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Nested unimodality

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Abstract

Let $P(x)$ be a unimodal polynomial of degree m with nonnegative coefficients and a mode n for nonnegative integers $n \leqslant m$. We study the unimodality of $P(x+z)$ for real numbers $z = 1$ or $z \geqslant 2$ and show that: if $z = 1$, $P(x+z)$ is unimodal provided that $m - n \leqslant 4$; if $z \geqslant 2$, then $P(x+z)$ is unimodal provided that $m - n \leqslant \lfloor 2z \rfloor + 1$; and we also show that the given conditions are best possible. Additionally, we explore the location of modes of $P(x+z)$, and show $P(x+z)$ has a mode $\lceil \frac{m-z}{z+1} \rceil$ or $\lceil \frac{m-z}{z+1} \rceil - 1$ or $\lceil \frac{m-z}{z+1} \rceil - 2$, which are reachable.

1 Introduction

A finite sequence of real numbers $\{a_0, a_1, \ldots, a_m\}$ is said to be *unimodal* if there exists an index k satisfying $0 \leqslant k \leqslant m$, called a *mode* of the sequence, such that a_i increases up to $i = k$ and decreases from then on; that is, $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_k$ and $a_k \geqslant a_{k+1} \geqslant \cdots \geqslant a_m$. It is said to be *logarithmically concave* (or *log-concave* for short) if $a_i^2 \geqslant a_{i-1}a_{i+1}$ for $i = 1, 2, \ldots, m-1$. It is said to have *no internal zeros* if whenever $a_i, a_k \neq 0$ and $0 \leqslant i < j < k \leqslant m$ then $a_j \neq 0$. A polynomial $P(x) = \sum_{i=0}^{m} a_i x^i$ is said to be *unimodal* (respectively, *log-concave*, with *no internal zeros*, *nondecreasing*) if the sequence $\{a_0, a_1, \ldots, a_m\}$ has the corresponding property. A mode of the sequence is also called a *mode* of the corresponding polynomial. In fact, a nonnegative log-concave sequence with no internal zeros is unimodal (see [9] for instance). Unimodal and log-concave polynomials arise often in combinatorics, geometry and algebra. The reader is referred to [4, 9] for surveys of the diverse techniques, problems, and results about unimodality and log-concavity.

It is well-known that if a polynomial $P(x)$ is log-concave with no internal zeros, then $P(x+1)$ is log-concave, which leads to the log-concavity of $P(x+z)$ for all positive integers z (see [4, Corollary 8.4] or [7, Theorem 2]). If $P(x)$ is nonnegative

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and nondecreasing, then $P(x+1)$ is unimodal in [3], which was implied by a result due to Chen, Yang, Zhou [6], and $P(x+n)$ is unimodal for any positive integer n [1]. Finally, Wang and Yeh [10] obtained a stronger result that $P(x+t)$ is unimodal for all real numbers $t > 0$. Llamas and Martínez-Bernal [8] proved that $P(x+t)$ is log-concave for all real numbers $t \geqslant 1$.

In this paper, we study an analogous problem: under what condition does a unimodal polynomial $P(x)$ guarantee the unimodality of $P(x+z)$ for positive real numbers z? It is obvious that if a polynomial $P(x)$ is unimodal, then $P(x+z)$ is not necessarily unimodal for positive real numbers z, even for positive integers z. For instance, $P(x) = 12 + x + x^2 + x^3 + x^4 + x^5$ is unimodal, but $P(x+1) = 17 + 15x + 20x^2 + 15x^3 + 6x^4 + x^5$ is not. Therefore, it is interesting to investigate the conditions mentioned above. We show that for a unimodal polynomial $P(x)$ of degree m with a mode n and real number z, if $z = 1$, $P(x+z)$ is unimodal provided that $m - n \leqslant 4$; if $z \geqslant 2$, then $P(x+z)$ is unimodal provided that $m - n \leqslant \lfloor 2z \rfloor + 1$, and we also give an example to prove that the given conditions are best possible. Additionally, we explore the location of modes of $P(x+z)$, and also show $P(x+z)$ has a mode $\lceil \frac{m-z}{z+1} \rceil$ or $\lceil \frac{m-z}{z+1} \rceil - 1$ or $\lceil \frac{m-z}{z+1} \rceil - 2$, all of which are reachable.

2 Mode of $P(x+z)$ for nondecreasing polynomial $P(x)$ and the real numbers $z = 1$ or $z \geqslant 2$

We first introduce a lemma.

