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On the structure of graphs having a unique k-factor

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Abstract

In this paper, we prove that there is no r-regular graph ($r \geq 2$) with a unique perfect matching. Also we show that a $2r$-regular graph of order n has at least $\left(\frac{(r-k)^{r-k}}{(r-k+1)^{r-k-1}}\right)^n$ $2k$-factors, where $k \leq r$. We investigate graphs with a unique $[a,b]$-factor and among other results, we prove that a connected graph with minimum degree at least 2 and a unique $[1,2]$-factor with regular components is an odd cycle.

1 Introduction

Throughout this paper all graphs are finite and simple. For a graph G, $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, and their cardinalities are called the *order* and the *size* of G, respectively. For a vertex v of G, $N_G(v)$ denotes the set of neighbors of v and $d_G(v)$ is the degree of v. By $\delta(G)$ and $\Delta(G)$ we denote the minimum and the maximum degree of G, respectively. For a subset A of $V(G)$, $E(A)$ denotes the set of edges whose end vertices are in A. The edge e is called a *cut edge* if $c(G-e) > c(G)$, where $c(G)$ is the number of components of G. A connected graph G is called 2-*edge-connected* if it has no cut edge.

A *factor* of a graph G, is a spanning subgraph of G. If f is a function assigning a non-negative integer to each vertex of G, then a factor F of G is called an f-*factor*, if $d_F(v) = f(v)$ for all $v \in V(G)$. Also, a *parity* f-*factor* of G is a factor F of G such that for every $v \in V(G)$, $d_F(v)$ and $f(v)$ have the same parity. An *even-factor* (resp *odd-factor*) is a factor such that all of its degrees are even (respectively odd).

An $[a,b]$-*factor* of G is a factor of G such that for each $v \in V(G)$, $a \leq d_F(v) \leq b$. A $[k,k]$-factor is simply called a k-*factor*. The following well-known result due to Petersen, guarantees the existence of an edge decomposition of a regular graph of even degree into 2-factors (see [2, Theorem 3.1]).

Theorem 1.1. *Let $r \geq 2$ be an even integer and G be an r-regular graph. Then the edges of G can be partitioned into 2-factors of G.*

A *perfect matching* is a 1-factor. A *perfect* $[a,b]$-*factor* is an $[a,b]$-factor whose components are regular.

If a bipartite graph has a perfect matching, then there are some lower bounds for the number of its perfect matchings which are stated in the following theorem.

Theorem 1.2. *Let G be a bipartite graph. Then the following hold:*

(i) [18, Theorem 1.6.1] *If G has a perfect matching, then G has at least $\delta(G)!$ perfect matchings.*

(ii) [15] *If G is r-regular of order $2n$, then G has at least $\left(\frac{(r-1)^{r-1}}{r^{r-2}}\right)^n$ perfect matchings.*

The next result due to Kötzig which was proved in [4, 11, 13, 14], shows that there is no 2-edge-connected graph with a unique perfect matching.

Theorem 1.3. *A connected graph with a unique perfect matching has a cut edge belonging to its unique perfect matching.*

Jackson and Whitty proved the following result which is a generalization of Theorem 1.3.

Theorem 1.4. [8] *Let G be a 2-edge-connected graph and $f : V(G) \to \mathbb{Z}^+$ be a function. If G has a unique f-factor, then there exists a vertex v such that $d_G(v) = f(v)$.*

The following corollary which was proved in [16] may be useful when the 2-edge connectivity condition is removed.

Corollary 1.5. *Let G be a graph with a unique f-factor F, where $f : V(G) \to \mathbb{Z}^+$ is a function such that $f(x) \geq 2$ for all $x \in V(G)$. Then some vertex x of G satisfies $d_G(x) = f(x)$ or there exist at least two vertices u_1 and u_2 such that $d_G(u_i) = f(u_i)+1$ for $i = 1, 2$.*

Theorems 1.1 and 1.2 are used in Section 2 to find lower bounds for the number of k-factors for an even k. Also using Theorem 1.3, we provide an upper bound for the number of edges of a graph with a unique perfect matching. Indeed, we prove the following.

Theorem 1.6. *Let G be a graph of order n and size m. If G has a unique perfect matching, then $m \leq (\frac{n-2}{2})\Delta(G)+1$. In particular, for $r>1$, there is no r-regular graph with a unique perfect matching.*

In Section 3, we investigate graphs with a unique $[a,b]$-factor and graphs with a unique perfect $[a,b]$-factor. In particular, we characterize all graphs with a unique perfect $[1,2]$-factor and prove the following.

