Edge-Kempe-equivalence graphs of class-1 regular graphs

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Abstract
We consider, for class-1, $\Delta$-regular graphs $G$, the associated graph $KE(G)$, where the vertices of $KE(G)$ are $\Delta$-edge colorings of $G$ and edges of $KE(G)$ are present where two $\Delta$-edge colorings of $G$ differ by a single edge-Kempe switch. We focus on the case of cubic graphs and determine various structural properties of $KE(G)$ and $KE_v(G)$, where the latter considers a fixed set of colors on the edges incident to the vertex $v$. Additionally, we consider the ways in which $KE_v(G)$ for any choice of $v$ must be very similar, as well as how they can differ.

1 Introduction and Summary

Consider a graph $G$ with a proper edge coloring. A maximal two-color alternating path or cycle of edges is called an edge-Kempe chain; switching the colors along such a chain is called an edge-Kempe switch. Two edge-colorings are Kempe equivalent if one can be obtained from the other by a sequence of edge-Kempe switches. In [1], we examined equivalence classes of edge-colorings of graphs, with a focus on cubic graphs. Here we examine the Kempe-equivalence graph, denoted $KE(G)$, and defined as follows. Let $G$ be a graph with maximum degree $\Delta$, associate to each $\Delta(G)$-edge coloring of $G$ a vertex in $KE(G)$, and two vertices of $KE(G)$ are adjacent when the colorings they represent differ by a single edge-Kempe switch. An equivalence class of colorings corresponds to a connected component of $KE(G)$. (In [1], we denoted the number of components of $KE(G)$ as $K'(G, \Delta(G))$.)

Kempe-equivalence graphs are an example of reconfiguration graphs. A reconfiguration graph has as its vertices all feasible solutions to a given problem, and two solutions are adjacent if and only if one can be obtained from the other by one application of a specific reconfiguration rule. There are several useful reconfiguration graphs for coloring that are currently being studied. One reason for this interest is the

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application to theoretical physics, where vertex-coloring graphs describe the Glauber dynamics of an anti-ferromagnetic Potts model at zero temperature [6, 9, 14]. In this situation, the set of all proper vertex $k$-colorings of a graph forms the state space for a Markov chain with iterations given by randomly recoloring a randomly selected vertex of $G$.

Motivated by the Markov chain connection, a graph $G$ is said to be $k$-mixing if the $k$-coloring graph, with vertices representing proper vertex $k$-colorings of $G$ and edges joining colorings that differ at a single vertex, is connected. The question of when $G$ is $k$-mixing as well as the computational complexity of answering that question have been extensively studied [3, 4]. For instance, it has been shown that if the chromatic number $\chi(G) = k \in \{2, 3\}$, then $G$ is not $k$-mixing. There are also examples of $G$ with $k_1 < k_2$ such that $G$ is $k_1$-mixing but not $k_2$-mixing. Mixing properties related to modifications of the $k$-coloring graph have also been studied (cf. [2, 8]). Recent work considers when the coloring graph and its modifications contain a Hamiltonian cycle [5, 8]. In each of the above references, the reconfiguration rule used was to allow recoloring of one vertex at a time. There is some literature that suggests that using Kempe changes is an appropriate model for some Glauber dynamics [13]. A recent paper by Feghali [7] et. al. shows that for almost all connected cubic graphs all 3-vertex colorings are Kempe equivalent.

In this paper we consider edge colorings, and two edge colorings will be adjacent if one is obtained from the other by switching the colors along one edge-Kempe chain. We investigate the structure of the Kempe-equivalence edge coloring graphs as has been done for other reconfiguration graphs. (We are not the first to do so; while Kaszonyi [10] has some similar results, these are for a very restricted class of cubic graphs and his language and notation are highly non-standard. Additionally, his proof methods are quite different from ours.) More recently, the Kempe equivalence of edge colorings was addressed in [12]. It turns out that while we can obtain some structural results, these are of limited use in understanding even relatively straightforward examples; the arguments used in proving structure theorems neither specialize nor extend in individual cases. We include a detailed example in Section 4 to show the limits of the theory and how cumbersome the calculation is.

Theorem 3.1 in [1] shows that given a vertex $v$ of $G$, we can reach any edge-Kempe equivalent coloring with the same colors on edges incident to $v$ without changing colors at $v$ in the process. If we fix the edge colors at a vertex $v$ of $G$ (so as to ignore colorings that differ only because of color permutation), we denote the resulting Kempe-equivalence graph by $KE_v(G)$. This is in contrast to $KE(G)$, which has as its vertex set all colorings of the graph $G$. We will discuss how choice of $v$ does and does not effect $KE_v(G)$.

We often assume that all graphs $G$ are $k$-regular (have all vertices of degree $k$) and class 1 (edge colorable with $k$ colors), although some of our results will hold more generally. A cubic graph is 3-regular and some of our results will only hold for cubic graphs. Colors are named as $\{1, 2, 3, \ldots\}$ or \{dash, gray, solid\} depending on which makes most expositional sense. We say that two edge colorings $c_1, c_2$ are isomorphic colorings if they partition the base graph $G$ into the same sets. That is, the colorings are isomorphic if they differ only by a permutation of the names of
the colors. Note that in a regular, class-1 graph, all Kempe chains will in fact be Kempe cycles. We use $S_k$ to refer to the symmetric group on $k$ letters (the group of permutations).

The main results of this paper are in Section 2 where we look at structural properties of $KE_v(G)$ and $KE(G)$ from several perspectives. We show that $KE(G)$ graphs can never contain triangles, but almost always contains hypercubes. We show that for any graph $G$, there is an infinite family of graphs with the same $KE$ graph as $G$. We also consider the ways in which $KE_v(G)$ for any choice of $v$ must be very similar, as well as how they can differ. The notation $KE_C(G)$ will denote the subgraph of $KE(G)$ where the colors on the edges in set $C$ are fixed and Section 3 provides descriptions of $KE(G)$ for some graph products that were also analyzed in [1]. Section 4 gives a sample calculation of $KE_v(G)$ for a particular graph, and Section 5 concludes the paper with some open questions.

2  Structural Results for $KE(G)$ and $KE_v(G)$

We begin with some simple observations that hold for all graphs, not just those that are regular and class 1.

