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Edge-Kempe-equivalence graphs of class-1 regular graphs

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Abstract

We consider, for class-1, Δ-regular graphs G, the associated graph $KE(G)$, where the vertices of $KE(G)$ are Δ-edge colorings of G and edges of $KE(G)$ are present where two Δ-edge colorings of G differ by a single edge-Kempe switch. We focus on the case of cubic graphs and determine various structural properties of $KE(G)$ and $KE_v(G)$, where the latter considers a fixed set of colors on the edges incident to the vertex v. Additionally, we consider the ways in which $KE_v(G)$ for any choice of v must be very similar, as well as how they can differ.

1 Introduction and Summary

Consider a graph G with a proper edge coloring. A maximal two-color alternating path or cycle of edges is called an *edge-Kempe chain*; switching the colors along such a chain is called an *edge-Kempe switch*. Two edge-colorings are Kempe equivalent if one can be obtained from the other by a sequence of edge-Kempe switches. In [1], we examined equivalence classes of edge-colorings of graphs, with a focus on cubic graphs. Here we examine the *Kempe-equivalence graph*, denoted $KE(G)$, and defined as follows. Let G be a graph with maximum degree Δ, associate to each $\Delta(G)$-edge coloring of G a vertex in $KE(G)$, and two vertices of $KE(G)$ are adjacent when the colorings they represent differ by a single edge-Kempe switch. An equivalence class of colorings corresponds to a connected component of $KE(G)$. (In [1], we denoted the number of components of $KE(G)$ as $K'(G, \Delta(G))$.)

Kempe-equivalence graphs are an example of reconfiguration graphs. A reconfiguration graph has as its vertices all feasible solutions to a given problem, and two solutions are adjacent if and only if one can be obtained from the other by one application of a specific reconfiguration rule. There are several useful reconfiguration graphs for coloring that are currently being studied. One reason for this interest is the

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application to theoretical physics, where vertex-coloring graphs describe the Glauber dynamics of an anti-ferromagnetic Potts model at zero temperature [6, 9, 14]. In this situation, the set of all proper vertex k-colorings of a graph forms the state space for a Markov chain with iterations given by randomly recoloring a randomly selected vertex of G.

Motivated by the Markov chain connection, a graph G is said to be *k-mixing* if the k-coloring graph, with vertices representing proper vertex k-colorings of G and edges joining colorings that differ at a single vertex, is connected. The question of when G is k-mixing as well as the computational complexity of answering that question have been extensively studied [3, 4]. For instance, it has been shown that if the chromatic number $\chi(G) = k \in \{2, 3\}$, then G is not k-mixing. There are also examples of G with $k_1 < k_2$ such that G is k_1-mixing but not k_2-mixing. Mixing properties related to modifications of the k-coloring graph have also been studied (cf. [2, 8]). Recent work considers when the coloring graph and its modifications contain a Hamiltonian cycle [5, 8]. In each of the above references, the reconfiguration rule used was to allow recoloring of one vertex at a time. There is some literature that suggests that using Kempe changes is an appropriate model for some Glauber dynamics [13]. A recent paper by Feghali [7] et. al. shows that for almost all connected cubic graphs all 3-vertex colorings are Kempe equivalent.

In this paper we consider edge colorings, and two edge colorings will be adjacent if one is obtained from the other by switching the colors along one edge-Kempe chain. We investigate the structure of the Kempe-equivalence edge coloring graphs as has been done for other reconfiguration graphs. (We are not the first to do so; while Kaszonyi [10] has some similar results, these are for a very restricted class of cubic graphs and his language and notation are highly non-standard. Additionally, his proof methods are quite different from ours.) More recently, the Kempe equivalence of edge colorings was addressed in [12]. It turns out that while we can obtain some structural results, these are of limited use in understanding even relatively straightforward examples; the arguments used in proving structure theorems neither specialize nor extend in individual cases. We include a detailed example in Section 4 to show the limits of the theory and how cumbersome the calculation is.

Theorem 3.1 in [1] shows that given a vertex v of G, we can reach any edge-Kempe equivalent coloring with the same colors on edges incident to v without changing colors at v in the process. If we fix the edge colors at a vertex v of G (so as to ignore colorings that differ only because of color permutation), we denote the resulting Kempe-equivalence graph by $KE_v(G)$. This is in contrast to $KE(G)$, which has as its vertex set *all* colorings of the graph G. We will discuss how choice of v does and does not effect $KE_v(G)$.

We often assume that all graphs G are k-regular (have all vertices of degree k) and class 1 (edge colorable with k colors), although some of our results will hold more generally. A cubic graph is 3-regular and some of our results will only hold for cubic graphs. Colors are named as $\{1, 2, 3, \dots\}$ or {dash, gray, solid} depending on which makes most expositional sense. We say that two edge colorings c_1, c_2 are *isomorphic colorings* if they partition the base graph G into the same sets. That is, the colorings are isomorphic if they differ only by a permutation of the names of the colors. Note that in a regular, class-1 graph, all Kempe chains will in fact be Kempe cycles. We use S_k to refer to the symmetric group on k letters (the group of permutations).

The main results of this paper are in Section 2, where we look at structural properties of $KE_v(G)$ and $KE(G)$ from several perspectives. We show that $KE(G)$ graphs can never contain triangles, but almost always contains hypercubes. We show that for any graph G, there is an infinite family of graphs with the same KE graph as G. We also consider the ways in which $KE_v(G)$ for any choice of v must be very similar, as well as how they can differ. The notation $KE_C(G)$ will denote the subgraph of $KE(G)$ where the colors on the edges in set C are fixed and Section 3 provides descriptions of $KE(G)$ for some graph products that were also analyzed in [1]. Section 4 gives a sample calculation of $KE_v(G)$ for a particular graph, and Section 5 concludes the paper with some open questions.

2 Structural Results for $KE(G)$ and $KE_v(G)$

We begin with some simple observations that hold for all graphs, not just those that are regular and class 1.

Proposition 2.1. *For any graph G, $KE(G)$ is triangle free.*

Proof. Suppose there were a 3-cycle in $KE(G)$. This means there is a sequence of exactly three edge-Kempe switches that returns a graph to its original coloring. Without loss of generality, let the first switch involve colors 1 and 2. If the second switch also involves colors 1 and 2, then the switched chains are disjoint and both cannot be un-switched using a third switch. If the second switch involves two different colors 3 and 4, the same problem occurs. If the second switch involves colors 2 and 3 (or colors 1 and 3), then either the chains are disjoint, causing the same problem as the previous case, or the chains intersect. In this case there is a subset of the edges with a color permutation that is not a transposition and therefore cannot be undone with a single edge-Kempe switch. □

Proposition 2.2. *For any graph G, $KE(G)$ has girth 4.*

Proof. Suppose $KE(G)$ has a vertex (coloring of G) in which there are at least two edge-disjoint 2-color edge-Kempe cycles. Switching the colors independently generates a 4-cycle in $KE(G)$, and so $KE(G)$ has girth 4.