Theorem 1.7. *A connected graph with minimum degree at least 2 and a unique perfect $[1,2]$-factor is an odd cycle.*

Section 4 deals with graphs which have a unique parity f-factor. We prove the following theorem about the graphs which have a unique even-factor with no isolated vertices.

Theorem 1.8. *If G has a unique even-factor with no isolated vertices, then $\delta(G)=2$.*

2 Unique k-Factors

There are some upper bounds for the size of a graph with a unique k-factor. Hetyei in 1964 proved that a graph of order n with a unique perfect matching cannot have more than $\frac{n^2}{4}$ edges (see [12, Corollary 1.6]). Also in [17], these kind of graphs are investigated. In [6, 9, 16], the maximum size of a graph with a unique k-factor, for some values of k was determined. In the following theorem, we present an upper bound for the size of a graph with a unique perfect matching depending on the maximum degree of the graph. Notice that when the maximum degree is at most $\frac{n}{2}+1$, our bound is better than the bound given by Hetyei.

Theorem 2.1. *Let G be a graph of order n and size m. If G has a unique perfect matching, then $m \leq (\frac{n-2}{2})\Delta(G)+1$. In particular, for $r>1$, there is no r-regular graph with a unique perfect matching.*

Proof. Clearly, it suffices to prove the theorem for connected graphs. The proof is by induction on n. Obviously, the result holds for $n=1,2$. Now, assume that $n>2$. By Theorem 1.3, there is a cut edge $e=xy$ which is contained in the unique perfect matching of G. Let H_1 and H_2 be the two subgraphs of $G-e$ such that $x \in V(H_1)$ and $y \in V(H_2)$. Now, consider the graphs $G_1=H_1-x$ and $G_2=H_2-y$. Since each G_i has a unique perfect matching, by the induction hypothesis on each component of G_i, we find that $|E(G_i)| \leq (\frac{|V(G_i)|-2}{2})\Delta(G_i)+1$, for $i=1,2$. Thus,

$$\begin{aligned} m &\leq 2\Delta(G)-1+\sum_{i=1}^{2}|E(G_i)| \\ &\leq 2\Delta(G)-1+\sum_{i=1}^{2}\left((\frac{|V(G_i)|-2}{2})\Delta(G)+1\right) \\ &= (\frac{n-2}{2})\Delta(G)+1, \end{aligned}$$

which completes the proof. □

Now, considering the following problem is notable.

Problem 2.2. *Improve the upper bound for the size of a graph with a unique perfect matching given in Theorem 2.1.*

Next, we consider graphs with a unique k-factor for $k \geq 2$. In the following corollary, a necessary condition for a graph to have a unique k-factor is stated.

Corollary 2.3. *Let G be a graph and k be a positive integer.*

(i) If $k \geq 2$ and $\delta(G) \geq k+2$, then G cannot have a unique k-factor.

(ii) If G is r-regular and $r \geq k+1$, then G cannot have a unique k-factor.

Proof. The first part is an immediate consequence of the Corollary 1.5. We prove the second part. If $r \geq k+2$, then by the first part, G cannot have a unique k-factor. Hence assume that $r = k+1$. On the contrary, assume that F is the unique k-factor of G. Thus $G \setminus F$ is the unique perfect matching of G. But by Theorem 2.1, G cannot have a unique perfect matching, a contradiction. □

Remark 2.4. *The graph in Figure* 1(*a*) *shows that Part (i) of Corollary 2.3 does not hold for $k = 1$. Also notice that the lower bound in Part (i) of Corollary 2.3 cannot be reduced to $k+1$. For example, the graph in Figure* 1(*b*) *has minimum degree* 3 *and a unique* 2*-factor.*

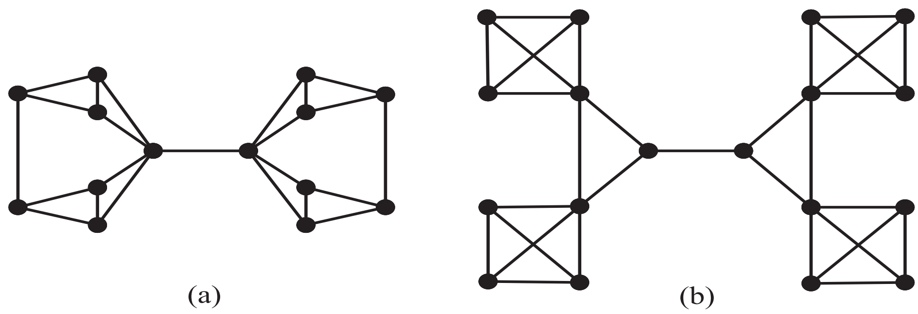

Figure 1: (a) A graph with minimum degree 3 and a unique 1-factor. (b) A graph with minimum degree 3 and a unique 2-factor.