**Proposition 2.1.** For any graph $G$, $KE(G)$ is triangle free.

**Proof.** Suppose there were a 3-cycle in $KE(G)$. This means there is a sequence of exactly three edge-Kempe switches that returns a graph to its original coloring. Without loss of generality, let the first switch involve colors 1 and 2. If the second switch also involves colors 1 and 2, then the switched chains are disjoint and both cannot be un-switched using a third switch. If the second switch involves two different colors 3 and 4, the same problem occurs. If the second switch involves colors 2 and 3 (or colors 1 and 3), then either the chains are disjoint, causing the same problem as the previous case, or the chains intersect. In this case there is a subset of the edges with a color permutation that is not a transposition and therefore cannot be undone with a single edge-Kempe switch.

**Proposition 2.2.** For any graph $G$, $KE(G)$ has girth 4.

**Proof.** Suppose $KE(G)$ has a vertex (coloring of $G$) in which there are at least two edge-disjoint 2-color edge-Kempe cycles. Switching the colors independently generates a 4-cycle in $KE(G)$, and so $KE(G)$ has girth 4.

Instead, suppose that $KE(G)$ has no such vertex (coloring of $G$). This implies that every edge-Kempe cycle in every coloring of $G$ is Hamiltonian, and that the only edges in $KE(G)$ correspond to transposing pairs of colors. Considering any three colors, say 1, 2, 3, then sequentially doing the color transpositions $(1, 2), (1, 3), (2, 3), (1, 3)$ forms a 4-cycle. Therefore $KE(G)$ has girth 4.

Moreover, any graph $G$ with some coloring that has many independent edge-Kempe cycles will have multiple copies of $C_4$ in $KE(G)$. More precisely we have the following.
Theorem 2.3. Suppose there exists a proper edge coloring, $c$, of a graph $G$ with $q_{ij}$ Kempe chains in the color pair $(i, j)$. Then for all $1 \leq i < j \leq k$, $KE(G)$ contains a $(q_{ij})$-cube. Further, all of these cubes intersect in exactly the one vertex of $KE(G)$ corresponding to the coloring $c$.

Proof. For each pair of colors, each Kempe switch generates an edge from the cube, and because when performing two such switches of the same color pair the order does not matter, we obtain a square in $KE(G)$. Thus the $(i, j)$ switches form a $(q_{ij})$-cube in $KE(G)$.

Another basic observation is that the $KE$ graph of a disjoint union of graphs is the cartesian product of the $KE$ graphs of the component parts.

Proposition 2.4. $KE(G_1 \sqcup G_2) = KE(G_1) \Box KE(G_2)$.

We now move to results that hold only for regular graphs. Recall that for a cubic graph the $\Delta - \gamma$ operation is defined as replacing a $K_3$ with a single vertex. This is shown in Figure 1. Next we show that for cubic graphs the $\Delta - \gamma$ operation has no effect on the $KE$ graph. Note that this proposition holds for multigraphs. We will discuss cubic multigraphs further in Section 2.1.

![Figure 1: Corresponding 3-edge colorings across a $\Delta - \gamma$ operation.](image)

Proposition 2.5. For any 3-edge colorable cubic (multi)graph $G$, and any $\Delta - \gamma$ operation on $G$, denote the resulting graph by $G^\Delta$. Then, $KE(G) = KE(G^\Delta)$.

Proof. Notice that there is a one-to-one correspondence between 3-edge colorings of $G$ and 3-edge colorings of $G^\Delta$, as shown in Figure 1. Moreover, each edge-Kempe switch between a pair of colorings on $G$ corresponds to one between the corresponding colorings of $G^\Delta$.

If $G$ is regular of degree $k > 3$, then a $\Delta - \gamma$ move does not preserve regularity. The generalization of these results for $k > 3$ is that replacing a $K_k$ by a vertex will not change the $KE$ graph.

For cubic $G$ adding (or removing) a pair of triangles that share an edge changes $KE(G)$, because it adds (or removes) a Kempe chain. Perhaps surprisingly, the existence of a pair of triangles in a cubic graph can produce different $KE_v(G)$ for different $v$. An example of this is given in Example 2.17 of Section 2.2. By a lone triangle (respectively lone $K_k$) we will mean a copy of $C_3$ (respectively $K_k$) that does not share an edge with another copy of $C_3$ (respectively $K_k$).
Corollary 2.6. In examining the structure and realizability of $KE(G)$ for cubic (respectively $k$-regular) graphs, it suffices to consider graphs without lone triangles (respectively $K_k$s).

Corollary 2.7. Given a $k$-regular graph $G$ with associated $KE(G)$, there is an infinite family of graphs $\mathcal{H}$ such that for all $H \in \mathcal{H}$, $KE(H) = KE(G)$.

Proof. This follows from Proposition 2.5 for cubic graphs, and more generally by the discussion above.

While the most naive method of removing triangles that share an edge can change $KE(G)$, it turns out that we can find an equivalent triangle-free graph if we are willing to consider cubic multigraphs.

2.1 Cubic Multigraphs

For cubic graphs, it turns out that the graphs that can be obtained as $KE(G)$ when $G$ has multiple edges are precisely the same as those obtainable for simple graphs. Let $G$ be a cubic multigraph with two edges $e, h$ both between the vertices $u, v \in V(G)$. Define the graph $G_{e|h}$, shown in Figure 2 by $V(G_{e|h}) = V(G) \cup \{u_1, u_2\}$;

![Figure 2: The construction of $G_{e|h}$ from $G$.](image)

and $E(G_{e|h}) = E(G) \setminus \{e, h\} \cup \{uu_1, uu_2, u_1u_2, u_1v, u_2v\}$.

Theorem 2.8. Let $G$ be a cubic multigraph with with two edges $e, h$ both between the vertices $u, v \in V(G)$, and $G_{e|h}$ as defined above. For any vertex $x \in V(G)$, we have $KE_x(G) = KE_x(G_{e|h})$, and $KE_u(G) = KE_{u_1}(G_{e|h}) = KE_{u_2}(G_{e|h}) = KE_v(G)$. Additionally, $KE(G) = KE(G_{e|h})$.

