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Facial rainbow colorings of trees

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In memory of Mirka Miller

Abstract

A vertex coloring of a plane graph G is a *facial rainbow coloring* if any two vertices of G connected by a facial path have distinct colors. The *facial rainbow number* of a graph G, denoted $\mathrm{vrb}(G)$, is the minimum number of colors that are necessary in any facial rainbow coloring. In the present note we investigate the facial rainbow coloring of trees. It is proved that for any nontrivial tree T the inequalities $L + 1 \leq \mathrm{vrb}(T) \leq \left\lceil \frac{5}{3}(L + 1) \right\rceil$ hold, where L is the length of a longest facial path in T. The upper bound is improved to $L + 6$ if T does not contain any internal vertices of degree 2.

1 Introduction

All graphs considered in this note are simple connected plane graphs provided that it is not stated otherwise. We use a standard graph theory terminology according to West [26]. However, we recall some important notions below.

A *plane* graph is a drawing of a planar graph in the Euclidean plane such that two edges may intersect only at endvertices. Let G be a connected plane graph with vertex set $V(G)$, edge set $E(G)$, and face set $F(G)$. The *boundary* of a face α is the boundary in the usual topological sense. It is a collection of all edges and vertices contained in the closure of α that can be organized into a closed walk in G traversing along a simple closed curve lying just inside the face α. This closed walk is unique up to the choice of the initial vertex and direction, and is called the *boundary walk* of the face α (see [15], p. 101).

Let α be a k-gonal face of G having a boundary walk $v_0 v_1 \dots v_{k-1} v_k$, where $v_k = v_0$ with $v_i \in V(G)$, $e_i \in E(G)$, and $e_i = v_i v_{i+1}$ for every $i = 0, 1, \dots, k-1$. A *facial path* of α is any path of the form $v_m v_{m+1} \dots v_{n-1} v_n$ (subscripts are calculated modulo k) which is a part of the boundary walk of α.

The *size* of a face $\alpha \in F(G)$ is the length of its boundary walk.

Two vertices (two edges or two faces) are *adjacent* if they are connected by an edge (have a common endvertex or their boundaries have a common edge, respectively). A vertex and an edge are *incident* if the vertex is an endvertex of the edge. A vertex (or an edge) and a face are *incident* if the vertex (or the edge) lies on the boundary of the face.

Let $\deg(v), \delta(G), \Delta(G), \Delta^*(G), \mathrm{diam}(G)$, and $L(G)$ denote the degree of a vertex v, the minimum degree, the maximum degree, the maximum face size, diameter, and the length of a longest facial path of G, respectively. For a path P let $|P|$ denote the length of P.

2 Cyclic and rainbow colorings

A cyclic coloring of a 2-connected plane graph G is a coloring of its vertices such that any two distinct vertices incident with the same face receive distinct colors. The cyclic chromatic number of a 2-connected plane graph G, denoted by $\chi_c(G)$, is the smallest number of colors used in a cyclic coloring of G. This graph invariant was introduced by Ore and Plummer [21]. Clearly, $\chi_c(G)$ is trivially bounded from below by the size $\Delta^*(G)$ of a largest face of G.

Ore and Plummer [21] proved the first upper bound $2\Delta^*$ for $\chi_c(G)$. Borodin [6] slightly improved this bound to $2\Delta^* - 3$ for $\Delta^* \geq 8$. Significant progress has been made recently. Borodin, Sanders, and Zhao [9] managed to prove the upper bound of $\left\lceil \frac{9}{5}\Delta^* \right\rceil$ and the currently best known general upper bound $\left\lceil \frac{5}{3}\Delta^* \right\rceil$ is due to Sanders and Zhao [25].

Better results are known for graphs with small maximum face sizes, i.e., for small values of Δ^*. The case of cyclic colorings of plane triangulations, i.e., $\Delta^* = 3$, is equivalent to the famous Four Color Theorem which was proved by Appel and Haken in [2], [3] and [4] (see also [24] for a refinement of its proof). Hence $\chi_c(G) \leq 4$ for $\Delta^* = 3$.

The case of $\Delta^* = 4$ is Ringel's problem [23]. The problem was solved and it was shown that $\chi_c(G) \leq 6$ by Borodin [8] (see [6] for a simpler proof). The case $\chi_c(G) \leq 9$ for $\Delta^* = 6$ was proved by Hebdige and Král' [17]. These bounds of $\chi_c(G)$ for $\Delta^* = 3$, $\Delta^* = 4$, and $\Delta^* = 6$ are the only ones which are currently known to be tight. The upper bounds 8 for $\Delta^* = 5$, [9], 11 for $\Delta^* = 7$, [16], and 13 for $\Delta^* = 8$, [26], were also proved.