Instead, suppose that $KE(G)$ has no such vertex (coloring of G). This implies that every edge-Kempe cycle in *every* coloring of G is Hamiltonian, and that the only edges in $KE(G)$ correspond to transposing pairs of colors. Considering any three colors, say $1, 2, 3$, then sequentially doing the color transpositions $(1, 2), (1, 3), (2, 3), (1, 3)$ forms a 4-cycle. Therefore $KE(G)$ has girth 4. □

Moreover, any graph G with some coloring that has many independent edge-Kempe cycles will have multiple copies of C_4 in $KE(G)$. More precisely we have the following.

Theorem 2.3. *Suppose there exists a proper edge coloring, c, of a graph G with* q_{ij} *Kempe chains in the color pair* (i, j)*. Then for all* $1 \le i < j \le k$*,* $KE(G)$ *contains a* (q_{ij})*-cube. Further, all of these cubes intersect in exactly the one vertex of* $KE(G)$ *corresponding to the coloring c.*

Proof. For each pair of colors, each Kempe switch generates an edge from the cube, and because when performing two such switches of the same color pair the order does not matter, we obtain a square in $KE(G)$. Thus the (i, j) switches form a (q_{ij})-cube in $KE(G)$. □

Another basic observation is that the KE graph of a disjoint union of graphs is the cartesian product of the KE graphs of the component parts.

Proposition 2.4. $KE(G_1 \sqcup G_2) = KE(G_1) \Box KE(G_2)$.

We now move to results that hold only for regular graphs. Recall that for a cubic graph the $\Delta - \curlyvee$ operation is defined as replacing a K_3 with a single vertex. This is shown in Figure 1. Next we show that for cubic graphs the $\Delta - \curlyvee$ operation has no effect on the KE graph. Note that this proposition holds for multigraphs. We will discuss cubic multigraphs further in Section 2.1.

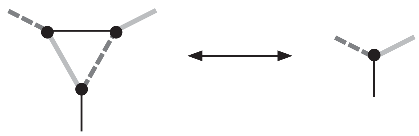

Figure 1: Corresponding 3-edge colorings across a $\Delta - \curlyvee$ operation.

Proposition 2.5. *For any 3-edge colorable cubic (multi)graph G, and any* $\Delta - \curlyvee$ *operation on G, denote the resulting graph by* G^Δ*. Then,* $KE(G) = KE(G^\Delta)$*.*

Proof. Notice that there is a one-to-one correspondence between 3-edge colorings of G and 3-edge colorings of G^Δ, as shown in Figure 1. Moreover, each edge-Kempe switch between a pair of colorings on G corresponds to one between the corresponding colorings of G^Δ. □

If G is regular of degree $k > 3$, then a $\Delta - \curlyvee$ move does not preserve regularity. The generalization of these results for $k > 3$ is that replacing a K_k by a vertex will not change the KE graph.

For cubic G adding (or removing) a pair of triangles that share an edge changes $KE(G)$, because it adds (or removes) a Kempe chain. Perhaps surprisingly, the existence of a pair of triangles in a cubic graph can produce different $KE_v(G)$ for different v. An example of this is given in Example 2.17 of Section 2.2. By a *lone triangle* (respectively *lone* K_k) we will mean a copy of C_3 (respectively K_k) that does not share an edge with another copy of C_3 (respectively K_k).

Corollary 2.6. *In examining the structure and realizability of $KE(G)$ for cubic (respectively k-regular) graphs, it suffices to consider graphs without lone triangles (respectively K_ks).*

Corollary 2.7. *Given a k-regular graph G with associated $KE(G)$, there is an infinite family of graphs $\mathcal{H}$ such that for all $H \in \mathcal{H}$, $KE(H) = KE(G)$.*

Proof. This follows from Proposition 2.5 for cubic graphs, and more generally by the discussion above. □

While the most naive method of removing triangles that share an edge can change $KE(G)$, it turns out that we can find an equivalent triangle-free graph if we are willing to consider cubic multigraphs.

2.1 Cubic Multigraphs

For cubic graphs, it turns out that the graphs that can be obtained as $KE(G)$ when G has multiple edges are precisely the same as those obtainable for simple graphs. Let G be a cubic multigraph with two edges e, h both between the vertices $u, v \in V(G)$. Define the graph $G_{e|h}$, shown in Figure 2 by $V(G_{e|h}) = V(G) \cup \{u_1, u_2\}$;

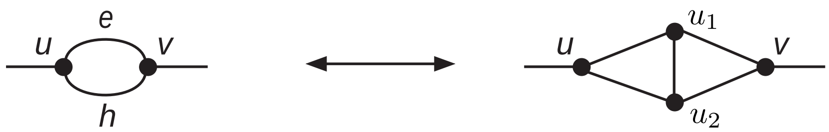

Figure 2: The construction of $G_{e|h}$ from G.

and $E(G_{e|h}) = E(G) \backslash \{e, h\} \cup \{uu_1, uu_2, u_1u_2, u_1v, u_2v\}$.

Theorem 2.8. *Let G be a cubic multigraph with with two edges e, h both between the vertices $u, v \in V(G)$, and $G_{e|h}$ as defined above. For any vertex $x \in V(G)$, we have $KE_x(G) = KE_x(G_{e|h})$, and $KE_u(G) = KE_{u_1}(G_{e|h}) = KE_{u_2}(G_{e|h}) = KE_v(G)$. Additionally, $KE(G) = KE(G_{e|h})$.*

Proof. First, there is a one-to-one correspondence between the proper edge colorings of G and $G_{e|h}$, as shown in Figure 3. Then notice that the edge-Kempe structure

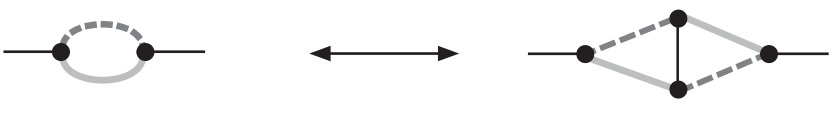

Figure 3: Corresponding 3-edge colorings between G and $G_{e|h}$.

of G is identical to that of $G_{e|h}$, because the three edge-Kempe chains correspond. Fixing the colors at any of the vertices u, u_1, u_2, v fixes the colors at the other three vertices, and this completes the proof. □

Corollary 2.9. *Given any cubic multigraph G there is a simple graph G^* such that $KE_v(G) = KE_v(G^*)$, for every $v \in V(G)$ and $KE(G) = KE(G^*)$. Moreover if $u \in V(G^*)$ then there exists $v \in V(G)$ such that $KE_v(G) = KE_u(G^*)$.*

Proof. This follows by repeated application of the operation in Theorem 2.8 until no multiple edges remain. □

Because the operation $G \to G_{e|h}$ is reversible, the reverse of Corollary 2.9 is also true.