Proof. First, there is a one-to-one correspondence between the proper edge colorings of $G$ and $G_{e|h}$, as shown in Figure 3. Then notice that the edge-Kempe structure

![Figure 3: Corresponding 3-edge colorings between $G$ and $G_{e|h}$.](image)

of $G$ is identical to that of $G_{e|h}$, because the three edge-Kempe chains correspond. Fixing the colors at any of the vertices $u, u_1, u_2, v$ fixes the colors at the other three vertices, and this completes the proof.
Corollary 2.9. Given any cubic multigraph \( G \) there is a simple graph \( G^* \) such that \( KE_v(G) = KE_v(G^*) \), for every \( v \in V(G) \) and \( KE(G) = KE(G^*) \). Moreover if \( u \in V(G^*) \) then there exists \( v \in V(G) \) such that \( KE_v(G) = KE_u(G^*) \).

Proof. This follows by repeated application of the operation in Theorem 2.8 until no multiple edges remain.

Because the operation \( G \rightarrow G_{e|h} \) is reversible, the reverse of Corollary 2.9 is also true.

Corollary 2.10. Suppose \( H \) is a cubic graph with \( u, v, u_1, u_2 \in V(H) \) and \( uu_1, uu_2, vu_1, vu_2, u_1u_2 \in V(H) \). Then there is a cubic multigraph \( G \) such that \( H = G_{e|h} \).

By combining this result with Proposition 2.6 we can restrict consideration to only triangle-free multigraphs.

Corollary 2.11. In examining the structure and realizability of \( KE(G) \) for cubic multigraphs, it suffices to consider triangle-free multigraphs.

2.2 \( KE_v(G) \) for the same \( G \) and different \( v \)

For any choice of \( v \in V(G) \) the graph \( KE_v(G) \) will have as its vertex set exactly one copy of each non-isomorphic coloring of \( G \). That is, there will be one vertex for each partition of the edges into \( \Delta \) independent sets. The assignment of color names to the parts of the partition will vary based on which \( v \) is chosen and the initial assignment of colors to the edges incident to \( v \). It turns out that the structure of \( KE_v(G) \) can depend on the choice of \( v \).

We start with some simple observations.

Remark 2.12. First, observe that if \( c \) and \( c' \) are isomorphic colorings of \( G \) then they are Kempe equivalent, since any permutation can be generated by transpositions.

Proposition 2.13. Suppose \( c_1 \) is a coloring of a class-1 regular graph \( G \) in which there are exactly two edge-Kempe chains in some color pair (say, gray-dash) and \( c_2, c'_2 \) the colorings that result from switching one or the other of the chains. Then \( c_2 \) and \( c'_2 \) are isomorphic colorings.

Proof. The colorings \( c_2, c'_2 \) partition the edges of \( G \) in the same way, where the dash edges in \( c_2 \) are precisely the gray edges in \( c'_2 \) and vice versa. All other color classes are identical.

Note that under the conditions of Proposition 2.13, there is an edge of \( KE_v(G) \) between colorings \( c_1 \) and \( c_2 \) (or an isomorphic copy of \( c_2 \)) for all choices of \( v \). We next give a sufficient condition for \( KE_v(G) \) to be isomorphic graphs for all \( v \in V(G) \). Recall that \( G \) is vertex transitive if for any pair of vertices \( v_1, v_2 \) there is a graph automorphism that takes \( v_1 \) to \( v_2 \).

Theorem 2.14. For a simple, regular graph \( G \), if \( |V(G)| < 12 \) or if \( G \) is vertex transitive, then for any vertices \( v_i, v_j \in V(G) \) the graphs \( KE_{v_i}(G) \) and \( KE_{v_j}(G) \) are isomorphic, and the isomorphism associates isomorphic colorings.
Proof. Certainly, if \( G \) is vertex transitive, then \( KE_{v_i}(G) \) and \( KE_{v_j}(G) \) are isomorphic for any vertices \( v_i, v_j \in V(G) \). By Proposition 2.13, if \( KE_{v_i}(G) \neq KE_{v_j}(G) \) then there must be a coloring \( c_1 \) that has three or more edge-Kempe chains in some color pair. Since each edge-Kempe chain must consist of at least 4 vertices (by Proposition 2.1), there must be at least 12 vertices in the graph.

We will show the above theorem is best possible by constructing a graph on 12 vertices that has two different \( KE_v \) graphs. Before constructing this example, we first give some results that help determine the structure of \( KE_v(G) \). The following results give some relationships between various parameters on \( KE_v(G) \), and \( KE(G) \). Indeed, they show that for a given graph \( G \), the \( KE_v(G) \) must be very similar for any choice of \( v \).

**Theorem 2.15.** Let \( G \) be a class-1 \( k \)-regular graph.

(a) The degree of a vertex \( c \), a coloring, in \( KE_v(G) \) is \( \deg_{KE(G)}(c) - \left( \frac{k}{2} \right) \), for any \( v \in V(G) \).

(b) The number of connected components of \( KE_v(G) \) is equal to the number of connected components of \( KE(G) \).

**Proof.** (a) The degree of a vertex \( c \) in \( KE_v(G) \) corresponds to the number of possible edge-Kempe switches that can be made in a coloring \( c \) of \( G \) with the edge colors at \( v \) fixed. This number is the same independent of the choice of \( v \in V(G) \) because \( v \) is on exactly \( \left( \frac{k}{2} \right) \) edge-Kempe chains.

(b) Suppose that \( c_1, c_2 \) are in the same connected component of \( KE_v(G) \). Then there exists a path between them in \( KE_v(G) \), and hence in \( KE(G) \) by subgraph inclusion. Now suppose that \( c_1, c_2 \) are in the same connected component of \( KE(G) \). If the edge colors of \( c_1, c_2 \) do not agree at \( v \), then there is a coloring \( c'_2 \), that is isomorphic to \( c_2 \) that has the edge colors of \( c_1 \) at \( v \). By Remark 2.12, \( c_2, c'_2 \) are in the same connected component of \( KE(G) \). Thus it suffices to assume that the edge colors of \( c_1, c_2 \) agree at \( v \). Then by Theorem 3.1 of [1], there is a sequence of edge-Kempe changes between \( c_1 \) and \( c_2 \) that avoids changing colors at \( v \), and so \( c_1, c_2 \) are in the same connected component of \( KE_v(G) \). \( \qed \)

The power of this result can be seen in the following immediate consequences of this theorem and its proof.

**Corollary 2.16.** Let \( G \) be a class-1 \( k \)-regular graph.

(i) If the coloring \( c \) is a leaf in \( KE_{v_i}(G) \) then it is also a leaf in \( KE_{v_j}(G) \) for any \( v_j \in V(G) \). Further, if \( c \) is a leaf then its unique neighbor is the same coloring (up to isomorphism) in all \( KE_{v_j}(G) \).