Lemma 2.1. *[10] Let $P(x)$ be a polynomial of degree m with nonnegative coefficients. Suppose that $P(x)$ is nondecreasing and z is a positive real number. Then $P(x+z)$ is unimodal.*

The locations of modes in Lemma 2.1, however, are uncertain [10]. If we restrict z to $z = 1$ or $z \geqslant 2$, then there is a result about the locations of modes. Before proving the result, we give a lemma.

Lemma 2.2. *Let m be a nonnegative integer and z a positive real number. We let $\overline{m}(z) = \lceil \frac{m-z}{z+1} \rceil$ and $\underline{m}(z) = \lfloor \frac{m}{z+1} \rfloor$. Then*

$$\overline{m}(z) - 1 \ \leqslant\ \underline{m}(z) \ \leqslant\ \overline{m}(z).$$

In particular, if z is a positive integer, then $\overline{m}(z) = \underline{m}(z)$.

Proof. First of all, note that $0 < \frac{m}{z+1} - \frac{m-z}{z+1} = \frac{z}{z+1} < 1$. If the closed interval between $\frac{m-z}{z+1}$ and $\frac{m}{z+1}$ contains an integer, then $\underline{m}(z) = \overline{m}(z)$; otherwise $\overline{m}(z) - 1 = \underline{m}(z)$. So $\overline{m}(z) - 1 \leqslant \underline{m}(z) \leqslant \overline{m}(z)$.

Suppose now that z is a positive integer.

Claim: $(z+1)\underline{m}(z) - 1 < m \leqslant (z+1)\underline{m}(z) + z$.

The definition of $\underline{m}(z)$ yields the inequalities

$$\frac{m}{z+1} - 1 < \underline{m}(z) \leqslant \frac{m}{z+1}$$

and it follows directly that $m \geqslant (z+1)\underline{m}(z) > (z+1)\underline{m}(z)-1$ and $m < (z+1)\underline{m}(z)+z+1$. Since z is an integer, $m \leqslant (z+1)\underline{m}(z)+z$.

By the Claim, we have

$$(z+1)(\underline{m}(z)-1) < m-z \leqslant (z+1)\underline{m}(z),$$

and after dividing the inequalities above by $z+1$, we obtain

$$\underline{m}(z)-1 < \frac{m-z}{z+1} \leqslant \underline{m}(z).$$

Hence

$$\overline{m}(z) = \lceil \frac{m-z}{z+1} \rceil = \underline{m}(z).$$

□

Lemma 2.3. *Let $P(x) = \sum_{k=0}^{m} a_k x^k$ be a polynomial of degree m with nonnegative and nondecreasing coefficients, and let z be a real number $z = 1$ or $z \geqslant 2$. Then the polynomial $P(x+z)$ is unimodal with mode $\overline{m}(z)$ or $\underline{m}(z)$, defined as in Lemma 2.2. In particular, if z is a positive integer, then $P(x+z)$ is unimodal with mode $\underline{m}(z)$.*

Proof. We give a similar proof as the proof of Theorem 2.3 in [1]. The binomial theorem yields

$$P(x+z) = \sum_{k=0}^{m} a_k \sum_{i=0}^{k} \binom{k}{i} z^{k-i} x^i.$$

Now we exchange the two sums and thus obtain

$$P(x+z) = \sum_{i=0}^{m} \Big(\sum_{k=i}^{m} a_k \binom{k}{i} z^{k-i} \Big) x^i = \sum_{i=0}^{m} q_i x^i.$$

So it is sufficient to prove the sequence $\{q_i = \sum_{k=i}^{m} a_k \binom{k}{i} z^{k-i}\}_{i=0}^{m}$ is unimodal with mode $\overline{m}(z)$ or $\underline{m}(z)$ for real numbers $z = 1$ or $z \geqslant 2$; i.e., by Lemma 2.2, to show that (a) $q_j - q_{j+1} \geqslant 0$ when $\overline{m}(z) \leqslant j \leqslant m-1$, i.e., $q_{\overline{m}(z)} \geqslant q_{\overline{m}(z)+1} \geqslant \cdots \geqslant q_m$ and (b) $q_{j+1} - q_j \geqslant 0$ when $0 \leqslant j \leqslant \underline{m}(z)-1$. Wang and Yeh have shown (a) [10, Lemma 2.2]. It is sufficient to show (b). Note that

$$\begin{aligned} (j+1)(q_{j+1}-q_j) &= (j+1)\Big(\sum_{k=j+1}^{m} a_k \binom{k}{j+1} z^{k-j-1} - \sum_{k=j}^{m} a_k \binom{k}{j} z^{k-j} \Big) \\ &= \sum_{k=j}^{m} a_k \binom{k}{j} z^{k-j-1} [k-j-(j+1)z]. \end{aligned} \quad (1)$$