Corollary 2.10. *Suppose H is a cubic graph with $u, v, u_1, u_2 \in V(H)$ and $uu_1, uu_2, vu_1, vu_2, u_1u_2 \in V(H)$. Then there is a cubic multigraph G such that $H = G_{e|h}$.*

By combining this result with Proposition 2.6 we can restrict consideration to only triangle-free multigraphs.

Corollary 2.11. *In examining the structure and realizability of $KE(G)$ for cubic multigraphs, it suffices to consider triangle-free multigraphs.*

2.2 $KE_v(G)$ for the same G and different v

For any choice of $v \in V(G)$ the graph $KE_v(G)$ will have as its vertex set exactly one copy of each non-isormorphic coloring of G. That is, there will be one vertex for each partition of the edges into Δ independent sets. The assignment of color names to the parts of the partition will vary based on which v is chosen and the initial assignment of colors to the edges incident to v. It turns out that the structure of $KE_v(G)$ can depend on the choice of v.

We start with some simple observations.

Remark 2.12. First, observe that if c and c' are isomorphic colorings of G then they are Kempe equivalent, since any permutation can be generated by transpositions.

Proposition 2.13. *Suppose c_1 is a coloring of a class-1 regular graph G in which there are exactly two edge-Kempe chains in some color pair (say, gray-dash) and c_2, c_2' the colorings that result from switching one or the other of the chains. Then c_2 and c_2' are isomorphic colorings.*

Proof. The colorings c_2, c_2' partition the edges of G in the same way, where the dash edges in c_2 are precisely the gray edges in c_2' and vice versa. All other color classes are identical. □

Note that under the conditions of Proposition 2.13, there is an edge of $KE_v(G)$ between colorings c_1 and c_2 (or an isomorphic copy of c_2) for all choices of v. We next give a sufficient condition for $KE_v(G)$ to be isomorphic graphs for all $v \in V(G)$. Recall that G is vertex transitive if for any pair of vertices v_1, v_2 there is a graph automorphism that takes v_1 to v_2.

Theorem 2.14. *For a simple, regular graph G, if $|V(G)| < 12$ or if G is vertex transitive, then for any vertices $v_i, v_j \in V(G)$ the graphs $KE_{v_i}(G)$ and $KE_{v_j}(G)$ are isomorphic, and the isomorphism associates isomorphic colorings.*

Proof. Certainly, if G is vertex transitive, then $KE_{v_i}(G)$ and $KE_{v_j}(G)$ are isomorphic for any vertices $v_i, v_j \in V(G)$. By Proposition 2.13, if $KE_{v_1}(G) \neq KE_{v_2}(G)$ then there must be a coloring c_1 that has three or more edge-Kempe chains in some color pair. Since each edge-Kempe chain must consist of at least 4 vertices (by Proposition 2.1), there must be at least 12 vertices in the graph. □

We will show the above theorem is best possible by constructing a graph on 12 vertices that has two different KE_v graphs. Before constructing this example, we first give some results that help determine the structure of $KE_v(G)$. The following results give some relationships between various parameters on $KE_v(G)$, and $KE(G)$. Indeed, they show that for a given graph G, the $KE_v(G)$ must be very similar for any choice of v.

Theorem 2.15. *Let G be a class-1 k-regular graph.*

(a) The degree of a vertex c, a coloring, in $KE_v(G)$ is $deg_{KE(G)}(c) - \binom{k}{2}$, for any $v \in V(G)$.

(b) The number of connected components of $KE_v(G)$ is equal to the number of connected components of $KE(G)$.

Proof. (a) The degree of a vertex c in $KE_v(G)$ corresponds to the number of possible edge-Kempe switches that can be made in a coloring c of G with the edge colors at v fixed. This number is the same independent of the choice of $v \in V(G)$ because v is on exactly $\binom{k}{2}$ edge-Kempe chains.

(b) Suppose that c_1, c_2 are in the same connected component of $KE_v(G)$. Then there exists a path between them in $KE_v(G)$, and hence in $KE(G)$ by subgraph inclusion. Now suppose that c_1, c_2 are in the same connected component of $KE(G)$. If the edge colors of c_1, c_2 do not agree at v, then there is a coloring c_2' , that is isomorphic to c_2 that has the edge colors of c_1 at v. By Remark 2.12, c_2, c_2' are in the same connected component of $KE(G)$. Thus it suffices to assume that the edge colors of c_1, c_2 agree at v. Then by Theorem 3.1 of [1], there is a sequence of edge-Kempe changes between c_1 and c_2 that avoids changing colors at v, and so c_1, c_2 are in the same connected component of $KE_v(G)$. □

The power of this result can be seen in the following immediate consequences of this theorem and its proof.

Corollary 2.16. *Let G be a class-1 k-regular graph.*

(i) If the coloring c is a leaf in $KE_{v_i}(G)$ then it is also a leaf in $KE_{v_j}(G)$ for any $v_j \in V(G)$. Further, if c is a leaf then its unique neighbor is the same coloring (up to isomorphism) in all $KE_{v_j}(G)$.

(ii) The degree sequences of $KE_{v_i}(G)$ and $KE_{v_j}(G)$ are equal.

(iii) If c_1, c_2 are colorings in the same connected component of $KE_{v_i}(G)$, then their isomorphic counterparts are in the same connected component of $KE_{v_j}(G)$.

We will now exhibit a simple connected cubic graph R on 12 vertices that has two non-isomorphic $KE_v(R)$ graphs.

Example 2.17. In order to have 3 disjoint Kempe cycles on 12 vertices, R must consist of three disjoint copies of C_4 with six additional edges, and by Corollary 2.6, R has no lone triangles. See Figure 4; note that R is planar, but not bipartite.