(ii) The degree sequences of \( KE_{v_i}(G) \) and \( KE_{v_j}(G) \) are equal.

(iii) If \( c_1, c_2 \) are colorings in the same connected component of \( KE_{v_i}(G) \), then their isomorphic counterparts are in the same connected component of \( KE_{v_j}(G) \).
We will now exhibit a simple connected cubic graph $R$ on 12 vertices that has two non-isomorphic $KE_v(R)$ graphs.

**Example 2.17.** In order to have 3 disjoint Kempe cycles on 12 vertices, $R$ must consist of three disjoint copies of $C_4$ with six additional edges, and by Corollary 2.6 $R$ has no lone triangles. See Figure 4: note that $R$ is planar, but not bipartite.

![Figure 4: Colorings of a graph $R$ with non-isomorphic $KE_v(R)$.](image)

There are eight nonisomorphic colorings of this graph, all shown in Figure 4. The coloring shown at top left has seven edge-Kempe chains, so will have degree 4 in $KE_v(R)$ for any $v$ (because three edge-Kempe chains will be fixed). Let $A$ denote the 4-cycle bounding the pair of triangles. Switching the colors of $A$ produces another coloring with seven edge-Kempe chains, shown at left in the middle row.

The coloring shown at top right has five edge-Kempe chains, so will have degree 2 in $KE_v(R)$ for any $v$. The remaining five colorings also have five edge-Kempe chains each. If the fixed vertex $v$ is not on the cycle $A$, then the two colorings of degree 4 are adjacent in $KE_v(R)$ (via switching the colors on $A$). If the fixed vertex is on the cycle $A$ then the two colorings of degree 4 are not adjacent in $KE_v(R)$. Thus there are (at least) two different graphs that occur as $KE_v(R)$.

A similar example on 12 vertices can be constructed that is triangle free (but not bipartite). A connected simple example that is both bipartite and planar requires 14 vertices; $G_{14}$ is shown in Figure 5. If we allow multigraphs then a smaller connected example can be constructed by replacing the adjacent triangles of the cycle $A$ of the graph $R$ in Example 2.17 with a pair of multiple edges as in Corollary 2.11. The smallest example of a (non-connected) multigraph with two different graphs occurring as $KE_v(G)$ is $D \cup P$, depicted in Figure 6.
2.3  \( KE_v(G) \) as a subgraph of \( KE(G) \)

In this section we describe how \( KE_v(G) \) is contained in \( KE(G) \). Many of the statements made about \( KE(G) \) have obvious implications for \( KE_v(G) \). For example Propositions 2.1 and 2.2 imply that for all \( G, v \), \( KE_v(G) \) must have girth at least 4.

**Proposition 2.18.** For any graph \( G \), and any \( v \in V(G) \), \( KE_v(G) \) has girth 4 if there are at least two disjoint edge-Kempe chains disjoint from \( v \).

We suspect that there is no upper bound on the girth of \( KE_v(G) \), for \( G \) cubic and class 1, but the largest girth we have observed is associated to the 14-vertex generalized Petersen graph shown in Figure 7; \( KE_v(GP) \cong C_7 \) and so has girth 7. (Some larger generalized Petersen graphs have smaller girths.)

In a class-1 \( k \)-regular graph, any vertex \( v \) will be in exactly one \((i, j)\) Kempe-chain for each pair of colors \( 1 \leq i < j \leq k \). Thus we get the following corollary to Theorem 2.3.
Corollary 2.19. If there exists a proper edge coloring, $c$, of a class-1 $k$-regular graph $G$ with $q_{ij}$ Kempe chains of colors $i, j$. Then for all $1 \leq i < j \leq k$, and any $v \in V(G)$, $KE_v(G)$ contains a $(q_{ij} - 1)$-cube. Further, all of these cubes intersect in exactly one vertex of $KE_v(G)$ corresponding to the coloring $c$.

For each vertex of $KE_v(G)$ there are $k!$ corresponding vertices in $KE(G)$. In fact, there are $k!$ disjoint copies of $KE_v(G)$ that partition the vertices of $KE(G)$. These copies of $KE_v(G)$ differ by the assignment of colors on the $k$ edges incident to $v$. In order to describe the relationships between the copies, we introduce a more precise notation: for $\sigma$ a permutation of the $k$ colors, define $KE_v(G)^\sigma$ to be the graph whose vertices are proper $k$-edge colorings of $G$, such that the order of the colors on the edges incident to $v$ is $\sigma$ with respect to a fixed ordering of the edges of $G$. The $k!$ disjoint $KE_v(G)^\sigma$ (one for each permutation $\sigma$) form a partition of the vertices of $KE(G)$. In any proper $k$-edge coloring of $G$ there are $\binom{k}{2}$ edge-Kempe chains incident to vertex $v$, and each of these changes the permutation of colors at $v$ by a different transposition. Thus each coloring in $KE_v(G)^\sigma$ is adjacent to $\binom{k}{2}$ other colorings, each of which belongs to a $KE_v(G)^{\sigma'}$ with a distinct $\sigma'$.

That $c \in V(KE_v(G)^\sigma)$ and $c' \in V(KE_v(G)^{\sigma'})$ are adjacent does not imply that $c, c'$ are isomorphic colorings. This is because moving from $KE_v(G)^\sigma$ to $KE_v(G)^{\sigma'}$ requires changing only, for example, the dash-solid edge-Kempe chain through $v$ and not all dash-solid edge-Kempe chains in $G$. Yet, when $\sigma$ and $\sigma'$ differ by a transposition, the edges between vertices of $KE_v(G)^\sigma$ and $KE_v(G)^{\sigma'}$ form a perfect matching between these sets.

It is clear that for any two permutations $\sigma, \sigma'$, the graphs $KE_v(G)^\sigma$ and $KE_v(G)^{\sigma'}$ are isomorphic; thus, in many cases the notation $KE_v(G)$ can be used without confusion.

Proposition 2.20. If $KE_v(G)$ has a Hamiltonian circuit for some $v \in V(G)$, then $KE(G)$ has a Hamiltonian path.