Next we propose the following problem.

Problem 2.5. *Is there a graph G with $\delta(G) = k+1$ and a unique k-factor, for $k \geq 3$?*

In [1], some upper bounds for the number of 2-factors in a family of directed complete bipartite graphs are obtained. For a given positive even integer k, we would like to find a lower bound for the number of k-factors of a graph which has at least one k-factor. Our main tool is Theorem 1.2 and a construction which relates 2-factors of a graph to the perfect matchings of a bipartite graph.

Theorem 2.6. *Let G be a graph and k be a positive integer.*

(i) If G has a $2k$-factor, then the number of $2k$-factors of G is at least $\lfloor \frac{\delta(G)-2k+2}{2} \rfloor !$.

(ii) If G is $2r$-regular of order n and $k \leq r$, then G has at least

$$\left(\frac{(r-k)^{r-k}}{(r-k+1)^{r-k-1}} \right)^n$$

$2k$-factors.

Proof. (i) First assume that $k = 1$ and F is a 2-factor of G. Construct a directed graph D with the underlying graph G as follows. Choose the orientation of the edges of F in such a way that F becomes a union of directed cycles. Next, choose an orientation for the edges of $G \setminus F$, such that the out-degree and the in-degree of each vertex differ by at most one (such orientation always exists, see for example [3, Theorem 11.5.4]). Let $V(D) = \{v_1, \ldots, v_n\}$ and H be a bipartite graph with bipartition $X = \{x_1, \ldots, x_n\}, Y = \{y_1, \ldots, y_n\}$ in which x_i is adjacent to y_j if and only if there is an arc in D from v_i to v_j. Note that the directed 2-factors of D are in one-to-one corresponding to the perfect matchings of H. Since $\delta(H) \geq \lfloor \frac{\delta(G)}{2} \rfloor$, Part(i) of Theorem 1.2 implies that H has at least $\lfloor \frac{\delta(G)}{2} \rfloor !$ perfect matchings. So the number of 2-factors of G is at least $\lfloor \frac{\delta(G)}{2} \rfloor !$, as desired. If $k \geq 2$, then let F be a $2k$-factor of G and let $F_1, \ldots, F_k$ be 2-factors of G which partition F (by Theorem 1.1 such 2-factors exist). Since $G \setminus (F_2 \cup \cdots \cup F_k)$ has a 2-factor, it has at least $\lfloor \frac{\delta(G)-2k+2}{2} \rfloor !$ 2-factors, each of which gives a $2k$-factor of G.

(ii) Let $F_1, \ldots, F_r$ be 2-factors of G which partition $E(G)$. Let $G' = G \setminus (F_1 \cup \cdots \cup F_{k-1})$. Since G' is $(2r-2k+2)$-regular, it has an orientation D' such that the in-degree and the out-degree of each vertex is $r-k+1$. Let $V(D') = \{v_1, \ldots, v_n\}$ and H' be a bipartite graph with bipartition $X = \{x_1, \ldots, x_n\}, Y = \{y_1, \ldots, y_n\}$ in which x_i is adjacent to y_j if and only if there is an arc in D' from v_i to v_j. Clearly, H' is $(r-k+1)$-regular, thus Part (ii) of Theorem 1.2 implies that H' has at least $\left(\frac{(r-k)^{r-k}}{(r-k+1)^{r-k-1}} \right)^n$ perfect matchings, each of which yields a 2-factor of G'. The union of every such 2-factor with $F_1 \cup \cdots \cup F_{k-1}$ gives a $2k$-factor of G. □

3 Unique $[a, b]$-Factors

In this section, we obtain some results on graphs with a unique $[a, b]$-factor or a unique perfect $[a, b]$-factor. First, we obtain a result similar to Theorem 1.4 for graphs which have a unique $[a, b]$-factor.