The best known lower bound $\left\lfloor \frac{3}{2}\Delta^* \right\rfloor$ for cyclic colorings is also conjectured to be the best possible upper bound. The conjecture is by Borodin [8]. For discussion on this see [20] by Jensen and Toft.

Conjecture 2.1 ([8])**.** Every 2-connected plane graph has a cyclic coloring with at most $\left\lceil \frac{3}{2}\Delta^* \right\rceil$ colors.

Amini, Esperet, and van den Heuvel [1] proved that for every $\varepsilon > 0$ there exists Δ_ε such that every 2-connected plane graph of maximum face size $\Delta^* \geq \Delta_\varepsilon$ admits a cylic coloring with at most $(\frac{3}{2} + \varepsilon)\Delta^*$ colors.

Restricting attention to 3-connected plane graphs, Plummer and Toft [22] proved that $\chi_c(G) \leq \Delta^* + 9$ and proposed the following conjecture.

Conjecture 2.2 ([22])**.** Every 3-connected plane graph has a cyclic coloring with at most $\Delta^* + 2$ colors.

This conjecture is true for 3-connected plane graphs with $\Delta^* \geq 16$. Horňák and Jendroľ [18] proved it for $\Delta^* \geq 24$, Horňák and Zlámalová [19] for $\Delta^* \geq 18$ and Dvořák et al. [12] for $16 \leq \Delta^* \leq 18$. Borodin [6] proved this conjecture for $\Delta^* = 4$, and Appel and Haken [2] for $\Delta^* = 3$. Enomoto, Horňák, and Jendroľ [14] obtained for $\Delta^* \geq 60$ even stronger results, namely $\chi_c \leq \Delta^* + 1$. Azarija et al. [5] proved the same bound for plane graphs in which all faces of size four or more are vertex-disjoint.

Conjecture 2.3 ([27])**.** Every 3-connected plane graph with $\Delta^* \neq 4$ has a cyclic coloring with at most $\Delta^* + 1$ colors.

The best known general upper bound is due to Enomoto and Horňák [13] who proved that $\chi_c(G) \leq \Delta^* + 5$ for every 3-connected plane graph G.

For more discussion about the problem see surveys in [7] or [8].

In this paper we introduce the notion of a facial rainbow coloring. A *facial rainbow coloring* of a connected plane graph G is a coloring of its vertices such that any two distinct vertices connected by a facial path receive distinct colors. The *facial rainbow number* of G, denoted by $\mathrm{vrb}(G)$, is the smallest number of colors used in a facial rainbow coloring of G.

Observe that the notion of the facial rainbow coloring extends the notion of the cyclic coloring also for all connected plane graphs. Therefore from now we will use this more general, newer, and more appropriate notion.

Motivated by the above mentioned papers and the paper by Brešar et al. [10], in this note, we investigate the facial rainbow coloring of trees with plane embeddings. Note that, in this paper, we identify a tree T with some plane embedding of T.

For other topics concerning facially restricted types of colorings of plane graphs see a recent survey [11] by Czap and Jendroľ.

3 Trees and their embeddings in the plane

Let graph G be a plane graph. The value of $\mathrm{vrb}(G)$ depends on the embedding of the graph G in the plane. Let $L(G)$ denote the length of a longest facial path in G. Then $\mathrm{vrb}(G) \geq L(G) + 1$. A *caterpillar* is a tree having the property that after deleting all leaves from it, the resulting graph is a path.

Let H_t be a caterpillar with $2t$ leaves all of whose internal vertices are of degree 4. The graph H_t can be embedded into the plane in several ways. Let H_t^* be such an embedding for which the longest facial path has a length of $L = t$. Let $\tilde{H}_t$ be the another embedding with the longest facial path of length $L = 3$. It is easy to show that $\text{vrb}(\tilde{H}_t) = 4$ and $\text{vrb}(H_t^*) = t + 1$ while $\text{diam}(\tilde{H}_t) = \text{diam}(H_t^*) = \text{diam}(T) = t$. Let T be a tree having $\Delta(T) \geq 3$. Then it has l leaves, $l \geq \Delta(T) \geq 3$. Those vertices of T which are not leaves are called *internal* vertices. Let $v_1, v_2, \ldots, v_l$ be leaves of T in an order given by the (unique) boundary walk of T. Denote by $P_{i,i+1}$ the unique $v_i - v_{i+1}$-path in T, where subscripts are calculated modulo l. A facial path is called the *maximal facial* one if it is not a proper subgraph of a longer facial path. A facial path is called a *maximum facial path* of T if it is maximal and is the longest one among all maximal facial paths of T. The length of a longest maximum facial path is denoted by $L(T)$.