Proof. A Hamiltonian path $H$ in $KE(G)$ is formed by concatenating Hamiltonian paths from $KE_v(G)^\sigma$ for each $\sigma$. It was shown in [11] that the Cayley graph of $S_n$ generated by transpositions has a Hamiltonian circuit. Suppose that the sequence of transpositions that accomplishes a Hamiltonian circuit in $S_n$ is $t_1, t_2, \ldots, t_p$, and let $t$ be the identity permutation.

Let $c_{11}, c_{12}$ be adjacent vertices in a Hamiltonian circuit in $KE_v(G)^t$. Let $H$ begin with the Hamiltonian path of $KE_v(G)^t$ that begins $c_{11}$ and ends $c_{12}$. Now, to coloring $c_{12}$ apply the $t_1$ edge-Kempe-switch that includes vertex $v$. The resulting coloring, say $c_{21}$, will be in $KE_v(G)^{t_1}$. Because each $KE_v(G)$ has a Hamiltonian circuit, there is a Hamiltonian path starting at $c_{21}$. Let $c_{22}$ be the last vertex of this path. Apply to $c_{22}$ the $t_2$ edge-Kempe-switch that includes vertex $v$, to get a coloring $c_{31}$ in $KE_v(G)^{t_2t_1}$. Proceed in a similar manner to traverse all the colorings of each $KE_v(G)^\sigma$. 

\[\square\]
2.4 More about the structure of $KE(G)$

Let $c_\sigma, c_{(ij)_\sigma}$ be two isomorphic colorings of $G$ that differ by a transposition $(ij)$ on color names. Then there is a path between $c_\sigma, c_{(ij)_\sigma}$ in $KE(G)$, each of whose edges corresponds to an edge-Kempe switch of colors $i$ and $j$ on some $i$-$j$ edge-Kempe chain of $c_\sigma$. There are in fact many such paths, as the switches can be made in any order. Let $c_\sigma \sim c_{(ij)_\sigma}$ denote one such path.

**Lemma 2.21.** For any $G$, $KE(G)$ always contains a subdivision of $K_{3,3}$.

**Proof.** Let $c$ be any coloring in $KE(G)$. For each of the six permutations of $\{1, 2, 3\}$ (fixing all other colors if $k > 3$) there corresponds a coloring isomorphic to $c$. Recall that the Cayley graph of the six permutations of $S_3$, generated by transpositions, forms a $K_{3,3}$. Thus, the subgraph of $KE(G)$ induced by the 9 paths $c_\sigma \sim c_{(ij)_\sigma}$, where $\sigma$ and $(ij)_\sigma$ fix all colors other than $\{1, 2, 3\}$ and permute (some of) $\{1, 2, 3\}$, is therefore a subdivision of $K_{3,3}$. \qed

In fact, associated to every vertex of $KE(G)$ are $k!$ vertices of $KE(G)$ (namely, the isomorphic colorings) forming a subgraph that is a subdivision of the Cayley graph with the transpositions of $S_k$ as the generators. Some easy corollaries follow from Lemma 2.21.

**Corollary 2.22.** No tree is realizable as $KE(G)$ for any $G$.

**Corollary 2.23.** No $KE(G)$ is planar.

**Corollary 2.24.** If $G$ is $k$-regular and is uniquely $k$-edge colorable, then $KE(G)$ is isomorphic to the Cayley graph of $S_k$ with the set of all transpositions as generators.

The graph $K_{3,3}$ has two nonisomorphic edge-colorings, but in both colorings every edge-Kempe chain is a Hamiltonian cycle. This observation produces a generalization of the above corollary. We use $\Gamma_k$ to denote the Cayley graph of $S_k$ with the set of all transpositions as generators.

**Corollary 2.25.** Suppose $G$ is $k$-regular and class 1 with exactly $h$ nonisomorphic $k$-edge colorings. If in each of the colorings of $G$ every edge-Kempe chain is a Hamiltonian cycle, then $KE(G) = \sqcup h \Gamma_k$.

3 Graph Products

It will be helpful to start by considering a variation on $KE_v(G)$. The graph $KE_v(G)$ was formed by fixing the colors of the edges incident to a single vertex $v \in V(G)$. We could instead fix the colors of a 3-edge cut $C$ of a cubic graph $G$, denoted $KE_C(G)$. A parity argument implies that a 3-edge cut of a properly colored class-1 cubic graph must contain exactly one edge of each color, so $KE_C(G)$ will contain one copy of each non-isomorphic edge coloring of $G$. Thus similar statements to those in Theorem 2.15 and other results in Section 2.2 hold for $KE_C(G)$ as well.
**Proposition 3.1.** Let $G$ be a class-1 cubic graph with $v \in V(G)$ and $C$ a 3-edge cut of $G$.

(i) The degree of a vertex $c$, a coloring, in $KE_C(G)$ is $\deg_{KE(G)}(c) - 3$.

(ii) The number of connected components of $KE_C(G)$ is equal to the number of connected components of $KE(G)$.

(iii) If the coloring $c$ is a leaf in $KE_v(G)$ then it is also a leaf in $KE_C(G)$. Further, if $c$ is a leaf then its unique neighbor is the same (up to isomorphism).

(iv) The degree sequences of $KE_v(G)$ and $KE_C(G)$ are equal.

The graph $KE_C(G)$ has a nice description when $G$ is formed by the following graph product, which was defined in [1]. Consider two cubic graphs $G_1, G_2$, and form $G_1 \gamma G_2$ by choosing vertices $v_1 \in V(G_1), v_2 \in V(G_2)$, removing $v_1, v_2$, and adding a matching of three edges joining the three neighbors of $v_1$ with the three neighbors of $v_2$. Of course there are many ways to choose $v_1, v_2$, and many ways to identify their incident edges, so the construction is not unique. The following theorem holds for any such choices.