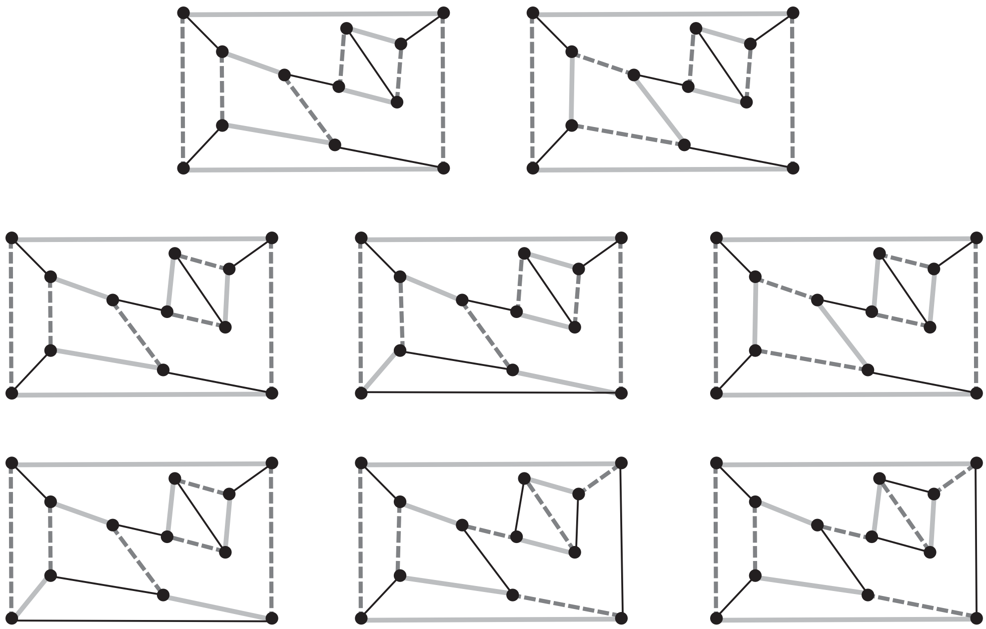

Figure 4: Colorings of a graph R with non-isomorphic $KE_v(R)$.

There are eight nonisomorphic colorings of this graph, all shown in Figure 4. The coloring shown at top left has seven edge-Kempe chains, so will have degree 4 in $KE_v(R)$ for any v (because three edge-Kempe chains will be fixed). Let A denote the 4-cycle bounding the pair of triangles. Switching the colors of A produces another coloring with seven edge-Kempe chains, shown at left in the middle row. The coloring shown at top right has five edge-Kempe chains, so will have degree 2 in $KE_v(R)$ for any v. The remaining five colorings also have five edge-Kempe chains each. If the fixed vertex v is *not* on the cycle A, then the two colorings of degree 4 are adjacent in $KE_v(R)$ (via switching the colors on A). If the fixed vertex *is* on the cycle A then the two colorings of degree 4 are not adjacent in $KE_v(R)$. Thus there are (at least) two different graphs that occur as $KE_v(R)$.

A similar example on 12 vertices can be constructed that is triangle free (but not bipartite). A connected simple example that is both bipartite and planar requires 14 vertices; G_{14} is shown in Figure 5. If we allow multigraphs then a smaller connected example can be constructed by replacing the adjacent triangles of the cycle A of the graph R in Example 2.17 with a pair of multiple edges as in Corollary 2.11. The smallest example of a (non-connected) multigraph with two different graphs occuring as $KE_v(G)$ is $D \cup P$, depicted in Figure 6.

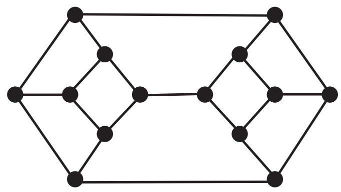

Figure 5: A cubic, planar, bipartite graph that has two different $KE_v(G_{14})$.

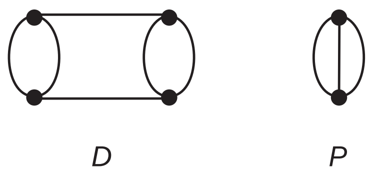

Figure 6: $D \cup P$ is a multigraph with two different $KE_v(D \cup P)$.

2.3 $KE_v(G)$ as a subgraph of $KE(G)$

In this section we describe how $KE_v(G)$ is contained in $KE(G)$. Many of the statements made about $KE(G)$ have obvious implications for $KE_v(G)$. For example Propositions 2.1 and 2.2 imply that for all G, v, $KE_v(G)$ must have girth at least 4.

Proposition 2.18. *For any graph G, and any $v \in V(G)$, $KE_v(G)$ has girth 4 if there are are at least two disjoint edge-Kempe chains disjoint from v.*

We suspect that there is no upper bound on the girth of $KE_v(G)$, for G cubic and class 1, but the largest girth we have observed is associated to the 14-vertex generalized Petersen graph shown in Figure 7; $KE_v(GP) \cong C_7$ and so has girth 7. (Some larger generalized Petersen graphs have smaller girths.)

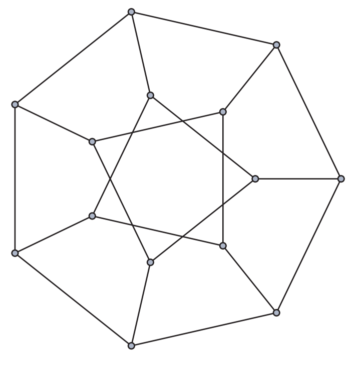

Figure 7: The generalized Petersen graph on 14 vertices.

In a class-1 k-regular graph, any vertex v will be in exactly one (i, j) Kempe-chain for each pair of colors $1 \leq i < j \leq k$. Thus we get the following corollary to Theorem 2.3.

Corollary 2.19. *If there exists a proper edge coloring, c, of a class-*1 *k-regular graph G with q_{ij} Kempe chains of colors i, j. Then for all $1 \leq i < j \leq k$, and any $v \in V(G)$, $KE_v(G)$ contains a $(q_{ij} - 1)$-cube. Further, all of these cubes intersect in exactly one vertex of $KE_v(G)$ corresponding to the coloring c.*

For each vertex of $KE_v(G)$ there are $k!$ corresponding vertices in $KE(G)$. In fact, there are $k!$ disjoint copies of $KE_v(G)$ that partition the vertices of $KE(G)$. These copies of $KE_v(G)$ differ by the assignment of colors on the k edges incident to v. In order to describe the relationships between the copies, we introduce a more precise notation: for σ a permutation of the k colors, define $KE_v(G)^\sigma$ to be the graph whose vertices are proper k-edge colorings of G, such that the order of the colors on the edges incident to v is σ with respect to a fixed ordering of the edges of G. The $k!$ disjoint $KE_v(G)^\sigma$ (one for each permutation σ) form a partition of the vertices of $KE(G)$. In any proper k-edge coloring of G there are $\binom{k}{2}$ edge-Kempe chains incident to vertex v, and each of these changes the permutation of colors at v by a different transposition. Thus each coloring in $KE_v(G)^\sigma$ is adjacent to $\binom{k}{2}$ other colorings, each of which belongs to a $KE_v(G)^{\sigma'}$ with a distinct σ'.

That $c \in V(KE_v(G)^\sigma)$ and $c' \in V(KE_v(G)^{\sigma'})$ are adjacent does not imply that c, c' are isomorphic colorings. This is because moving from $KE_v(G)^\sigma$ to $KE_v(G)^{\sigma'}$ requires changing only, for example, the dash-solid edge-Kempe chain through v and not *all* dash-solid edge-Kempe chains in G. Yet, when σ and σ' differ by a transposition, the edges between vertices of $KE_v(G)^\sigma$ and $KE_v(G)^{\sigma'}$ form a perfect matching between these sets.

