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Colourings with bounded monochromatic components in graphs of given circumference

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Abstract

We prove that every graph with circumference at most k is $O(\log k)$-colourable such that every monochromatic component has size at most k. The $O(\log k)$ bound on the number of colours is best possible, even in the setting of colourings with bounded monochromatic degree.

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In a vertex-coloured graph, a *monochromatic component* is a connected component of the subgraph induced by all the vertices of one colour. As a relaxation of proper colouring, recent work has focused on graph colourings with monochromatic components of bounded size (so-called *clustered colourings* [1, 3, 8, 13, 14, 16, 18, 19, 22]) or bounded monochromatic degree (so-called *defective colourings* [2, 4–7, 9–12, 15, 17, 21, 22]).

The *circumference* of a graph G is the length of the longest cycle if G contains a cycle, and is 2 if G is a forest. This paper studies colourings of graphs of given circumference with monochromatic components of bounded size. Our primary goal is to minimize the number of colours, while reducing the order of the monochromatic components is a secondary objective.

Let $g(k)$ be the minimum integer c for which there exists an integer d such that every graph with circumference at most k has a c-colouring in which every monochromatic component has order at most d. Our main result is that $g(k) = \Theta(\log k)$. First we prove the upper bound.

Theorem 1. *For every integer $k \geqslant 2$, every graph G with circumference at most k is $\lfloor 3\log_2 k\rfloor$-colourable such that every monochromatic component has order at most k.*

This result is implied by the following lemma with $C = \emptyset$. A *clique* is a set of pairwise adjacent vertices.

Lemma 2. *For every integer $k \geqslant 2$, for every graph G with circumference at most k and for every pre-coloured clique C of size at most 2 in G, there is a $\lfloor 3\log_2 k\rfloor$-colouring of G such that every monochromatic component has order at most k and every monochromatic component that intersects C is contained in C.*

Proof. We proceed by induction on $k + |V(G)|$. The result is trivial if $|V(G)| \leqslant 2$. Now assume $|V(G)| \geqslant 3$.

First suppose that $k = 2$. Then G is a forest, which is properly 2-colourable. If $|C| \leqslant 1$ or $|C| = 2$ and two colours are used on C, we obtain the desired colouring (with $2 < \lfloor 3\log_2 k\rfloor$ colours). Otherwise, $|C| = 2$ with the same colour on the vertices in C. Contract the edge C and 2-colour the resulting forest by induction, to obtain the desired colouring of G. Now assume that $k \geqslant 3$.

Suppose that G is not 3-connected. Then G has a minimal separation (G_1, G_2) with $S := V(G_1 \cap G_2)$ of size at most 2. If $|S| = 2$, then add the edge on S if the edge is not already present. Consider both G_1 and G_2 to contain this edge. Observe that since the separation is minimal, there is a path in each G_j ($j = 1, 2$) between the two vertices of S. Therefore, adding the edge does not increase the circumference of G. Also note that any valid colouring of the augmented graph will be valid for the original graph. Since C is a clique, we may assume that $C \subseteq V(G_1)$. By induction, there is a $\lfloor 3\log_2 k\rfloor$-colouring of G_1, with C precoloured, such that every monochromatic component of G_1 has order at most k and every monochromatic component of G_1 that intersects C is contained in C. This colours S. By induction, there is a $\lfloor 3\log_2 k\rfloor$-colouring of G_2, with S precoloured, such that every monochromatic component of

G_2 has order at most k and every monochromatic component of G_2 that intersects S is contained in S. By combining the two colourings, every monochromatic component of G has order at most k and every monochromatic component of G that intersects C is contained in C, as required. Now assume that G is 3-connected.

Menger's theorem implies that every 3-connected graph contains a cycle of length at least 4. Thus $k \geqslant 4$.

If G contains no cycle of length k, then apply the induction hypothesis for $k-1$; thus we may assume that G contains a cycle Q of length k. Let $\mathcal{A}$ be the set of cycles in G of length at least $\lceil \frac{1}{2}(k-5) \rceil$. Suppose that a cycle $A \in \mathcal{A}$ is disjoint from Q. Since G is 3-connected, there are three disjoint paths between A and Q. It follows that G contains three cycles with total length at least $2(|A| + |Q| + 3) > 3k$. Thus G contains a cycle of length greater than k, which is a contradiction. Hence, every cycle in $\mathcal{A}$ intersects Q.

Let $S := V(Q) \cup C$. As shown above, $G' := G - S$ contains no cycle of length at least $\lceil \frac{1}{2}(k-5) \rceil$. Then G' has circumference at most $\max\{2, \lceil \frac{1}{2}(k-7) \rceil\}$, which is at most $\lfloor \frac{1}{2}k \rfloor$, which is at least 2. By induction (with no precoloured vertices), there is a $\lfloor 3\log_2 \lfloor \frac{1}{2}k \rfloor \rfloor$-colouring of G' such that every monochromatic component of G' has order at most $\lfloor \frac{1}{2}k \rfloor$. Use a set of colours for G' disjoint from the (at most two) preassigned colours for C. Use one new colour for $S \setminus C$, which has size at most k. In total, there are at most $\lfloor 3\log_2 \lfloor \frac{1}{2}k \rfloor \rfloor + 3 \leqslant \lfloor 3\log_2 k \rfloor$ colours. Every monochromatic component of G has order at most k, and every monochromatic component of G that intersects C is contained in C. $\square$

Note that if h is the function defined by the recurrence,

$$h(k) := \begin{cases} 2 & \text{if } k = 2 \\ 5 & \text{if } 3 \leqslant k \leqslant 11 \\ h(\lceil \frac{1}{2}(k-7) \rceil) + 3 & \text{if } k \geqslant 12, \end{cases}$$

then $\lfloor 3\log_2 k \rfloor$ can be replaced by $h(k)$ in Theorem 1.