**Proposition 3.2.** Let $G_1 \gamma G_2$ be formed from $G_1, G_2$ using vertices $v_1 \in V(G_1), v_2 \in V(G_2)$, so that $C$ is the edge cut formed. Then

$$KE_C(G_1 \gamma G_2) = KE_{v_1}(G_1) \square KE_{v_2}(G_2).$$

*Proof.* Every coloring of $G_1 \gamma G_2$ can be written as an ordered pair of colorings $(c_1, c_2)$ where $c_1$ is a coloring of $G_1$ and $c_2$ is a coloring of $G_2$. If the colors on $C$ are fixed, then no edge-Kempe chain in $G_1 \gamma G_2$ can cross $C$. Therefore, any edge in $KE_C(G_1 \gamma G_2)$ corresponds to an edge-Kempe chain in exactly one of $G_1$ and $G_2$. This is the definition of $KE_{v_1}(G_1) \square KE_{v_2}(G_2)$. □

Examining $KE_v(G_1 \gamma G_2)$ shows that changing $v$ may produce nonisomorphic graphs, even when there is only one possibility for $KE_v(G_1)$. Consider $Q_3 \gamma Q_3$ (shown in Figure 3), where $Q_3$ is the cube. Because $Q_3$ is vertex transitive, all $KE_v(Q_3)$ are isomorphic, but direct computation shows that there are at least two different graphs that occur as $KE_v(Q_3 \gamma Q_3)$. It is true that no matter the choices made in making the $\gamma$ product, and on which vertex or cut in the resulting product we fix the colors, the number of edges in and degree sequence of $KE_v(Q_3 \gamma Q_3)$ or $KE_C(Q_3 \gamma Q_3)$ will be the same.

**Proposition 3.3.** Let $G_1, G_2$ be class-1 cubic graphs, with $v_1 \in V(G_1), v_2 \in V(G_2)$ the vertices used in creating $G_1 \gamma G_2$ and $x \neq v_1 \in V(G_1)$. Then the vertex sets $V(KE_x(G_1 \gamma G_2)) \simeq V(KE_x(G_1) \square KE_{v_2}(G_2))$, corresponding vertices have the same degree, and $E(KE_x(G_1 \gamma G_2)) \supseteq E(KE_x(G_1) \square KE_{v_2}(G_2))$. 
Proof. The set of colorings in $KE_x(G_1 \gamma G_2)$ may be indexed as $(c_i, c_j)$, where $c_i$ is a coloring of $G_1$ and $c_j$ is a coloring of $G_2$ with colors permuted to match $c_i$ on the edges that were incident to $v_1, v_2$. Thus each $c_i$ of $KE_x(G_1)$ can only be paired with all colorings from $KE_{v_2}(G_2)^\sigma$ for exactly one $\sigma$. The particular $\sigma$ depends on the colors that $c_i$ assigns to the edges incident to $v_1$. Then, by the parity lemma each edge in a 3-edge cut of a cubic graph must receive a different color; hence, every coloring of $G_1 \gamma G_2$ induces a coloring of $G_1$ and of $G_2$. Thus, $V(KE_x(G_1 \gamma G_2)) \simeq V(KE_x(G_1) \square KE_{v_2}(G_2))$.

Let $(c_i, c_j) \in V(KE_x(G_1 \gamma G_2))$ and the corresponding coloring be $(c_1, c_2) \in V(KE_x(G_1) \square KE_{v_2}(G_2))$. The coloring $(c_i, c_j) \in V(KE_x(G_1) \square KE_{v_2}(G_2))$ has three kinds of edge-Kempe chains not incident to $x$, namely (i) entirely within $G_1$ (and not incident to $v_1$), (ii) entirely in $G_2$ (not incident to $v_2$), or (iii) containing edges previously incident to $v_1, v_2$. The first kind are in one-to-one correspondence with edge-Kempe chains in $KE_x(G_1)$ that are not incident to $v_1$. The second kind are in one-to-one correspondence with edge-Kempe chains in $KE_{v_2}(G_2)$ that are not incident to $v_2$. The third kind are in one-to-one correspondence with the edge-Kempe chains in $KE_x(G_1)$ that are incident to the vertex $v_1$. This shows that the degrees of the vertices are the same. To see that the edge inclusion holds, note further that (i) and (iii) correspond to edges in $KE_x(G_1)$ and (ii) correspond exactly to the edges in $KE_{v_2}(G_2)$.

Observe that while the degrees are the same for corresponding vertices in $KE_x(G_1 \gamma G_2)$ and $KE_x(G_1) \square KE_{v_2}(G_2)$, the edges of $KE_x(G_1 \gamma G_2)$ do not join corresponding pairs of colorings in $KE_x(G_1) \square KE_{v_2}(G_2)$. Specifically, the second kind of edge-Kempe chain in the proof changes colors on edges in both $G_1$ and $G_2$ and that never happens in the edge-Kempe chains represented in $KE_x(G_1) \square KE_{v_2}(G_2)$.

A similar correspondence occurs with $KE(G_1 \gamma G_2)$.

**Proposition 3.4.** Let $G_1, G_2$ be class-1 cubic graphs. Then

(a) $V(KE(G_1 \gamma G_2)) \simeq V(KE(G_1) \square KE_{v_2}(G_2))$ and

(b) $E(KE(G_1 \gamma G_2)) \supseteq \sqcup_\sigma E(KE_{v_1}(G_1)^\sigma \square KE_{v_2}(G_2))$.

The proof is almost identical to that of Proposition 3.3.

Note that $G \gamma K_4$ is simply the $\Delta - \gamma$ operation. We can now generalize by considering this product with any uniquely 3-edge colorable graph.

**Proposition 3.5.** If $U$ is a uniquely 3-edge colorable cubic graph, and $G$ is any class-1 cubic graph, then $KE(G \gamma U) = KE(G)$ and $KE_v(G \gamma U) = KE_{v'}(G)$, where $v' := v$ if $v \in V(G)$ is not used in creating the product, and $v'$ is a vertex remaining in $U$ otherwise. More generally, if $H$ is cubic with exactly $h$ nonisomorphic 3-edge colorings, and in every edge-Kempe chain is a Hamiltonian cycle, then $KE(G \gamma H) = \sqcup_h KE(G)$, and $KE_v(G \gamma H) = \sqcup_h KE_{v'}(G)$, with $v'$ defined as before.

Another product, also introduced in [1], is natural when considering $KE$ graphs. Let $G_1, G_2$ be graphs with edges $e_1, e_2$ in $G_1, G_2$ respectively. Consider $v_1, w_1$ endpoints of $e_1$ and $v_2, w_2$ endpoints of $e_2$. Form $G_1 \pm G_2$ by removing $e_1, e_2$ and adding
two edges connecting \( v_1 \) to \( v_2 \) and \( w_1 \) to \( w_2 \). (While in \[1\] this product was defined only for cubic graphs, it generalizes directly to \( k \)-regular graphs.) As with the \( \gamma \) product, the choice of edges to cut and the ways to pair them up mean this construction is not unique. Nonetheless, the following analysis holds for all choices.