It is clear that for any two permutations σ, σ', the graphs $KE_v(G)^\sigma$ and $KE_v(G)^{\sigma'}$ are isomorphic; thus, in many cases the notation $KE_v(G)$ can be used without confusion.

Proposition 2.20. *If $KE_v(G)$ has a Hamiltonian circuit for some $v \in V(G)$, then $KE(G)$ has a Hamiltonian path.*

Proof. A Hamiltonian path H in $KE(G)$ is formed by concatenating Hamiltonian paths from $KE_v(G)^\sigma$ for each σ. It was shown in [11] that the Cayley graph of S_n generated by transpositions has a Hamiltonian circuit. Suppose that the sequence of transpositions that accomplishes a Hamiltonian circuit in S_n is $t_1, t_2, \ldots t_p$, and let ι be the identity permutation.

Let c_{11}, c_{12} be adjacent vertices in a Hamiltonian circuit in $KE_v(G)^\iota$. Let H begin with the Hamiltonian path of $KE_v(G)^\iota$ that begins c_{11} and ends c_{12}. Now, to coloring c_{12} apply the t_1 edge-Kempe-switch that includes vertex v. The resulting coloring, say c_{21}, will be in $KE_v(G)^{t_1}$. Because each $KE_v(G)$ has a Hamiltonian circuit, there is a Hamiltonian path starting at c_{21}. Let c_{22} be the last vertex of this path. Apply to c_{22} the t_2 edge-Kempe-switch that includes vertex v, to get a coloring c_{31} in $KE_v(G)^{t_2 t_1}$. Proceed in a similar manner to traverse all the colorings of each $KE_v(G)^\sigma$. □

2.4 More about the structure of $KE(G)$

Let $c_\sigma, c_{(ij)\sigma}$ be two isomorphic colorings of G that differ by a transposition (ij) on color names. Then there is a path between $c_\sigma, c_{(ij)\sigma}$ in $KE(G)$, each of whose edges corresponds to a edge-Kempe switch of colors i and j on some i-j edge-Kempe chain of c_σ. There are in fact many such paths, as the switches can be made in any order. Let $c_\sigma \sim c_{(ij)\sigma}$ denote one such path.

Lemma 2.21. *For any G, $KE(G)$ always contains a subdivision of $K_{3,3}$.*

Proof. Let c be any coloring in $KE(G)$. For each of the six permutations of $\{1, 2, 3\}$ (fixing all other colors if $k > 3$) there corresponds a coloring isomorphic to c. Recall that the Cayley graph of the six permutations of S_3, generated by transpositions, forms a $K_{3,3}$. Thus, the subgraph of $KE(G)$ induced by the 9 paths $c_\sigma \sim c_{(ij)\sigma}$, where σ and $(ij)\sigma$ fix all colors other than $\{1, 2, 3\}$ and permute (some of) $\{1, 2, 3\}$, is therefore a subdivision of $K_{3,3}$. □

In fact, associated to every vertex of $KE(G)$ are $k!$ vertices of $KE(G)$ (namely, the isomorphic colorings) forming a subgraph that is a subdivision of the Cayley graph with the transpositions of S_k as the generators. Some easy corollaries follow from Lemma 2.21.

Corollary 2.22. *No tree is realizable as $KE(G)$ for any G.*

Corollary 2.23. *No $KE(G)$ is planar.*

Corollary 2.24. *If G is k-regular and is uniquely k-edge colorable, then $KE(G)$ is isomorphic to the Cayley graph of S_k with the set of all transpositions as generators.*

The graph $K_{3,3}$ has two nonisomorphic edge-colorings, but in both colorings every edge-Kempe chain is a Hamiltonian cycle. This observation produces a generalization of the above corollary. We use Γ_k to denote the Cayley graph of S_k with the set of all transpositions as generators.

Corollary 2.25. *Suppose G is k-regular and class 1 with exactly h nonisomorphic k-edge colorings. If in each of the colorings of G every edge-Kempe chain is a Hamiltonian cycle, then $KE(G) = \sqcup_h \Gamma_k$.*

3 Graph Products

It will be helpful to start by considering a variation on $KE_v(G)$. The graph $KE_v(G)$ was formed by fixing the colors of the edges incident to a single vertex $v \in V(G)$. We could instead fix the colors of a 3-edge cut C of a cubic graph G, denoted $KE_C(G)$. A parity argument implies that a 3-edge cut of a properly colored class-1 cubic graph must contain exactly one edge of each color, so $KE_C(G)$ will contain one copy of each non-isomorphic edge coloring of G. Thus similar statements to those in Theorem 2.15 and other results in Section 2.2 hold for $KE_C(G)$ as well.

Proposition 3.1. *Let G be a class-1 cubic graph with $v \in V(G)$ and C a 3-edge cut of G.*

(i) The degree of a vertex c, a coloring, in $KE_C(G)$ is $deg_{KE(G)}(c) - 3$.

(ii) The number of connected components of $KE_C(G)$ is equal to the number of connected components of $KE(G)$.

(iii) If the coloring c is a leaf in $KE_v(G)$ then it is also a leaf in $KE_C(G)$. Further, if c is a leaf then its unique neighbor is the same (up to isomorphism).

(iv) The degree sequences of $KE_v(G)$ and $KE_C(G)$ are equal.

The graph $KE_C(G)$ has a nice description when G is formed by the following graph product, which was defined in [1]. Consider two cubic graphs G_1, G_2, and form $G_1 \curlyvee G_2$ by choosing vertices $v_1 \in V(G_1), v_2 \in V(G_2)$, removing v_1, v_2, and adding a matching of three edges joining the three neighbors of v_1 with the three neighbors of v_2. Of course there are many ways to choose v_1, v_2, and many ways to identify their incident edges, so the construction is not unique. The following theorem holds for any such choices.

Proposition 3.2. *Let $G_1 \curlyvee G_2$ be formed from G_1, G_2 using vertices $v_1 \in V(G_1)$, $v_2 \in V(G_2)$, so that C is the edge cut formed. Then*

$$KE_C(G_1 \curlyvee G_2) = KE_{v_1}(G_1) \Box KE_{v_2}(G_2).$$

Proof. Every coloring of $G_1 \curlyvee G_2$ can be written as an ordered pair of colorings (c_1, c_2) where c_1 is a coloring of G_1 and c_2 is a coloring of G_2. If the colors on C are fixed, then no edge-Kempe chain in $G_1 \curlyvee G_2$ can cross C. Therefore, any edge in $KE_C(G_1 \curlyvee G_2)$ corresponds to an edge-Kempe chain in exactly one of G_1 and G_2. This is the definition of $KE_{v_1}(G_1) \Box KE_{v_2}(G_2)$. ☐

Examining $KE_v(G_1 \curlyvee G_2)$ shows that changing v may produce nonisomorphic graphs, even when there is only one possibility for $KE_v(G_i)$. Consider $Q_3 \curlyvee Q_3$ (shown in Figure 5), where Q_3 is the cube. Because Q_3 is vertex transitive, all $KE_v(Q_3)$ are isomorphic, but direct computation shows that there are at least two different graphs that occur as $KE_v(Q_3 \curlyvee Q_3)$. It is true that no matter the choices made in making the $\curlyvee$ product, and on which vertex or cut in the resulting product we fix the colors, the number of edges in and degree sequence of $KE_v(Q_3 \curlyvee Q_3)$ or $KE_C(Q_3 \curlyvee Q_3)$ will be the same.