It is easy to see that the set of all maximal facial paths is exactly the following one: $\{P_{i,i+1} : i = 1, \ldots, l, \text{ subscripts modulo } l\}$. Then $L(T) = L = \max\{|P_{i,i+1}| : i = 1, \ldots, l, \text{ subscripts modulo } l\}$.

4 Facial rainbow number and diameter

For a given nontrivial tree T, let l denote the number of its leaves. Denote by $S_{l,r}$ the generalized star with a central vertex v_0 of degree l and l paths of length r starting at v_0. It is easy to see that $S_{l,r}$ has $rl + 1$ vertices, diameter $2r$ and facial rainbow number $2r + 1$ if l is even, or $3r + 1$ if l is odd.

Theorem 4.1. *Let T be a tree of diameter* $\text{diam}(T)$*. Then*

$$\text{vrb}(T) \leq 1 + \left\lfloor \frac{3}{2}\,\text{diam}(T) \right\rfloor .$$

Moreover, the bound is tight.

Proof. The proof is by induction on the number of the leaves l.

If $l = 2$, then T is a path on $\text{diam}(T) + 1$ vertices. The theorem is true. Let $l \geq 3$ and let $v_1, v_2, \ldots, v_l$ be consecutive leaves in an order given by the (unique up to the orientation and initial vertex) boundary walk. Consider the leaves v_1, v_2 and v_3. Let $P_{i,j}$ be a $v_i - v_j$-path, $1 \leq i < j \leq 3$. Let x be the (unique) common vertex of these three paths. Let Q_i be the $v_i - x$-path. Put $a_i = |Q_i|$, the length of Q_i. Then $a_i + a_j = |P_{i,j}| \leq \text{diam}(T)$, $1 \leq i < j \leq 3$. This gives $a_1 + a_2 + a_3 \leq \frac{3}{2}\,\text{diam}(T)$. Now delete the vertices of Q_2 from T except the vertex x. The resulting tree T^* has $l - 1$ leaves and $\text{diam}(T^*) \leq \text{diam}(T)$. By the induction hypothesis, T^* has a facial rainbow coloring with at most $1 + \left\lfloor \frac{3}{2}\,\text{diam}(T) \right\rfloor$ colors. This coloring can be extended from T^* to T by coloring a_2 uncolored vertices of Q_2. This is always possible because we have at least a_2 available colors for coloring these vertices.

To see that the bound $1 + \left\lfloor \frac{3}{2}\,\text{diam}(T) \right\rfloor$ is tight, consider the graph $S_{l,r}$ for l odd, $l \geq 3$. □

Theorem 4.2. *Let T be a tree of diameter* $\operatorname{diam}(T)$ *and without internal vertices of odd degree. Then*

$$\operatorname{vrb}(T) \leq 1 + \operatorname{diam}(T)\,.$$

Moreover, this bound is tight.

Proof. It is easy to see that our tree T has an even number $l = 2t$ of leaves. The proof is by induction according to t.

If $t = 1$, then T is a path and the theorem trivially holds.

Let $t \geq 2$ and let $v_1, v_2, \ldots, v_{2t}$ be the leaves of T in an order given by the boundary walk. Consider the leaves v_1, v_2, v_3, and v_4. Let $P_{i,j}$ be a $v_i - v_j$-path, $1 \leq i < j \leq 4$. Let x be the common vertex of all of these paths. Let Q_i be a $v_i - x$-path, $i = 1, 2, 3, 4$, having length $|Q_i| = a_i$. Then the path $P_{i,j}$ is a concatenation of the paths Q_i and Q_j and $|Q_i| + |Q_j| = a_i + a_j = |P_{i,j}| \leq \operatorname{diam}(T)$. This yields $a_1 + a_2 + a_3 + a_4 \leq 2\operatorname{diam}(T)$. Moreover, at most one of a_1, a_2, a_3, and a_4 is bigger than $\frac{1}{2}\operatorname{diam}(T)$.

Now delete from T the vertices of the path $P_{2,3}$ except the vertex x. The resulting tree T^* has $l - 2 = 2(t-1)$ leaves and $\operatorname{diam}(T^*) \leq \operatorname{diam}(T)$. Hence T^* has a facial rainbow coloring with $1+\operatorname{diam}(T)$ colors. This coloring can be extended to a coloring of T by coloring $a_2 + a_3$ uncolored vertices of the path $P_{2,3}$. This is always possible because for the coloring of the vertices of Q_2 (Q_3) we can also use the colors already used in the coloring of the vertices of Q_4 (Q_1, respectively), and furthermore, we know $a_1 + a_2 + a_3 + a_4 \leq 2\operatorname{diam}(T)$.