We can combine a coloring \( c \) of \( G_1 \) with a coloring \( d \) on \( G_2 \) to get a coloring of \( G_1 \circ G_2 \) if the colors agree on \( e_1, e_2 \). Note that by parity, in any coloring of \( G_1 \circ G_2 \) the same color will be assigned to both \( v_1v_2 \) and \( w_1w_2 \). In \( KE_{v_1}(G_1 \circ G_2) \) we have fixed the colors on all edges incident to \( v_1 \), but only on one edge incident to \( v_2 \). Therefore, for each vertex of \( KE_{v_1}(G_1) \) we have \( |V(KE_{v_2}(G_2))| \) copies of each of the \((k-1)!\) isomorphic colorings of \( G_2 \), corresponding to the \((k-1)!\) permutations of the colors on the other edges incident to \( v_2 \). This means that there are \((k-1)!|V(KE_{v_1}(G_1))| \cdot |V(KE_{v_2}(G_2))| \) vertices in \( KE_{v_1}(G_1 \circ G_2) \). Note that because the edge colors are fixed at \( v_1 \), no edge-Kempe chains cross the 2-edge cut formed in the construction \( G_1 \circ G_2 \); that is, colors change in \( G_1 \) or in \( G_2 \) but not both. There are therefore three types of edges in \( KE_{v_1}(G_1 \circ G_2) \): those that correspond to edges in \( KE_{v_1}(G_1) \), those that correspond to edges in \( KE_{v_2}(G_2) \), and those that connect vertices from different copies of \( KE_{v_2}(G_2)^\sigma \). Extending our notation in a natural way, we now define \( KE_e(G) \) to be the subgraph of \( KE(G) \) where the color on the edge \( e \) is fixed. Note that if \( e \) is incident to \( v \), then \( KE_e(G) = \cup_{\sigma} KE_v(G)^\sigma \) (with the union taken over the \((k-1)!\) permutations that fix the color on \( e \)), with additional edges \( d_1d_2 \) if \( d_1 \in V(KE_v(G)^\sigma), d_2 \in V(KE_v(G)^{(i)}), \) and \( d_1, d_2 \) agree on the coloring of all edges except those on the \((i, j)\) edge-Kempe chain that passes through \( v \) and does not use the edge \( e \). This proves the following.

**Lemma 3.6.** Let \( G_1, G_2 \) be class-1 \( k \)-regular graphs. If \( v_i \in V(G_i) \) and \( e_i \in E(G_i) \) are the vertices and edges involved in forming \( G_1 \circ G_2 \), then

(a) \( KE_{v_1}(G_1 \circ G_2) = KE_{v_1}(G_1)\square KE_{e_2}(G_2) \) and

(b) \( KE_{v_2}(G_1 \circ G_2) = KE_{v_1}(G_1)\square KE_{v_2}(G_2) \).

The above result requires that \( v_i \in V(G_i) \) are vertices incident to the cut edge. One might hope that a result for general \( v \) would also be possible, at least in the case of joining a uniquely edge-colorable graph. Unfortunately, Example 2.17 shows that is not true. That graph is \( Q_3 \circ K_4 \), where \( Q_3 \) is the cube. As the cube is vertex transitive, only one graph can occur as \( KE_v(Q_3) \), and \( K_4 \) is uniquely colorable. Yet we have seen that there are \( v_i, v_j \in V(Q_3 \circ K_4) \) such that \( KE_v(Q_3 \circ K_4) \neq KE_{v_i}(Q_3 \circ K_4) \).

The result about \( KE(G_1 \circ G_2) \) is similar to that for \( KE(G_1 \gamma G_2) \).

**Proposition 3.7.** Let \( G_1, G_2 \) be class-1 \( k \)-regular graphs. If \( v_i \in V(G_i) \) and \( e_i \in E(G_i) \) are the vertices and edges involved in forming \( G_1 \circ G_2 \), then

(a) \( V(KE(G_1 \circ G_2)) = V(KE(G_1)\square KE_{e_2}(G_2)) \) and

(b) \( E(KE(G_1 \circ G_2)) \supseteq \cup_{\sigma} E(KE_{e_1}(G_1)^\sigma \square KE_{e_2}(G_2)) \).
Proof. Consider a coloring \( c \) of \( G_1 \). Form \( G_1 \bigoplus G_2 \) and use \( c \) for the edges originally in \( G_1 \). For the remaining edges, we can use any coloring of \( G_2 \) up to permutation of the colors so that the color on \( e_2 \) matches that of \( e_1 \) in \( c \). Therefore the vertices of \( KE(G_1 \bigoplus G_2) \) may be indexed by ordered pairs of colorings from \( G_1 \) and \( G_2 \).

An edge-Kempe chain in \( G_1 \bigoplus G_2 \) may be in \( G_1 \setminus \{e_1\} \), in \( G_2 \setminus \{e_2\} \), or may involve the edges formed from \( e_1, e_2 \). The first kind are in one-to-one correspondence with the edge-Kempe chains in \( \bigcup KE_{e_1}(G_1)^\sigma \), the second kind are in one-to-one correspondence with the edge-Kempe chains in \( KE_{e_2}(G_2) \), and the third kind do not appear in \( \bigcup_{\sigma} E(KE_{e_1}(G_1)^\sigma \square KE_{e_2}(G_2)) \).

Again, even when \( G_2 \) is uniquely colorable there is no simple description of \( KE(G_1 \bigoplus G_2) \). Let \( P \) and \( D \) be defined as in Figure 6. Note that \( D = P \bigoplus P \). As for any uniquely colorable graph, \( KE(P) = K_{3,3} \), \( KE_e(P) = K_1 \) for any vertex \( v \in V(P) \), and \( KE_e(P) = K_2 \) for any edge \( e \in E(P) \). By Proposition 3.7, \( KE(D) \) has 12 vertices. Taking \( \bigcup_{\sigma} E(KE_{v_1}(P)^\sigma \square KE_{e_2}(P)) \) we see that the 12 vertices form 3 disjoint 4-cycles. Each of \( K_1, K_2, K_{3,3} \) are bipartite and the \( \square \) product preserves the property of being bipartite. However, the additional edges of \( KE(D) \) not in \( \bigcup_{\sigma} E(KE_{v_1}(P)^\sigma \square KE_{e_2}(P)) \) cause \( KE(D) \) to not be bipartite. In Figure 8 a 5-cycle in \( KE(D) \) is shown. The bold 5-cycle edges correspond to edge-Kempe chains that use edges of both copies of \( P \). Note that by Corollary 2.11, an identical analysis holds for \( K_4 \bigoplus K_4 \), which is a simple planar graph (with triangles).