Proposition 3.3. *Let G_1, G_2 be class-1 cubic graphs, with $v_1 \in V(G_1), v_2 \in V(G_2)$ the vertices used in creating $G_1 \curlyvee G_2$ and $x \neq v_1 \in V(G_1)$. Then the vertex sets $V(KE_x(G_1 \curlyvee G_2)) \simeq V(KE_x(G_1) \Box KE_{v_2}(G_2))$, corresponding vertices have the same degree, and $E(KE_x(G_1 \curlyvee G_2)) \supseteq E(KE_x(G_1) \Box KE_{v_2}(G_2))$.*

Proof. The set of colorings in $KE_x(G_1 \curlyvee G_2)$ may be indexed as (c_i, c_j), where c_i is a coloring of G_1 and c_j is a coloring of G_2 with colors permuted to match c_i on the edges that were incident to v_1, v_2. Thus each c_i of $KE_x(G_1)$ can only be paired with all colorings from $KE_{v_2}(G_2)^\sigma$ for exactly one σ. The particular σ depends on the colors that c_i assigns to the edges incident to v_1. Then, by the parity lemma each edge in a 3-edge cut of a cubic graph must receive a different color; hence, every coloring of $G_1 \curlyvee G_2$ induces a coloring of G_1 and of G_2. Thus, $V(KE_x(G_1 \curlyvee G_2)) \simeq V(KE_x(G_1) \square KE_{v_2}(G_2))$.

Let $(c_i, c_j) \in V(KE_x(G_1 \curlyvee G_2))$ and the corresponding coloring be $(c_i, \hat{c_j}) \in V(KE_x(G_1) \square KE_{v_2}(G_2))$. The coloring $(c_i, c_j) \in V(KE_x(G_1 \curlyvee G_2))$ has three kinds of edge-Kempe chains not incident to x, namely (i) entirely within G_1 (and not incident to v_1), (ii) entirely in G_2 (not incident to v_2), or (iii) containing edges previously incident to v_1, v_2. The first kind are in one-to-one correspondence with edge-Kempe chains in $KE_x(G_1)$ that are not incident to v_1. The second kind are in one-to-one correspondence with edge-Kempe chains in $KE_{v_2}(G_2)$ that are not incident to v_2. The third kind are in one-to-one correspondence with the edge-Kempe chains in $KE_x(G_1)$ that are incident to the vertex v_1. This shows that the degrees of the vertices are the same. To see that the edge inclusion holds, note further that (i) and (iii) correspond to edges in $KE_x(G_1)$ and (ii) correspond exactly to the edges in $KE_{v_2}(G_2)$. □

Observe that while the degrees are the same for corresponding vertices in $KE_x(G_1 \curlyvee G_2)$ and $KE_x(G_1) \square KE_{v_2}(G_2)$, the edges of $KE_x(G_1 \curlyvee G_2)$ do not join corresponding pairs of colorings in $KE_x(G_1) \square KE_{v_2}(G_2)$. Specifically, the second kind of edge-Kempe chain in the proof changes colors on edges in both G_1 and G_2 and that never happens in the edge-Kempe chains represented in $KE_x(G_1) \square KE_{v_2}(G_2)$.

A similar correspondence occurs with $KE(G_1 \curlyvee G_2)$.

Proposition 3.4. *Let G_1, G_2 be class-1 cubic graphs. Then*

(a) $V(KE(G_1 \curlyvee G_2)) \simeq V(KE(G_1) \square KE_{v_2}(G_2))$ *and*

(b) $E(KE(G_1 \curlyvee G_2)) \supseteq \cup_\sigma E(KE_{v_1}(G_1)^\sigma \square KE_{v_2}(G_2))$.

The proof is almost identical to that of Proposition 3.3.

Note that $G \curlyvee K_4$ is simply the $\Delta - \curlyvee$ operation. We can now generalize by considering this product with any uniquely 3-edge colorable graph.

Proposition 3.5. *If U is a uniquely 3-edge colorable cubic graph, and G is any class-1 cubic graph, then $KE(G \curlyvee U) = KE(G)$ and $KE_v(G \curlyvee U) = KE_{v'}(G)$, where $v' := v$ if $v \in V(G)$ is not used in creating the product, and v' is a vertex remaining in U otherwise. More generally, if H is cubic with exactly h nonisomorphic 3-edge colorings, and in every edge-Kempe chain is a Hamiltonian cycle, then $KE(G \curlyvee H) = \sqcup_h KE(G)$, and $KE_v(G \curlyvee H) = \sqcup_h KE_{v'}(G)$, with v' defined as before.*

Another product, also introduced in [1], is natural when considering KE graphs. Let G_1, G_2 be graphs with edges e_1, e_2 in G_1, G_2 respectively. Consider v_1, w_1 endpoints of e_1 and v_2, w_2 endpoints of e_2. Form $G_1 \mp G_2$ by removing e_1, e_2 and adding two edges connecting v_1 to v_2 and w_1 to w_2. (While in [1] this product was defined only for cubic graphs, it generalizes directly to k-regular graphs.) As with the $\curlyvee$ product, the choice of edges to cut and the ways to pair them up mean this construction is not unique. Nonetheless, the following analysis holds for all choices.