Theorem 3.1. *Let G be a graph and $a < b$ be two positive integers. If G has a unique $[a, b]$-factor containing a vertex of degree a, then G has a vertex of degree a.*

Proof. Let F be the unique $[a,b]$-factor of G. Let $A = \{v \in V(G) | d_F(v) = a\}$ and $B = V(G) \setminus A$. By uniqueness of the $[a,b]$-factor, $E(A) \cap (E(G) \setminus F) = E(B) \cap F = \varnothing$. Now, assume on the contrary that $\delta(G) > a$. Let $v_1 \in A$. Since $d_F(v_1) = a < d_G(v_1)$, there is an edge $v_1v_2 \in E(G) \setminus F$ such that $v_2 \in B$. Also, since $d_F(v_2) > 0$, there is an edge $v_2v_3 \in F$ such that $v_3 \in A$. By continuing this procedure, we obtain a walk whose edges are alternately in $E(G) \setminus F$ and F. Since the order of G is finite, after some steps we find a cycle C whose edges are alternately in $E(G) \setminus F$ and F. Now, $(F \setminus E(C)) \cup (E(C) \setminus F)$ is another $[a,b]$-factor of G, a contradiction. The proof is complete. □

In [7], it is proved that if a graph G has a unique perfect $[1,k]$-factor F, then F is a perfect $[1,2]$-factor of G. In the next corollary we generalize this result. Before giving the proof, we state the following theorem on the existence of perfect $[k-1,k]$-factors of regular graphs.

Theorem 3.2. [2, Theorem 4.37] *Let $r \geq 3$ be an odd integer and k an integer such that $1 \leq k \leq \frac{2r}{3}$. Then every r-regular graph has a perfect $[k-1,k]$-factor.*

Corollary 3.3. *Let G be a graph, $a \leq b$ be two positive integers and F be the unique perfect $[a,b]$-factor of G. Then F is a perfect $[a, \lceil \frac{3a+1}{2} \rceil]$-factor of G, all of its even degrees are at most $a+1$.*

Proof. Assume on the contrary that F has an r-regular component H, where $r \geq \frac{3(a+1)}{2}$. If r is even, then H has an $(r-2)$-factor which gives another perfect $[a,b]$-factor of G, a contradiction. If r is odd, then by Theorem 3.2, H has a perfect $[a,a+1]$-factor which contradicts the uniqueness of F. Also, if F has an r-regular component H, where r is an even integer more than $a+1$, then by removing a 2-factor from H, we construct another perfect $[a,b]$-factor of G which is impossible and the proof is complete. □

As a consequence of the previous corollary, one can see that if a graph has a unique perfect $[1,k]$-factor (respectively $[2,k]$-factor) F, then F is a perfect $[1,2]$-factor (respectively $[2,3]$-factor).

In the sequel, we determine all graphs with a unique perfect $[1,2]$-factor. We require the next simple lemma and we omit its proof.

Lemma 3.4. *Each of the following graphs has at least two perfect $[1,2]$-factors:*
(i) Two odd cycles connected by a path.
(ii) Two odd cycles whose intersection is a path.
(iii) An odd cycle with a chord.

A subgraph H of a graph G is called a *forbidden subgraph*, if H has at least two perfect $[1,2]$-factors and $G \setminus V(H)$ has a perfect $[1,2]$-factor. Note that a graph with a unique perfect $[1,2]$-factor cannot have a forbidden subgraph.

Theorem 3.5. *A connected graph with minimum degree at least 2 and a unique perfect $[1,2]$-factor is an odd cycle.*

Proof. Let G be a graph with $\delta(G) \geq 2$ and let F be the unique perfect $[1, 2]$-factor of G. Clearly, the cycle components of F are odd cycles. We claim that each component of F is a cycle. Assume on the contrary that F has a 1-regular component and let $M = \{x_1y_1, \ldots, x_ky_k\}$ be the edges of all 1-regular components of F. Let P be an M-alternating path of maximum length which is started and terminated with the edges of M. Without the loss of generality, assume that $P = x_1y_1x_2y_2 \cdots x_ry_r$. Since $d_G(x_1), d_G(y_r) \geq 2$, x_1 and y_r should be adjacent to a vertex other than y_1 and x_r, respectively. Note that $(N_G(x_1) \cup N_G(y_r)) \cap \{x_{r+1}, \ldots, x_k, y_{r+1}, \ldots, y_k\} = \varnothing$. Now, consider the following cases. In each case, we construct a forbidden subgraph of G, a contradiction.