Assume now that $0 \leqslant j \leqslant \overline{m}(z)-1$. To show that $q_{j+1}-q_j \geqslant 0$, we divide the sum (1) into two parts: one part includes all negative terms, (denote the inverse of the part by T_1) and the other includes nonnegative terms, denoted by T_2. Then it is sufficient to prove that $T_1 \leqslant T_2$.

Now we analyze the sign of terms in the sum (1). Since a_k, z^{k-j-1} and the binomial coefficient $\binom{k}{j}$ are nonnegative, the term $a_k\binom{k}{j}z^{k-j-1}[k-j-(j+1)z] \geqslant 0$ if and only if $k-j-(j+1)z \geqslant 0$, i.e., $k \geqslant (j+1)z+j$. For the sake of simplicity, we let $c = \lceil (j+1)z+j \rceil$. Note that $c < m$. (Since $(z+1)\underline{m}(z) = (z+1)\lfloor \frac{m}{z+1} \rfloor \leqslant m$, $(z+1)(\underline{m}(z)-1)+z \leqslant m-1$. Combined with $j \leqslant \underline{m}(z)-1$, we have $(j+1)z+j = (z+1)j+z \leqslant m-1$. So $c = \lceil (j+1)z+j \rceil < m$.) Then

$$\begin{aligned} T_1 = & -\sum_{k=j}^{c-1} a_k \binom{k}{i} z^{k-j-1}[k-j-(j+1)z] \\ = & \sum_{k=j}^{c-1} a_k \binom{k}{j} z^{k-j-1}[(j+1)z+j-k] \end{aligned}$$

and

$$T_2 = \sum_{k=c}^{m} a_k \binom{k}{j} z^{k-j-1}[k-j-(j+1)z)]. \tag{2}$$

In what follows, we estimate the values of T_1.

Observe that

$$\begin{aligned} T_1 &= \sum_{k=j}^{c-1} a_k \binom{k}{j} z^{k-j-1}[((j+1)z+j-k] \\ &\leqslant a_{c+1} \sum_{k=j}^{c-1} \binom{k}{j} z^{k-j-1}[(j+1)z+j-k] \\ &\leqslant a_{c+1} z^{c-j-2} \sum_{k=j}^{c-1} \binom{k}{j} [(j+1)z+j-k] \\ &\leqslant a_{c+1} z^{c-j-2} \sum_{k=j}^{c-1} \binom{k}{j} (c-k) \\ &= a_{c+1} z^{c-j-2} \binom{c+1}{j+2}. \end{aligned} \tag{3}$$

The monotonicity of the coefficients of $P(x)$ was used in the first inequality and the definition of c was used in the last inequality. The last equality can be proved as follows: $\sum_{k=j}^{c-1} \binom{k}{j}(c-k)$ can be written as $\sum_{i=1}^{c-j}\sum_{k=j}^{c-i} \binom{k}{j}$. Using the formula (see [2, Theorem 4.5]) $\sum_{i=a}^{b} \binom{i}{a} = \binom{b+1}{a+1}$ twice, we obtain the result.

Claim:
$$\frac{1}{z^2}\binom{c+1}{j+2} \leqslant \binom{c+1}{j}[(c+1)-(j+(j+1)z)]. \tag{4}$$

We first show that the claim holds for $z \geqslant 2$.

$$\begin{aligned} \frac{1}{z^2}\frac{\binom{c+1}{j+2}}{\binom{c+1}{j}} &= \frac{1}{z^2}\frac{\frac{(c+1)!}{(j+2)!(c-j-1)!}}{\frac{(c+1)!}{j!(c-j+1)!}} \\ &= \frac{(c-j+1)(c-j)}{z(j+2)z(j+1)}. \end{aligned} \tag{5}$$

By the definition of c,

$$\begin{aligned} c-j+1 &= \lceil (j+1)z+j \rceil - j + 1 \\ &< (j+1)z + j + 1 - j + 1 \\ &= zj + z + 2 \quad \leqslant \quad z(j+2). \end{aligned} \tag{6}$$

The last inequality follows from the fact that $z \geqslant 2$. Substituting Eq. (6) into Eq. (5), we have

$$\begin{aligned} \frac{1}{z^2}\frac{\binom{c+1}{j+2}}{\binom{c+1}{j}} &< \frac{c-j}{z(j+1)} \\ &= \frac{c-j-z(j+1)}{z(j+1)} + 1 \\ &< c-j-z(j+1)+1 \\ &= (c+1)-(j+(j+1)z). \end{aligned}$$

Hence the claim holds for $z \geqslant 2$.