To see that the bound $1 + \operatorname{diam}(T)$ is tight, consider the graph $S_{l,r}$ with l even, $l \geq 2$. □

5 Facial rainbow numbers and longest facial paths

As we can see on the example of the graph $\tilde{H}_t$, the difference between the diameter $\operatorname{diam}(\tilde{H}_t)$ and the facial rainbow number $\operatorname{vrb}(\tilde{H}_t)$ of it can be arbitrarily large. Therefore, another parameter can be chosen for estimating the facial rainbow numbers. In this section we will consider the relation between the facial rainbow number $\operatorname{vrb}(T)$ of a given tree T and the length $L(T)$ of a longest facial path of T. Our first result is the following.

Theorem 5.1. *Let T be a tree with $\Delta(T) \geq 3$ and let L be the length of a longest facial path in T. Then* $L + 1 \leq \operatorname{vrb}(T) \leq \left\lceil \frac{5}{3}(L+1) \right\rceil$.

Proof. Let $v_1, v_2, \ldots, v_l$ be leaves of T in an order given by the boundary walk of T. We associate a plane graph $S(T)$ with the tree T of $\Delta(T) \geq 3$. The plane graph $S(T)$ is constructed from T in the following way. First we insert new edges $v_i v_{i+1}$ for every $i = 1, \ldots, l$; subscripts are calculated modulo l. The resulted graph $H(T)$ has $l + 1$ faces: $\alpha_1, \ldots, \alpha_l$ and β, where α_i is bounded by the path $P_{i,i+1}$ and the edge $v_i v_{i+1}$. The remaining l-gonal face β is bounded by the inserted edges $v_i v_{i+1}$; $i = 1, \ldots, l$, subscripts modulo l. The graph $S(T)$ is obtained from the graph $H(T)$ by inserting $l-3$ diagonals into β to get $l-2$ triangular faces instead of β. Note that if all internal vertices of T are of degree at least three, then the graph $S(T)$ is a 3-connected plane graph. The maximum face size $\Delta^*(S(T)) = L+1$. By a theorem of Sanders and Zhao [25] it has a facial rainbow coloring with at most $\lceil \frac{5}{3}\Delta^*(S(T)) \rceil = \lceil \frac{5}{3}(L+1) \rceil$ colors. If we delete all inserted edges from $S(T)$ we obtain the tree T having a facial rainbow coloring with at most the stated number of colors. □

By the graph $S_{l,r}$, r odd, we know that there are trees that need at least $\lfloor \frac{3}{2}L \rfloor + 1$ colors in any facial rainbow coloring. We strongly believe that the following weaker version of Conjecture 2.1 (see [8]) holds.

Conjecture 5.2. If T is a tree with a longest facial path of length L, then

$$\mathrm{vrb}(T) \leq \left\lceil \frac{3}{2}(L+1) \right\rceil .$$

Theorem 5.3. *Let T be a tree without any internal vertices of degree 2 and let L be the length of a longest facial path in T. Then*

(i) $\mathrm{vrb}(T) \leq L+2$ *if* $L \geq 59$ *or* $L = 2$

(ii) $\mathrm{vrb}(T) \leq L+3$ *if* $L \geq 15$ *or* $L \in \{3,4\}$

(iii) $\mathrm{vrb}(T) \leq L+4$ *if* $L \in \{5,6\}$

(iv) $\mathrm{vrb}(T) \leq L+5$ *if* $L = 7$, *and*

(v) $\mathrm{vrb}(T) \leq L+6$ *in all other cases.*

Proof. We proceed analogously as in the proof of Theorem 5.1. Now the graph $S(T)$ is 3-connected. Next we apply the result of Enomoto, Horňák, and Jendroľ [14] if $L \geq 59$, Horňák and Zlámalová [19] if $L \geq 17$, Dvořák et al. [12] if $L \geq 15$, the Four Color Theorem if $L = 2$, Borodin [6] if $L = 3$ or Enomoto and Horňák [13] in all other cases. The rest of the proof is analogous to the proof of the previous theorem. □

6 Two more problems

As we could see, a tree T can have several nonisomorphic embeddings in the plane. So we can define two parameters for a given tree:

$$\mathrm{vrb}^+(T) = \max\Big\{ \mathrm{vrb}(T^*) : T^* \text{ is an embedding of } T \text{ in the plane} \Big\}$$

and

$$\mathrm{vrb}^-(T) = \min\Big\{ \mathrm{vrb}(T^*) : T^* \text{ is an embedding of } T \text{ in the plane} \Big\}.$$

Because no facial path in T can be longer than $\mathrm{diam}(T)$, from Theorems 4.1 and 4.2 we immediately have the following:

Theorem 6.1. *For every tree T the following hold:*

(i) $\mathrm{vrb}^+(T) \leq 1 + \lfloor \frac{3}{2} \mathrm{diam}(T) \rfloor$

(ii) $\mathrm{vrb}^+(T) \leq 1 + \mathrm{diam}(T)$, *if T does not contain any internal vertices of odd degree.*

Problem 6.2. Determine $\mathrm{vrb}^-(T)$ for any tree T.