Case 1. For some $i \in \{2, \ldots, r\}$, $x_1y_i \in E(G)$, or for some $j \in \{1, \ldots, r-1\}$ $y_rx_j \in E(G)$. If $x_1y_i \in E(G)$, then $x_1y_1 \cdots x_iy_ix_1$ is an even cycle which is a forbidden subgraph of G. If $y_rx_j \in E(G)$, then the proof is similar.

Case 2. For some $i, j \in \{1, \ldots, r\}$, $x_1x_i, y_ry_j \in E(G)$. Let H be the subgraph of G consisting two odd cycles $x_1y_1 \cdots x_{i-1}y_{i-1}x_ix_1$, $y_rx_r \cdots y_{j+1}x_{j+1}y_jy_r$. If $i > j$, then the intersection of these two cycles is the path $\{y_jx_{j+1}y_{j+1} \cdots x_{i-1}y_{i-1}x_i\}$. If $i \leq j$, then add the path $x_iy_i \cdots x_jy_j$ to H. Now, Lemma 3.4 implies that H is a forbidden subgraph of G.

In the sequel, let x_1 be adjacent to a vertex z in an odd cycle C of F.

Case 3. Suppose that $y_ry_j \in E(G)$ for some $j \in \{1, \ldots, r-1\}$. Let H be the subgraph G consisting two odd cycles C and $y_jx_{j+1}y_{j+1} \cdots x_ry_ry_j$ which are connected by the path $zx_1y_1 \cdots x_jy_j$. By Part (i) of Lemma 3.4, H is a forbidden subgraph of G.

Case 4. Assume that y_r is adjacent to a vertex w in an odd cycle C' of F different from C. Let H be the subgraph of G consisting C, C' and the path $zx_1y_1 \ldots x_ry_rw$. By Part (i) of Lemma 3.4, H is a forbidden subgraph of G.

Case 5. Assume that y_r is adjacent to a vertex w in C. If $w = z$, let H be the subgraph G consists of two odd cycles C and $zx_1y_1 \ldots x_ry_rz$ which have a vertex z in common. By Part (ii) of Lemma 3.4, H is a forbidden subgraph of G. Now, let $w \neq z$. Let Q be a wz-path in C of even length. Assume that H is the subgraph G consisting two odd cycles C and $zx_1y_1 \ldots x_ry_rwQz$ whose intersection is Q. By Part (ii) of Lemma 3.4, H is a forbidden subgraph of G.

Thus every component of F is an odd cycle. Note that by Part(i) of Lemma 3.4, there is no edge between two cycles of F. Since G is connected, it has a Hamilton cycle of odd length. Now, Part (iii) of Lemma 3.4 implies that this Hamilton cycle has no chord and so G is an odd cycle. □

Remark 3.6. *Let G be a graph with a unique perfect $[1, 2]$-factor. If G has a vertex of degree one, say u, whose neighbor is v, then $G \setminus \{u, v\}$ is a graph with a unique* *perfect* $[1,2]$*-factor. By repeating this procedure, we reach the empty graph or a graph which is a disjoint union of finitely-many odd cycles.*

The next theorem provides an upper bound for the size of a graph with a unique perfect $[1,2]$-factor.

Theorem 3.7. *Let G be a graph of order n with a unique perfect $[1,2]$-factor F. Then $|E(G)| \leq n(k+1) - k(k+2)$, where k is the number of 1-regular components of F.*

Proof. Let $H_1, \dots, H_k$ be the 1-regular components and $C_1, \dots, C_t$ be the cycle components of F. Since G has a unique perfect $[1,2]$-factor, it has no forbidden subgraph. So by Parts (ii) and (iii) of Lemma 3.4, all of the cycle components of F are induced odd cycles of G and there is no edge between these components. Let $u \in V(C_i)$. If u is adjacent to two vertices of H_j, say x, y, then, using Part (ii) of Lemma 3.4, $C_i \cup uxy$ is a forbidden subgraph of G, which is impossible. Thus each vertex of a cycle in F is adjacent to at most one vertex of H_i, for $i = 1, \dots, k$. Also, note that the induced subgraph on $V(H_1) \cup \cdots \cup V(H_k)$ has a unique perfect matching. So by the Corollary 1.6 in [12], it has at most $\frac{(2k)^2}{4} = k^2$ edges. Therefore we find the following,

$$|E(G)| \leq k^2 + (n-2k) + k(n-2k) = n(k+1) - k(k+2)$$

as desired. □

Note that for an odd cycle, the equality holds in the previous theorem. The following corollary was first proved in [7]. It is not hard to see that it is a consequence of Theorem 3.7.