If $z = 1$, we easily obtain $c = 2j+1$. So

$$\begin{aligned} \binom{c+1}{j}[(c+1)-(j+(j+1)z)] &= \binom{2j+2}{j} \\ &= \binom{2j+2}{j+2} \\ &= \frac{1}{z^2}\binom{c+1}{j+2}. \end{aligned}$$

Hence the claim holds for $z = 1$.

Combining Eqs. (3) and (4), we have

$$\begin{aligned} T_1 &\leqslant a_{c+1} z^{c-j}\binom{c+1}{j}[(c+1)-(j+(j+1)z)] \\ &= a_{c+1} z^{(c+1)-j-1}\binom{c+1}{j}[(c+1)-(j+(j+1)z)]. \end{aligned} \tag{7}$$

The term in Eq. (7) is exactly the second term in the sum (2) by substituting k for $c+1$. So $T_1 \leqslant T_2$.

In particular, if z is a positive integer, by Lemma 2.2, $\overline{m}(z) = \underline{m}(z)$. So $P(x+z)$ has a mode $\underline{m}(z)$. $\square$

Two possible distinct values of modes in Lemma 2.3 are available as in the following examples.

Example 2.4. Let $P(x) = x+x^2+x^3$, $z = 2.5$. Since $m = 3$, $\overline{m}(z) = 1$ and $\underline{m}(z) = 0$, the mode of $P(x+2.5) = 24.375+24.75x+8.5x^2+x^3$ is $\overline{m}(z)$, If $P(x) = 1+x+x^2+x^3$, then the mode of $P(x+2.5) = 25.375 + 24.75x + 8.5x^2 + x^3$ is $\underline{m}(z)$.

Corollary 2.5. *[1] Let $P(x) = \sum_{i=0}^{m} a_i x^i$ be a polynomial of degree m with nonnegative and nondecreasing coefficients, and let z be a positive integer. Then the polynomial $P(x+z)$ is unimodal with mode $\lfloor \frac{m}{z+1} \rfloor$.*

In fact, a similar result as Lemma 2.3 was obtained by Wang and Yeh [10, Corollary 4.1]: $P(x + z)$ has at most two modes $\overline{m}(z)$ and $\overline{m}(z) + 1$ if $P(x) = ax^m$ for some positive real number a, or $\overline{m}(z) - 1$ and $\overline{m}(z)$ otherwise. But there is a small difference between them. For example, if $\overline{m}(z) = \underline{m}(z)$, it follows from Lemma 2.3 that $\overline{m}(z)$ must be a mode of $P(x + z)$.

3 Layer decomposition and main results

First, we give a new notion about unimodal polynomials: layer decomposition.

Definition Let $P_1(x) = \sum_{i=i_1}^{j_1} a_{1i} x^i$ be a unimodal polynomial with positive coefficients and mode n for nonnegative integers $i_1 \leqslant j_1$. Let $\alpha_1 = \min\{a_{1i_1}, a_{1j_1}\}$ and $P_2(x) = P_1(x) - \alpha_1 \sum_{i=i_1}^{j_1} x^i$. Obviously, $P_2(x)$ is still unimodal with nonnegative coefficients and mode n. If $P_2(x)$ is nonzero, suppose $P_2(x) = \sum_{i=i_2}^{j_2} a_{2i} x^i$ with $a_{2i} > 0$ for $i_2 \leqslant i \leqslant j_2$. Note that $i_1 \leqslant i_2 \leqslant n \leqslant j_2 \leqslant j_1$. Likewise, let $\alpha_2 = \min\{a_{2i_2}, a_{2j_2}\}$ and $P_3(x) = P_2(x) - \alpha_2 \sum_{i=i_2}^{j_2} x^i$. We can do such decomposition until we reach the zero polynomial. So $P_1(x)$ can be decomposed as the form $P_1(x) = \alpha_1 \sum_{i=i_1}^{j_1} x^i + \alpha_2 \sum_{i=i_2}^{j_2} x^i + \cdots + \alpha_k \sum_{i=i_k}^{j_k} x^i$ for some integer k with $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant n \leqslant j_k \leqslant \cdots \leqslant j_2 \leqslant j_1$. We call such a decomposition the *layer decomposition* of a unimodal polynomial $P_1(x)$ with positive coefficients.