Next we consider this problem for caterpillars. As defined above, a caterpillar is a tree whose internal vertices induce a path. Let $T = K_{1,k}$, $k \geq 3$, be a star. It is easy to see that $\mathrm{vrb}^-(K_{1,k}) = 3$ or 4 if k is even or odd, respectively.
In general we have the following.

Theorem 6.3. *Let T be a caterpillar with at least two vertices of degree at least three and let t be the order of a longest path induced by the vertices of degree 2 in T. Then*

(i) $\mathrm{vrb}^-(T) = t + 4$, *if T does not contain any vertex of degree 3 and*

(ii) $t + 4 \leq \mathrm{vrb}^-(T) \leq 2t + 5$, *otherwise.*

Moreover, the bounds $t + 4$ and $2t + 5$ are tight.

Proof. Let $B = (v_1, ..., v_k)$ be the backbone path induced by the internal vertices of T. Clearly $k \geq 2$. Denote by $d_i = \deg_T(v_i)$ for any $i = 1, ..., k$. We consider three cases.

Case 1. Let $d_i \geq 4$ for any $i = 1, ..., k$. Then it is easy to see that there is an embedding of T in the plane such that any facial path of this embedding consists of at most four vertices. Let v_0 be a leaf adjacent to v_1 but not on the facial path together with v_2. We obtain the required 4-coloring by coloring v_0 with color a, v_1 with color b and then we continue using color a and b alternatively along the path B. Let x be a leaf adjacent to v_1 and on the same facial path as v_0 is. Then the required 4-coloring is completed by the alternative use of colors c and d on leaves while going from x along the boundary walk of T in the direction avoiding the leaf v_0.

Case 2. Let $d_i \geq 3$ for any $i = 1, ..., k$. Then we can easily find an embedding of T in the plane such that any facial path of this embedding has at most five vertices. We obtain the required 5-coloring by coloring a leaf v_0 adjacent to v_1 using color a, then we continue color vertices $v_1, ..., v_k$ with colors b, c, a, b, c,... Let x be a leaf adjacent to v_1 and on the same facial path as vertex v_0. Then the required 5-coloring is completed by alternative coloring of leaves of T with colors d and e, starting at vertex x and following the boundary walk of T.

Case 3. Let $d_i \geq 2$ for any $i = 1, ..., k$. Let T_0 be a caterpillar homeomorphic with T having no vertices of degree 2. (Recall that two graphs are homeomorphic if they are obtainable from the same graph by subdividing edges with vertices of degree 2.)

Case 3.1. If T does not contain any vertices of degree 3, then T_0 does not have such a vertex. Then we color T_0 as in Case 1. This coloring induces a partial coloring of T

in which only vertices of degree 2 are not colored. The required coloring is finished by coloring the vertices of any path induced by vertices of degree 2 of T with different colors from the set $\{c_1, ..., c_t\}$.

Case 3.2. In this case we color the vertices of T_0 as in Case 2. This coloring induces a partial coloring of T which is completed analogously as in Case 3.1. using alternatively disjoint color sets $\{c_1, ..., c_t\}$ and $\{c_1', ..., c_t'\}$ for consecutive paths of 2-vertices.

To see the tightness of the lower bound $t + 4$ consider a caterpillar with the backbone path $B = (v_1, ..., v_{t+2}, v_{t+3})$ having $\deg(v_1) = \deg(v_{t+3}) = 3$ and $\deg(v_i) = 2$ for $i = 2, ..., t + 2$.

The tightness of the upper bound $2t + 5$ of the theorem can be easily seen from the suitable embedding of the caterpillar with a backbone path $B = (v_1, ..., v_{2t+3})$ having $\deg(v_1) = \deg(v_{t+2}) = \deg(v_{2t+3}) = 3$ and $\deg(v_i) = 2$ for $i = 2, ..., 2t + 2$, $i \neq t + 2$. □

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Note added in proof

Conjecture 5.2 has been recently proved in [A].

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