We can combine a coloring c of G_1 with a coloring d on G_2 to get a coloring of $G_1 \mathbin{\mathrm{I}} G_2$ if the colors agree on e_1, e_2. Note that by parity, in any coloring of $G_1 \mathbin{\mathrm{I}} G_2$ the same color will be assigned to both v_1v_2 and w_1w_2. In $KE_{v_1}(G_1 \mathbin{\mathrm{I}} G_2)$ we have fixed the colors on all edges incident to v_1, but only on one edge incident to v_2. Therefore, for each vertex of $KE_{v_1}(G_1)$ we have $|V(KE_{v_2}(G_2))|$ copies of each of the $(k-1)!$ isomorphic colorings of G_2, corresponding to the $(k-1)!$ permutations of the colors on the other edges incident to v_2. This means that there are $(k-1)!|V(KE_{v_1}(G_1))| \cdot |V(KE_{v_2}(G_2))|$ vertices in $KE_{v_1}(G_1 \mathbin{\mathrm{I}} G_2)$. Note that because the edge colors are fixed at v_1, no edge-Kempe chains cross the 2-edge cut formed in the construction $G_1 \mathbin{\mathrm{I}} G_2$; that is, colors change in G_1 or in G_2 but not both. There are therefore three types of edges in $KE_{v_1}(G_1 \mathbin{\mathrm{I}} G_2)$: those that correspond to edges in $KE_{v_1}(G_1)$, those that correspond to edges in $KE_{v_2}(G_2)$, and those that connect vertices from different copies of $KE_{v_2}(G_2)^\sigma$. Extending our notation in a natural way, we now define $KE_e(G)$ to be the subgraph of $KE(G)$ where the color on the edge e is fixed. Note that if e is incident to v, then $KE_e(G) \supseteq \cup_\sigma KE_v(G)^\sigma$ (with the union taken over the $(k-1)!$ permutations that fix the color on e), with additional edges d_1d_2 if $d_1 \in V(KE_v(G)^\sigma), d_2 \in V(KE_v(G)^{(ij)\sigma})$, and d_1, d_2 agree on the coloring of all edges except those on the (i,j) edge-Kempe chain that passes through v and does not use the edge e. This proves the following.

Lemma 3.6. *Let G_1, G_2 be class-1 k-regular graphs. If $v_i \in V(G_i)$ and $e_i \in E(G_i)$ are the vertices and edges involved in forming $G_1 \mathbin{\mathrm{I}} G_2$, then*

(a) $KE_{v_1}(G_1 \mathbin{\mathrm{I}} G_2) = KE_{v_1}(G_1)\Box KE_{e_2}(G_2)$ *and*

(b) $KE_{v_2}(G_1 \mathbin{\mathrm{I}} G_2) = KE_{e_1}(G_1)\Box KE_{v_2}(G_2)$.

The above result requires that $v_i \in V(G_i)$ are vertices incident to the cut edge. One might hope that a result for general v would also be possible, at least in the case of joining a uniquely edge-colorable graph. Unfortunately, Example 2.17 shows that is not true. That graph is $Q_3 \mathbin{\mathrm{I}} K_4$, where Q_3 is the cube. As the cube is vertex transitive, only one graph can occur as $KE_v(Q_3)$, and K_4 is uniquely colorable. Yet we have seen that there are $v_i, v_j \in V(Q_3 \mathbin{\mathrm{I}} K_4)$ such that $KE_{v_i}(Q_3 \mathbin{\mathrm{I}} K_4) \neq KE_{v_j}(Q_3 \mathbin{\mathrm{I}} K_4)$.

The result about $KE(G_1 \mathbin{\mathrm{I}} G_2)$ is similar to that for $KE(G_1 \curlyvee G_2)$.

Proposition 3.7. *Let G_1, G_2 be class-1 k-regular graphs. If $v_i \in V(G_i)$ and $e_i \in E(G_i)$ are the vertices and edges involved in forming $G_1 \mathbin{\mathrm{I}} G_2$, then*

(a) $V(KE(G_1 \mathbin{\mathrm{I}} G_2)) = V(KE(G_1)\Box KE_{e_2}(G_2))$ *and*

(b) $E(KE(G_1 \mathbin{\mathrm{I}} G_2)) \supseteq \cup_\sigma E(KE_{e_1}(G_1)^\sigma \Box KE_{e_2}(G_2))$.

Proof. Consider a coloring c of G_1. Form $G_1 \mathbin{\mp} G_2$ and use c for the edges originally in G_1. For the remaining edges, we can use any coloring of G_2 up to permutation of the colors so that the color on e_2 matches that of e_1 in c. Therefore the vertices of $KE(G_1 \mathbin{\mp} G_2)$ may be indexed by ordered pairs of colorings from G_1 and G_2.

An edge-Kempe chain in $G_1 \mathbin{\mp} G_2$ may be in $G_1 \setminus \{e_1\}$, in $G_2 \setminus \{e_2\}$, or may involve the edges formed from e_1, e_2. The first kind are in one-to-one correspondence with the edge-Kempe chains in $\cup KE_{e_1}(G_1)^\sigma$, the second kind are in one-to-one correspondence with the edge-Kempe chains in $KE_{e_2}(G_2)$, and the third kind do not appear in $\cup_\sigma E(KE_{v_1}(G_1)^\sigma \square KE_{e_2}(G_2))$. □

Again, even when G_2 is uniquely colorable there is no simple description of $KE(G_1 \mathbin{\mp} G_2)$. Let P and D be defined as in Figure 6. Note that $D = P \mathbin{\mp} P$. As for any uniquely colorable graph, $KE(P) = K_{3,3}$, $KE_v(P) = K_1$ for any vertex $v \in V(P)$, and $KE_e(P) = K_2$ for any edge $e \in E(P)$. By Proposition 3.7, $KE(D)$ has 12 vertices. Taking $\cup_\sigma E(KE_{v_1}(P)^\sigma \square KE_{e_2}(P))$ we see that the 12 vertices form 3 disjoint 4-cycles. Each of $K_1, K_2, K_{3,3}$ are bipartite and the $\square$ product preserves the property of being bipartite. However, the additional edges of $KE(D)$ not in $\cup_\sigma E(KE_{v_1}(P)^\sigma \square KE_{e_2}(P))$ cause $KE(D)$ to not be bipartite. In Figure 8 a 5-cycle in $KE(D)$ is shown. The bold 5-cycle edges correspond to edge-Kempe chains that

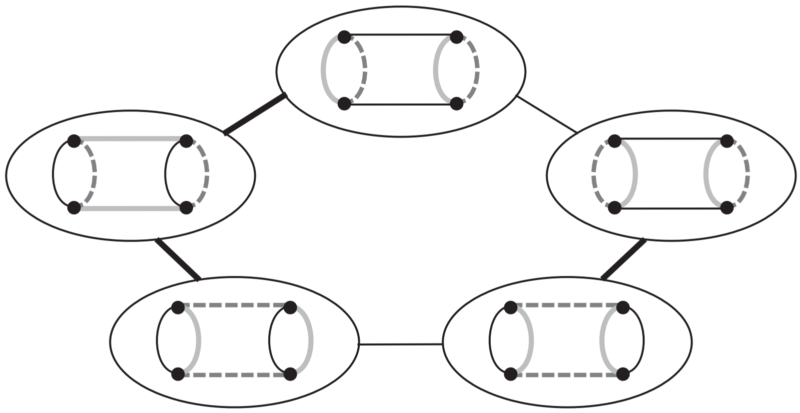

Figure 8: Five edge-colorings of D that form a 5-cycle in $KE(D)$.

use edges of both copies of P. Note that by Corollary 2.11, an identical analysis holds for $K_4 \mathbin{\mp} K_4$, which is a simple planar graph (with triangles).