It is obvious that every unimodal polynomial has a unique layer decomposition. An example follows.

Example 3.1. Let $P(x) = 2 + 5x + 7x^2 + 8x^3 + 8x^4 + 2x^5 + x^6 + x^7$. Then the layer decomposition of $P(x)$ is

$$P(x) = \sum_{i=0}^{7} x^i + \sum_{i=0}^{5} x^i + 3\sum_{i=1}^{4} x^i + 2\sum_{i=2}^{4} x^i + \sum_{i=3}^{4} x^i.$$

Theorem 3.2. *Let $P(x) = \sum_{i=0}^{m} a_i x^i$ be a unimodal polynomial of degree m with nonnegative coefficients and mode n. Suppose $P(x+z) = \sum_{i=0}^{m} b_i x^i$ for real numbers $z = 1$ or $z \geqslant 2$. Then $b_0 \leqslant b_1 \leqslant \cdots \leqslant b_{\lfloor \frac{n}{z+1} \rfloor}$ and $b_{\lceil \frac{m-z}{z+1} \rceil} \geqslant b_{\lceil \frac{m-z}{z+1} \rceil + 1} \geqslant \cdots \geqslant b_m$.*

Proof. Suppose the layer decomposition of $P(x)$ is

$$P(x) = \alpha_1 \sum_{i=i_1}^{j_1} x^i + \alpha_2 \sum_{i=i_2}^{j_2} x^i + \cdots + \alpha_k \sum_{i=i_k}^{j_k} x^i$$

for some k with $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant n \leqslant j_k \leqslant \cdots \leqslant j_2 \leqslant j_1 = m$. By Lemma 2.3, for any $1 \leqslant l \leqslant k$ and any real number z satisfying $z = 1$ or $z \geqslant 2$, a mode of $\alpha_l \sum_{i=i_l}^{j_l} (x+z)^i$ is $\overline{j_l}(z)$ or $\underline{j_l}(z)$, which are not less than $\underline{n}(z) = \lfloor \frac{n}{z+1} \rfloor$ and not greater than $\overline{m}(z) = \lceil \frac{m-z}{z+1} \rceil$. So $b_0 \leqslant b_1 \leqslant \cdots \leqslant b_{\lfloor \frac{n}{z+1} \rfloor}$ and $b_{\lceil \frac{m-z}{z+1} \rceil} \geqslant b_{\lceil \frac{m-z}{z+1} \rceil + 1} \geqslant \cdots \geqslant b_m$. □

From Theorem 3.2, we can obtain some corollaries as follows.

Corollary 3.3. *Let $P(x)$ be a unimodal polynomial of degree m with nonnegative coefficients and mode n, $z = 1$ or $z \geqslant 2$. If $z \geqslant m - n$, then $P(x+z)$ is unimodal with mode $\overline{m}(z)$ or $\underline{n}(z)$.*

Proof. By $z \geqslant m-n$, $n \geqslant m-z$, and further $\frac{n}{z+1} \geqslant \frac{m-z}{z+1}$. So $\lceil \frac{m-z}{z+1} \rceil - \lfloor \frac{n}{z+1} \rfloor \leqslant 1$. By Theorem 3.2, $P(x+z)$ is unimodal with mode $\overline{m}(z)$ or $\underline{n}(z)$. □

Corollary 3.4. *Let $P(x)$ be a unimodal polynomial of degree m with mode n and nonnegative coefficients. Then, for any positive integer $z \geqslant m-n-1$, $P(x+z)$ is unimodal with mode $\lfloor \frac{m}{z+1} \rfloor$ or $\lfloor \frac{n}{z+1} \rfloor$.*

Proof. By $z \geqslant m-n-1$, $\frac{m-n}{z+1} \leqslant 1$ and further $\lfloor \frac{m}{z+1} \rfloor - \lfloor \frac{n}{z+1} \rfloor \leqslant 1$. Combining with Lemma 2.2, we have $\lceil \frac{m-z}{z+1} \rceil - \lfloor \frac{n}{z+1} \rfloor = \lfloor \frac{m}{z+1} \rfloor - \lfloor \frac{n}{z+1} \rfloor \leqslant 1$. It follows from Theorem 3.2 that $P(x+z)$ is unimodal with mode $\lfloor \frac{m}{z+1} \rfloor$ or $\lfloor \frac{n}{z+1} \rfloor$. □

In fact, we can show a stronger result than Corollaries 3.3 and 3.4. First, we give a lemma.

