x^4 & x^4 & 0 & 0 \\ 0 & 0 & 0 & x+x^4 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ x & 0 & 0 & 0 & x \end{pmatrix},$$

where $(\boldsymbol{B}(x))_{ij} = \alpha x^r + \beta x^s$, with $\alpha, \beta \in \{0,1\}$ and $r, s \in \mathbb{Z}_5$, means that there are arcs from vertex (i,p) to vertices $(j, p+r)$ or/and $(j, p+s)$ if and only if $\alpha = 1$ or/and $\beta = 1$. More generally, it can be shown that $(\boldsymbol{B}(x)^{\ell})_{ij} = \alpha_4 x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0$ if and only if there are α_h walks on length ℓ from vertex (i,p) to vertex $(j, p+h)$, for $h = 0, \ldots, 4$.

For example, the first row of $\boldsymbol{I} + \boldsymbol{B}(x) + \boldsymbol{B}(x)^2 + \boldsymbol{B}(x)^3 + \boldsymbol{B}(x)^4$ has entries: $3+x+x^2+x^3+x^4$, $1+x+x^2+x^3+2x^4$, $1+x+x^2+x^3+2x^4$, $1+x+x^2+x^3+2x^4$, $2+x+x^2+x^3+x^4$. Note that all coefficients α_i, for $i = 0, \ldots, 4$, of the above polynomials are non-zero, since Γ^{α} has diameter four. Notice also that the quotient matrix of Γ^{α} is $\boldsymbol{B}(1)$.

By reading as columns, this means that if $\boldsymbol{A}$ is the adjacency matrix of the Alegre digraph, then, the first row of $\boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2 + \boldsymbol{A}^3 + \boldsymbol{A}^4$ is

$$3, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 1.$$

The following result shows that the powers of $\boldsymbol{B}(x)$ yield the same information as the powers of the adjacency matrix of the lifted digraph Γ^{α}.

Lemma 4.1. *Let* $(\boldsymbol{B}(x)^{\ell})_{uv} = \beta_0 + \beta_1 x + \cdots + \beta_{m-1}x^{m-1}$. *Then, for every* $i = 0, \ldots, m-1$, *the coefficient* β_i *equals the number of walks of length* ℓ *in the lifted digraph* Γ^{α}, *from vertex* (u, h) *to vertex* $(v, h+i)$ *for every* $h \in G$.

Proof. The result is clear for $\ell = 0, 1$. Then, the result follows easily by using induction. □

Note that in the above result, the products of the entries (polynomials) of $\boldsymbol{B}(x)$ must be understood in the ring $\mathbb{R}^c_{m-1}$.

References

[1] I. Alegre, M. A. Fiol and J. L. A. Yebra, Some large graphs with given degree and diameter, *J. Graph Theory* **10** (1986), 219–224.

[2] E. T. Baskoro, L. Branković, M. Miller, J. Plesník, J. Ryan and J. Širáň, Large digraphs with small diameter: A voltage assigment approach, *J. Combin. Math. Combin. Comput.* **24** (1997), 161–176.

[3] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, third ed., Chapman and Hall, London, 1996.

[4] P. J. Davis, *Circulant Matrices*, John Wiley & Sons, New York, 1979.

[5] R. Diestel, *Graph Theory* (4th ed.), Graduate Texts in Mathematics **173**, Springer-Verlag, Heilderberg, 2010.

[6] M. A. Fiol, J. L. A. Yebra and I. Alegre, Line digraph iterations and the (d, k) digraph problem, *IEEE Trans. Comput.* **C-33** (1984), 400–403.

[7] C. D. Godsil, *Algebraic Combinatorics*, Chapman and Hall, New York, 1993.

[8] J. L. Gross, Voltage graphs, *Discrete Math.* **9** (1974), 239–246.

[9] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Wiley, New York, 1987.

[10] M. Miller and J. Širáň, Moore graphs and beyond: A survey of the degree/diameter problem, *Electron. J. Combin.* **20(2)** (2013), #DS14v2.

(Received 11 Oct 2016; revised 23 June 2017)