4 A sample calculation

In this section we consider a specific base graph with nice structure and calculate its KE_v graph. This is evidence that even an example expected to be simple can require a cumbersome argument.

Definition 4.1. Consider two $2k$-cycles with vertices $A = \{a_i : 1 \leq i \leq 2k\}, B = \{b_i : 1 \leq i \leq 2k\}$ respectively. For $i = 1, \ldots k$ add the edges $a_{2i}b_{2i+1}$ and $b_{2i}a_{2i+1}$. The result is called the *crossed prism* graph CPr_k on $4k$ vertices.

Note that CPr_k is vertex-transitive; thus, $KE_v(CPr_k)$ is the same independent of choice of v.

Theorem 4.2. *Let CPr_k be the crossed prism graph with $4k$ vertices.*

(a) For k even, $KE_v(CPr_k)$ is a $(k-1)$-cube with 2 leaves on each vertex of one of the parts (of a bipartition of the cube).

(b) For k odd, $KE_v(CPr_k)$ is a $(k-1)$-cube with one leaf on each vertex.

Proof. Consider the coloring of CPr_k shown in Figure 9. The A and B cycles form dash-solid edge-Kempe chains. Each cross (pair of a_ib_j edges) is colored in gray. Without loss of generality, fix the colors at the upper-left-most vertex, a_1.

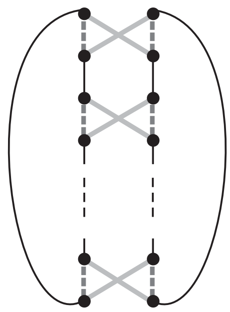

Figure 9: A coloring of CPr_k.

Since the colors at vertex a_1 are fixed, there are now $k-1$ gray-dash edge-Kempe cycles (each of length 4) that can be switched, and one gray-solid cycle. Note that these edge-Kempe switches are independent, and that making any two different switches can be done in either order to form a square. Considering only the gray-dash cycles, we see that $KE_v(CPr_k)$ contains a $(k-1)$-cube $\mathcal{C}$. We can represent each coloring in $\mathcal{C}$ by a binary string $(x_1,\ldots,x_k)$ of length k, where 0 represents that $a_{2i-1}a_{2i}$, $b_{2i-1}b_{2i}$ are the dash edges, and 1 represents that $a_{2i-1}b_{2i}$, $b_{2i-1}a_{2i}$ are the dash edges (so the coloring in Figure 9 is denoted by $(0,0,0,\ldots,0)$). Note that for ease of exposition we use a k-tuple, but all allowable colorings will have $x_1=0$ as we are considering $KE_{a_1}(CPr_k)$.

Now consider a gray-solid edge-Kempe cycle starting at b_1 on a particular coloring $(x_1,\ldots x_k)\in\mathbb{Z}_2^k$ in $\mathcal{C}$. For each $x_i=0$ for $i=1,\ldots,k$, the cycle will cross from an A vertex to a B vertex (or back). For each $x_i=1$ the cycle remains on the same part. Thus, a coloring with an even number of 0s will have two gray-solid edge-Kempe cycles, one including a_1 and the other including b_1, and a coloring with an odd number of of 0s will have a Hamiltonian gray-solid edge-Kempe cycle. The situation is similar for solid-dash edge-Kempe cycles on the set of colorings of $\mathcal{C}$: a coloring with an even number of 1s will have two solid-dash edge-Kempe cycles, one including a_1 and the other including b_1, and a coloring with an odd number of of 1s will have a Hamiltonian solid-dash edge-Kempe cycle.

When k is odd, an even number of 1's leaves an odd number of 0s. Thus, each coloring on the $(k-1)$-cube has exactly one edge-Kempe cycle that uses solid and does not use vertex a_1.

When k is even, an even number of 1s leaves an *even* number of 0s. Thus in one part (of a bipartition) of the $(k-1)$-cube, there are no edge-Kempe cycles that use solid edges and do not use a_1, and in the other part each coloring has two edge-Kempe cycles using solid edges and not using a_1, one each of solid-dash and gray-solid.

It remains to show that the colorings resulting from solid-dash and gray-solid edge-Kempe cycle switches are leaves in $KE_v(CPr_k)$. (We know from Theorem 4.12 in [1] that $KE_v(CPr_k)$ has only one component, so this exhausts the possible vertices.) For any coloring in $\mathcal{C}$, the only allowable switch (if any) using a solid edge will involve the edge b_1b_{2k}. Suppose making the switch results in the edge b_1b_{2k} becoming gray. For each $i = 1, \ldots k$, a gray-dash edge-Kempe chain will traverse all 4 vertices in the set $\{a_{2i-1}, a_{2i}, b_{2i-1}, b_{2i}\}$ in some order before moving to the next group of 4. (Precisely, the order is $\{a_{2i-1}, a_{2i}, b_{2i-1}, b_{2i}\}$ if $x_i = 0$, and $\{b_{2i-1}, a_{2i}, a_{2i-1}, b_{2i}\}$ if $x_i = 1$.) This means that any gray-dash edge-Kempe chain will proceed through all $4k$ vertices before closing. Thus there is only one gray-dash edge-Kempe chain. The same argument is true for the solid-dash edge-Kempe chains and for the edge-Kempe chains when the edge b_1b_{2k} is dash. □

Corollary 4.3. *$KE_v(CPr_k)$ has 2^k vertices and is bipartite with girth* 4.

The *prism* graph Pr_k, is defined similarly to CPr_k; start with $2k$-cycles with vertices $A = \{a_j\}, B = \{b_j\}$, and add the additional edges $a_{2i-1}b_{2i-1}$ and $a_{2i}b_{2i}$ for $i = 1, \ldots k$. It is simple to calculate the structure of $KE_v(Pr_k)$ for small k and almost immediate to conjecture its general structure (according to parity). Despite the fact that this seems to be a simpler graph, a proof of the exact form of $KE_v(Pr_k)$ is harder to come by.

5 Open Questions and New Directions

There are a wealth of questions to be addressed about $KE(G)$ and $KE_v(G)$. What properties must $KE_v(G)$ have for various restrictions on G such as having maximum degree 3, or being bipartite? Under what conditions is $KE_v(G)$ 2-connected? Or Hamiltonian? What can its diameter be? How many connected components can it have? In addition to degree and number of components, which other graph parameters must $KE_{v_1}(G)$ and $KE_{v_2}(G)$ share? For example, if one is k-connected then is the other as well? Will they have the same girth? Is there a way to characterize the *best* choice for v with respect to some property for $KE_v(G)$? For example, which choice of v gives the highest connectivity or girth?

We suspect that some of these questions will be as confounding as are many issues in graph edge colorings and in reconfiguration graphs, but others may be attainable.

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