Lemma 3.5. *[10] Suppose that the polynomial $P(x)$ is unimodal and positive real number z. Then $(x+z)P(x)$ is unimodal.*

In what follows we give the main result.

Theorem 3.6. *Let $P(x)$ be a unimodal polynomial of degree m with nonnegative coefficients and mode n. Then $P(x+z)$ is unimodal with a mode $\overline{m}(z)$ or $\overline{m}(z) - 1$ or $\overline{m}(z) - 2$ provided that either*
(1) $z \geqslant 2$ and $m - n \leqslant \lfloor 2z \rfloor + 1$; or
(2) $z = 1$ and $m - n \leqslant 4$.

Proof. Let $P(x)=\sum_{i=0}^{m} a_ix^i$ be a unimodal polynomial of degree m with nonnegative coefficients and mode n, and $B(x)=\sum_{i=0}^{m-1} a_{i+1}x^i$. Then $P(x)=a_0+xB(x)$. For notational simplicity, we let $P(x+z)=\sum_{i=0}^{m} d_ix^i$.

We first prove the unimodality of $P(x+z)$ under the conditions by induction on n.

First of all, we prove the result under condition (1). It is sufficient to prove that, for a nonnegative integer n and $z\geqslant 2$, a unimodal polynomial $P(x)$ with nonnegative coefficients and mode n, of degree m satisfying

$$m\leqslant n+\lfloor 2z\rfloor+1, \tag{8}$$

$P(x+z)$ is unimodal.

The initial step: If $n=0$, from the condition $m\leqslant n+\lfloor 2z\rfloor+1$, we get $m\leqslant\lfloor 2z\rfloor+1$ and therefore $\lceil\frac{m-z}{z+1}\rceil\leqslant\lceil\frac{2z+1-z}{z+1}\rceil=1$. It follows from Theorem 3.2 that $P(x+z)$ is unimodal.

The inductive step: Now we assume that the result holds for less than n and prove it for $n(\geqslant 1)$.

Case 1. $(1\leqslant)n\leqslant\lceil z\rceil$.

In this case $m\leqslant n+\lfloor 2z\rfloor+1\leqslant 3z+2$, so $\lceil\frac{m-z}{z+1}\rceil\leqslant 2$. By Theorem 3.2, we have

$$d_2\geqslant d_3\geqslant\cdots\geqslant d_m. \tag{9}$$

By the definition of $B(x)$, $B(x)$ is a unimodal polynomial of degree $m-1$ with mode $n-1$. By the condition $m\leqslant n+\lfloor 2z\rfloor+1$, we get $(m-1)\leqslant(n-1)+\lfloor 2z\rfloor+1$ satisfying Eq. (8) for $B(x)$. So $B(x+z)=\sum_{i=0}^{m-1} b_ix^i$ is unimodal by the inductive hypothesis. Since $m\leqslant 3z+2$, we have $\lceil\frac{m-1-z}{z+1}\rceil\leqslant 2$. Combining with Theorem 3.2, we get $b_2\geqslant b_3\geqslant\cdots\geqslant b_{m-1}$. Hence, either

$$b_1\geqslant b_0$$

or

$$b_0>b_1\geqslant b_2.$$

Since $P(x+z)=a_0+(x+z)B(x+z)$,

$$d_2=b_1+zb_2, \tag{10}$$
$$d_1=b_0+zb_1, \tag{11}$$
$$d_0=a_0+zb_0. \tag{12}$$

Subcase 2.1 $b_1\geqslant b_0$.

Since $n\geqslant 1$, $a_1\geqslant a_0$. Therefore $b_0=B(z)=\sum_{i=0}^{m-1} a_{i+1}z^i=\sum_{i=1}^{m-1} a_{i+1}z^i+a_1\geqslant a_1\geqslant a_0$. Combining with Eqs. (11) and (12), we have $d_1\geqslant d_0$. Hence we prove that $P(x+z)$ is unimodal by Eq. (9).

Subcase 2.2. $b_0 > b_1 \geqslant b_2$.

In this case, by Eqs. (10) and (11), $d_1 = b_0 + zb_1 > b_1 + zb_2 = d_2$. Combining with Eq. (9), we get $P(x+z)$ is unimodal regardless of relative magnitude of d_0 and d_1.