We now show that the $O(\log k)$ bound in Theorem 1 is within a constant factor of optimal even in the setting of colourings of bounded monochromatic degree. The following result is implicit in [21]. We include the proof for completeness.

Proposition 3. *For any integers $k, d \geqslant 1$ there is a graph $G_{k,d}$ with circumference at most 2^k, such that every k-colouring of $G_{k,d}$ contains a vertex of monochromatic degree at least d.*

Proof. We proceed by induction on $k \geqslant 1$ with d fixed (and thus write G_k instead of $G_{k,d}$), and with the additional property that G_k contains no path of order 2^{k+1}. For the base case, $k = 1$, let G_1 be the star $K_{1,d}$, which has circumference 2 and no path of order 4. Every 1-colouring of G_1 contains a vertex of monochromatic degree d. Now assume that $k \geqslant 2$ and there is a graph G_{k-1} with circumference 2^{k-1} and no path of order 2^k, such that every $(k-1)$-colouring of G_{k-1} contains a vertex of monochromatic degree at least d. Let G_k be obtained from d copies of G_{k-1} by adding one new dominant vertex v.

If C is a cycle in G_k with length at least 2^k+1, then C is contained in one copy of G_{k-1} plus v, and thus G_{k-1} contains a path of order 2^k, which is a contradiction. Thus G_k has circumference at most 2^k.

If G_k contains a path P of order 2^{k+1}, then v is in P, otherwise P is contained in some copy of G_{k-1}. Hence $P-v$ includes a path component of order $\lceil \frac{1}{2}(2^{k+1}-1)\rceil = 2^k$ contained in a copy of G_{k-1}, which is a contradiction. Hence G_k contains no path of order 2^{k+1}.

Finally, consider a k-colouring of G_k. Say v is blue. If every copy of G_{k-1} contains a blue vertex, then v has monochromatic degree d, and we are done. Otherwise, some copy of G_{k-1} contains no blue vertex, in which case G_{k-1} is $(k-1)$-coloured, and thus G contains a monochromatic vertex of degree at least d. $\square$

Let $f(k)$ be the minimum integer c for which there exists an integer d such that every graph with circumference at most k has a c-colouring in which every monochromatic component has maximum degree at most d. In the language of Ossona de Mendez et al. [21], $f(k)$ is the defective chromatic number of the class of graphs with circumference at most k. Obviously, bounded size implies bounded degree, so $f(k) \leqslant g(k)$. Theorem 1 and Proposition 3 imply that

$$\lfloor \log_2 k \rfloor + 1 \leqslant f(k) \leqslant g(k) \leqslant \lfloor 3 \log_2 k \rfloor. \tag{1}$$

We conclude this paper by placing our results in the context of a conjecture of Ossona de Mendez et al. [21]. The *closure* of a rooted tree T is the graph obtained from T by adding an edge between each ancestor and descendant. The *tree-depth* of a connected graph H, denoted $\mathrm{td}(H)$, is the minimum depth of a rooted tree for which H is a subgraph of the closure of T, where the *depth* of a rooted tree T is the maximum number of vertices in a root-to-leaf path. For a graph H, let $f(H)$ be the minimum integer c such that there exists an integer d such that every H-minor-free graph has a c-colouring in which every monochromatic component has maximum degree at most d. Ossona de Mendez et al. [21] proved that $f(H) \geqslant \mathrm{td}(H)-1$ for every connected graph H, and conjectured that

$$f(H) = \mathrm{td}(H) - 1. \tag{2}$$

A graph has circumference at most k if and only if it contains no C_{k+1} minor; thus $f(k) = f(C_{k+1})$. It is easily seen that

$$\mathrm{td}(C_{k+1}) = 1 + \lceil \log_2(k+1) \rceil = 2 + \lfloor \log_2 k \rfloor.$$

Thus the lower bound $f(H) \geqslant \mathrm{td}(H)-1$, in the case of cycles, is equivalent to the lower bound on $f(k)$ in (1). And conjecture (2), in the case of cycles, asserts that equality holds. That is,

$$f(k) = f(C_{k+1}) = \mathrm{td}(C_{k+1}) - 1 = \lceil \log_2(k+1) \rceil.$$

Hence Theorem 1, which proves that $f(k) \leqslant \lfloor 3 \log_2 k \rfloor$, is within a factor 3 of conjecture (2) for excluded cycles. The best previous upper bound was linear in k.

We obtain similar results for graph classes excluding a fixed path, which were identified by Ossona de Mendez et al. [21] as a key case for which their bounds on f were far apart. Let P_k be the path on k vertices. Then $\text{td}(P_k) = \lceil \log_2(k+1) \rceil$; see [20]. Of course, a graph contains a P_k minor if and only if it contains a P_k subgraph. Thus conjecture (2), in the case of paths, asserts that

$$f(P_{k+1}) = \text{td}(P_{k+1}) - 1 = \lceil \log_2(k+2) \rceil - 1.$$

Every graph with no P_{k+1}-minor has circumference at most k. Thus Theorem 1 implies that $f(P_{k+1}) \leqslant \lfloor 3 \log_2 k \rfloor$, which is within a factor of 3 of conjecture (2) for excluded paths. The best previous upper bound was linear in k.

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