Corollary 3.8. *Let G be a graph of order n and size m with a unique perfect $[1,2]$-factor. Then,* $m \leq \begin{cases} \frac{n^2}{4}; & \text{if } n \text{ is even} \\ \frac{n^2}{4} + \frac{3}{4}; & \text{if } n \text{ is odd} \end{cases}$.

4 Unique Parity Factors

In this section, we investigate parity f-factors of graphs. Note that if M is the incidence matrix of G and the rows and columns of M are indexed by $V(G)$ and $E(G)$, respectively, then parity f-factors of G are in one-to-one correspondence to the solutions of the equation $Mx = \bar{f}$ in $\mathbb{Z}_2$, where $\bar{f}$ is the vector corresponding to f.

Theorem 4.1. *Let G be a graph of order n and size m which has c components and let $f : V(G) \to \mathbb{Z}$ be a function. If G has a parity f-factor, then G has 2^{m-n+c} parity f-factors. In particular, G has a unique parity f-factor if and only if G is a forest and it has at least one parity f-factor.*

Proof. Since the rank of M over $\mathbb{Z}_2$ is $n-c$ (see [5, Proposition 14.15.1]), the null space of M has 2^{m-n+c} vectors. So the number of solutions of the non-homogeneous equation $Mx = \bar{f}$ in $\mathbb{Z}_2$ is zero or 2^{m-n+c}. □

As a consequence of Theorem 4.1 and the fact that a connected graph G has an odd-factor if and only if its order is even (see [10, Lemma 16.4]), we have the following corollary.

Corollary 4.2. *The only graphs with a unique odd-factor are forests whose components have even number of vertices.*

Lovász proved that every 2-edge-connected graph with minimum degree at least 3 has one even-factor with no isolated vertices (see [18, Theorem 2.4.7]). We close this paper with the following result, in which we prove that there is no graph with minimum degree at least 3 which has a unique even-factor with no isolated vertices.

Theorem 4.3. *If G has a unique even-factor with no isolated vertices, then $\delta(G) = 2$.*

Proof. By induction on $\sum_{v\in V(G)} |d_G(v) - 3|$, we prove that if $\delta(G) \geq 3$ and G has an even-factor with no isolated vertices, then G has at least two such factors. If G is cubic, then even-factors of G with no isolated vertices are 2-factors of G and Part(ii) of Corollary 2.3 yields the result. Now, let F be an even-factor of G with no isolated vertices and $v \in V(G)$ has degree at least 4. First, suppose that $d_F(v) \geq 4$. Split v into two vertices v_1 and v_2. Join v_1 to two vertices $u_1, u_2 \in N_F(v)$ and join v_2 to every vertex in $N_G(v)\setminus\{u_1, u_2\}$. Also add an edge $e = v_1v_2$, and denote the resulting graph by G'. Clearly, $\delta(G') \geq 3$. Let F' be a factor of G' that contains the edges of F which are not incident with v, together with v_1u_1, v_1u_2 and $\{v_2x : vx \in F \setminus \{vu_1, vu_2\}\}$. Obviously, F' is an even-factor for G' with no isolated vertices. Thus by the induction hypothesis it has at least two such factors. By contracting e in G', these two factors give two distinct even-factors of G with no isolated vertices. Next, suppose that $d_F(v) = 2$ and let $N_F(v) = \{u_1, u_2\}$. Split v into two vertices v_1 and v_2. Let $w \in N_G(v)\setminus\{u_1, u_2\}$. Join v_1 to u_1 and w, and join v_2 to all vertices in $N_G(v)\setminus\{u_1, w\}$. Also add an edge $e = v_1v_2$, and denote the resulting graph by G'. Note that F yields a factor F' of G', all of its degrees are even integers except $d_{F'}(v_1)$ and $d_{F'}(v_2)$ which are 1. Now, $F' + e$ is an even factor of G' with no isolated vertices. Thus by the induction hypothesis, G' has at least two such factors. By contracting e in G', these two factors yield two distinct even-factors of G with no isolated vertices. □

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