Case 2. $n \geqslant \lceil z \rceil + 1$.

In this case $\lfloor \frac{n}{z+1} \rfloor \geqslant \lfloor \frac{z+1}{z+1} \rfloor = 1$. By Theorem 3.2, $d_0 \leqslant d_1$. Similar to the proof in Case 1, we can show that $B(x)$ is a unimodal polynomial with mode $n-1$ of degree $m-1$ satisfying Eq. (8). By the inductive hypothesis, $B(x+z)$ is unimodal. Combining with Lemma 3.5, $(x+z)B(x+z) = \sum_{i=0}^{m} c_i x^i$ is unimodal. Note that $d_0 = a_0 + c_0, d_i = c_i$ for $1 \leqslant i \leqslant m$. It follows from $d_0 \leqslant d_1$ that $c_0 \leqslant c_1$. So there is some positive integer $1 \leqslant k \leqslant m$ such that $c_0 \leqslant c_1 \leqslant \cdots \leqslant c_k \geqslant c_{k+1} \geqslant \cdots \geqslant c_m$ by the unimodality of $(x+z)B(x+z)$. Therefore $d_1 \leqslant \cdots \leqslant d_k \geqslant d_{k+1} \geqslant \cdots \geqslant d_m$. Combining with $d_0 \leqslant d_1$, we get $d_0 \leqslant d_1 \leqslant \cdots \leqslant d_k \geqslant d_{k+1} \geqslant \cdots \geqslant d_m$. Hence $P(x+z)$ is unimodal.

Now, we prove $P(x+z)$ is unimodal under the condition (2): $z = 1$ and $m-n \leqslant 4$. Similarly, it is sufficient to prove that, for nonnegative integer n and a unimodal polynomial $P(x)$ with mode n of degree m satisfying $m \leqslant n+4$, $P(x+1)$ is unimodal. i.e., it is sufficient to prove $P(x+z)$ is unimodal provided that $m \leqslant n + 2z + 2$ and $z = 1$. In order to reduce the proof by repeating the proof above, we make this treatment.

The initial step. If $n = 0$, then $m \leqslant 4$. If $m \leqslant 3$, then $\lceil \frac{m-z}{z+1} \rceil \leqslant \lceil \frac{2}{2} \rceil = 1$. By Theorem 3.2, $P(x+z) = P(x+1)$ is unimodal with mode 0 or 1. If $m = 4$, then $P(x) = \sum_{i=0}^{4} a_i x^i = a_4 \sum_{i=0}^{4} x^i + C(x)$, where $C(x)$ is a unimodal polynomial of degree $\leqslant 3$ with mode $n = 0$. Similar to the proof above in the case $m \leqslant 3$, $C(x+z) = C(x+1)$ is unimodal with mode 0 or 1. Combining with $a_4 \sum_{i=0}^{4} (x+1)^i = a_4(5 + 10x + 10x^2 + 5x^3 + x^4)$, we get $P(x+1)$ is unimodal.

The induction step. We can give the parallel proof as the case under condition (1) by substituting $\lfloor 2z \rfloor + 1$ for $2z + 2 = 4$, and $n + \lfloor 2z \rfloor + 1$ in Eq. (8) for $n + \lfloor 2z \rfloor + 2 = n + 4$.

We now prove the locations of modes of $P(x+z)$. If the condition (1) is satisfied, then $\frac{m-z}{z+1} - \frac{n}{z+1} \leqslant \frac{\lfloor 2z \rfloor + 1 - z}{z+1} \leqslant 1$ and further $\lceil \frac{m-z}{z+1} \rceil - \lfloor \frac{n}{z+1} \rfloor \leqslant 2$ by simple analysis. Hence $P(x+z)$ has a mode $\overline{m}(z)$ or $\overline{m}(z) - 1$ or $\overline{m}(z) - 2$ by Theorem 3.2. Likewise, if the condition (2) is satisfied, then $\frac{m}{2} - \frac{n}{2} \leqslant 2$. Therefore $\lfloor \frac{m}{2} \rfloor - \lfloor \frac{n}{2} \rfloor \leqslant 2$. Note that $\overline{m}(1) = \lfloor \frac{m}{2} \rfloor$ and $\underline{n}(1) = \lfloor \frac{n}{2} \rfloor$ in this case. Hence $P(x+1)$ has a mode $\overline{m}(z)$ or $\overline{m}(z) - 1$ or $\overline{m}(z) - 2$ by Theorem 3.2. □

In fact, the condition given in Theorem 3.6 is sharp, i.e., if $z = 1$ and $m - n = 5$, or $z \geqslant 2$ and $m - n = \lfloor 2z \rfloor + 2$, we cannot guarantee that $P(x+z)$ is unimodal.