4 A sample calculation

In this section we consider a specific base graph with nice structure and calculate its \( KE_v \) graph. This is evidence that even an example expected to be simple can require a cumbersome argument.

Definition 4.1. Consider two \( 2k \)-cycles with vertices \( A = \{a_i : 1 \leq i \leq 2k\}, B = \{b_i : 1 \leq i \leq 2k\} \) respectively. For \( i = 1, \ldots, k \) add the edges \( a_{2i+1}b_{2i+1} \) and \( b_{2i}a_{2i+1} \). The result is called the crossed prism graph \( CPr_k \) on \( 4k \) vertices.

Figure 8: Five edge-colorings of \( D \) that form a 5-cycle in \( KE(D) \).
Note that $CPr_k$ is vertex-transitive; thus, $KE_v(CPr_k)$ is the same independent of choice of $v$.

**Theorem 4.2.** Let $CPr_k$ be the crossed prism graph with $4k$ vertices.

(a) For $k$ even, $KE_v(CPr_k)$ is a $(k-1)$-cube with 2 leaves on each vertex of one of the parts (of a bipartition of the cube).

(b) For $k$ odd, $KE_v(CPr_k)$ is a $(k-1)$-cube with one leaf on each vertex.

**Proof.** Consider the coloring of $CPr_k$ shown in Figure 9. The $A$ and $B$ cycles form dash-solid edge-Kempe chains. Each cross (pair of $a_ib_j$ edges) is colored in gray. Without loss of generality, fix the colors at the upper-left-most vertex, $a_1$.

![Figure 9: A coloring of $CPr_k$.](image)

Since the colors at vertex $a_1$ are fixed, there are now $k - 1$ gray-dash edge-Kempe cycles (each of length 4) that can be switched, and one gray-solid cycle. Note that these edge-Kempe switches are independent, and that making any two different switches can be done in either order to form a square. Considering only the gray-dash cycles, we see that $KE_v(CPr_k)$ contains a $(k-1)$-cube $C$. We can represent each coloring in $C$ by a binary string $(x_1, \ldots, x_k) \in \mathbb{Z}_2^k$ in $C$. For each $x_i = 0$ for $i = 1, \ldots, k$, the cycle will cross from an $A$ vertex to a $B$ vertex (or back). For each $x_i = 1$ the cycle remains on the same part. Thus, a coloring with an even number of 0s will have two gray-solid edge-Kempe cycles, one including $a_1$ and the other including $b_1$, and a coloring with an odd number of 0s will have a Hamiltonian gray-solid edge-Kempe cycle. The situation is similar for solid-dash edge-Kempe cycles on the set of colorings of $C$: a coloring with an even number of 1s will have two solid-dash edge-Kempe cycles, one including $a_1$ and the other including $b_1$, and a coloring with an odd number of of 1s will have a Hamiltonian solid-dash edge-Kempe cycle.
When $k$ is odd, an even number of 1’s leaves an odd number of 0s. Thus, each coloring on the $(k - 1)$-cube has exactly one edge-Kempe cycle that uses solid and does not use vertex $a_1$.

When $k$ is even, an even number of 1s leaves an even number of 0s. Thus in one part (of a bipartition) of the $(k - 1)$-cube, there are no edge-Kempe cycles that use solid edges and do not use $a_1$, and in the other part each coloring has two edge-Kempe cycles using solid edges and not using $a_1$, one each of solid-dash and gray-solid.

It remains to show that the colorings resulting from solid-dash and gray-solid edge-Kempe cycle switches are leaves in $KE_v(CPr_k)$. (We know from Theorem 4.12 in [1] that $KE_v(CPr_k)$ has only one component, so this exhausts the possible vertices.) For any coloring in $C$, the only allowable switch (if any) using a solid edge will involve the edge $b_1b_{2k}$. Suppose making the switch results in the edge $b_1b_{2k}$ becoming gray. For each $i = 1, \ldots, k$, a gray-dash edge-Kempe chain will traverse all 4 vertices in the set $\{a_{2i-1}, a_{2i}, b_{2i-1}, b_{2i}\}$ in some order before moving to the next group of 4. (Precisely, the order is $\{a_{2i-1}, a_{2i}, b_{2i-1}, b_{2i}\}$ if $x_i = 0$, and $\{b_{2i-1}, a_{2i}, a_{2i-1}, b_{2i}\}$ if $x_i = 1$.) This means that any gray-dash edge-Kempe chain will proceed through all 4$k$ vertices before closing. Thus there is only one gray-dash edge-Kempe chain. The same argument is true for the solid-dash edge-Kempe chains and for the edge-Kempe chains when the edge $b_1b_{2k}$ is dash.

Corollary 4.3. $KE_v(CPr_k)$ has $2^k$ vertices and is bipartite with girth 4.

The prism graph $Pr_k$, is defined similarly to $CPr_k$: start with $2k$-cycles with vertices $A = \{a_j\}$, $B = \{b_j\}$, and add the additional edges $a_{2i-1}b_{2i-1}$ and $a_{2i}b_{2i}$ for $i = 1, \ldots, k$. It is simple to calculate the structure of $KE_v(Pr_k)$ for small $k$ and almost immediate to conjecture its general structure (according to parity). Despite the fact that this seems to be a simpler graph, a proof of the exact form of $KE_v(Pr_k)$ is harder to come by.

5 Open Questions and New Directions

There are a wealth of questions to be addressed about $KE(G)$ and $KE_v(G)$. What properties must $KE_v(G)$ have for various restrictions on $G$ such as having maximum degree 3, or being bipartite? Under what conditions is $KE_v(G)$ 2-connected? Or Hamiltonian? What can its diameter be? How many connected components can it have? In addition to degree and number of components, which other graph parameters must $KE_v(G)$ and $KE_{v_j}(G)$ share? For example, if one is $k$-connected then is the other as well? Will they have the same girth? Is there a way to characterize the best choice for $v$ with respect to some property for $KE_v(G)$? For example, which choice of $v$ gives the highest connectivity or girth?

We suspect that some of these questions will be as confounding as are many issues in graph edge colorings and in reconfiguration graphs, but others may be attainable.
References


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