Example 3.7. Let $P(x) = 12 + x + x^2 + x^3 + x^4 + x^5$, which is unimodal with $m = 5, n = 0$. Then $P(x+1) = 17 + 15x + 20x^2 + 15x^3 + 6x^4 + x^5$ is not unimodal.

Lemma 3.8. *[10] Let $P(x) = \sum_{i=0}^{m} x^i$ for some nonnegative integer m and $z > 1$. If $z\overline{m}(z)$ is an integer, then $P(x+z)$ has the unique mode $\overline{m}(z)$.*

Example 3.9. Let $P(x) = (c+1)+x+x^2+\cdots+x^{\lfloor 2z \rfloor+2} = c+B(x)$ for nonnegative real numbers c and $z \geqslant 2$. Obviously, $P(x)$ is unimodal of degree $m = \lfloor 2z \rfloor + 2$ with the unique mode $n = 0$. Suppose $2z$ is an integer. Then $z\overline{m}(z) = z\lceil \frac{\lfloor 2z \rfloor+2-z}{z+1} \rceil = z\lceil \frac{z+2}{z+1} \rceil = 2z$. By Lemma 3.8, $B(x+z)$ has the unique mode $\overline{m}(z) = 2$. It follows that $P(x+z) = c + B(x+z)$ is not unimodal for a sufficient number c.

In addition, three possible modes of $P(x+z)$ in Theorem 3.6 are reached.

Example 3.10. Suppose $z \geqslant 2$ is an integer and d is a positive integer. Let $P(x) = a \sum_{i=0}^{(d+2)(z+1)} x^i + b \sum_{i=0}^{(d+1)(z+1)} x^i + c \sum_{i=0}^{d(z+1)+1} x^i$ for $a, b, c > 0$. It is obvious that $P(x)$ is unimodal of degree $m = (d+2)(z+1)$ with a mode $n = d(z+1)+1$. Then $\overline{m}(z) = \lceil \frac{m-z}{z+1} \rceil = \lceil \frac{(d+2)(z+1)-z}{z+1} \rceil = d+2$ and $m-n = 2z+1 = \lfloor 2z \rfloor + 1$. It follows from Theorem 3.6 that $P(x+z)$ is unimodal. By Lemma 3.8, $\sum_{i=0}^{(d+2)(z+1)} (x+z)^i$, $\sum_{i=0}^{(d+1)(z+1)} (x+z)^i$, $\sum_{i=0}^{d(z+1)+1} (x+z)^i$ have the unique modes $d+2, d+1, d$, respectively. Note that $\overline{(d+1)(z+1)}(z) = d+1, \overline{d(z+1)+1}(z) = d$. Hence $P(x+z)$ has a unique mode $\overline{m}(z) = d+2$ for fixed b, c and sufficient large a. Similarly, $P(x+z)$ has a unique mode $\overline{m}(z) - 1 = d+1$ for fixed a, c and sufficient large b, $P(x+z)$ has a unique mode $\overline{m}(z) - 2 = d$ for fixed a, b and sufficient large c.

In addition, from Theorem 3.6, we can directly obtain the following corollary.

Corollary 3.11. *Let $P(x)$ be a unimodal polynomial of degree m with nonnegative coefficients and mode n. If $m - n \leqslant 4$, then for any positive integer z, $P(x+z)$ is unimodal with a mode $\overline{m}(z)$ or $\overline{m}(z) - 1$ or $\overline{m}(z) - 2$.*

4 Conclusions

If $P(x)$ is a polynomial with nonnegative and nondecreasing coefficients, then for any positive real number z, $P(x+z)$ is unimodal. Does this fact generalize to a unimodal polynomial $P(x)$ with nonnegative coefficients? Unfortunately, the result does not hold. In this paper we investigate under what conditions $P(x+z)$ is unimodal. If the real number $z = 1$ or $z \geqslant 2$, then we give respective sharp conditions for completely answering this problem (i.e. Theoerem 3.6), and we also locate a mode of $P(x+z)$. Hence there is an open question which is worthy of further exploration: is there a corresponding result similar to Theorem 3.6 for real numbers $0 < z < 1$ and $1 < z < 2$?

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