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Revisiting the intersection problem for minimum coverings of complete graphs with triples

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Abstract

The intersection problem for minimum covers of K_n with triples has previously been settled in the case where the paddings are required to be equal. In this paper we extend this result by considering two generalizations: the paddings need not be equal, and the paddings are required to be simple.

1 Introduction

A *Steiner triple system* of order n, STS(n), is a pair $(S, \mathcal{T})$, where $\mathcal{T}$ is a set of edge-disjoint *triangles* (or *triples*) which partitions the edge set of K_n (the complete undirected graph on n vertices) with vertex set S. It is well known that the spectrum for Steiner triple systems is precisely the set of all $n \equiv 1$ or $3 \pmod 6$ [3], and that if $(S, \mathcal{T})$ is a triple system of order n then $|\mathcal{T}| = n(n-1)/6$; define $\tau = n(n-1)/6$. Denote by $I(n) = \{0, 1, 2, \ldots, \tau\} \setminus \{\tau - 1, \tau - 2, \tau - 3, \tau - 5\}$ and by $J(n) = \{k \mid$ there exists a pair of triple systems of order n having exactly k triples in common$\}$.

The following theorem gives a complete solution of the intersection problem for triple systems.

Theorem 1.1 (C. C. Lindner, A. Rosa [7]) *Let $n \equiv 1$ or $3 \pmod 6$. Then $J(n) = I(n)$ if $n \neq 9$ and $J(9) = I(9) \setminus \{5, 8\}$.*

When $n \not\equiv 1$ or $3 \pmod 6$ there does not exist a triple system; so the intersection problem for maximum packings and minimum coverings suggests itself. Much of the intersection problem for maximum packings has been solved in [2, 8]. The object of this paper is to generalize the intersection problem for *minimum coverings* of K_n with triples. We need to be a bit more precise.

A *covering* of K_n with triples is a triple $(S, \mathcal{T}, \mathcal{P})$, where S is the vertex set of K_n, $\mathcal{P}$ is a multiset edges on the vertex set S (called the *padding*) and $\mathcal{T}$ is an edge-disjoint multiset of triples which partitions the edge set $K_n \cup \mathcal{P}$. If $|\mathcal{P}|$ is as small as possible we say that $(S, \mathcal{T}, \mathcal{P})$ is a *minimum covering* of K_n with triples (MCT(n)). If $|\mathcal{P}|$ is as small as possible with the added requirement that $\mathcal{P}$ contains no repeated edges then we say that $(S, \mathcal{T}, \mathcal{P})$ is a *simple minimum covering* of K_n with triples (SMCT(n)). So a Steiner triple system is a minimum covering of K_n with padding $\mathcal{P} = \emptyset$. It turns out that if $n \equiv 5 \pmod 6$ then $\mathcal{P}$ is a doubled edge in any MCT(n), and is a 5-cycle in any SMCT(n) (both paddings are listed in Table 1).

The following easy to read table gives paddings for the minimum covering of order n for all n. When $n \equiv 5 \pmod 6$, both the MCT(n) and the SMCT(n) are provided.

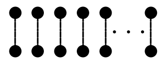
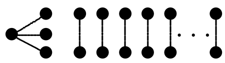
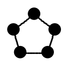
K_n	padding
$n \equiv 0 \pmod 6$	 one-factor
$n \equiv 2$ or $4 \pmod 6$	 tripole
$n \equiv 5 \pmod 6$	double edge  5-cycle

Table 1.

Most of the intersection problem for minimum coverings of K_n with triples is settled in the following result. For the rest of the paper, let $I(n) = \{1, 2, \ldots, \tau(n)\}$, where $\tau(n)$ is the number of triples in an MCT(n) or SMCT(n), whichever is being considered. (In what follows, when the value of n is clear, then we simply write τ instead of $\tau(n)$.)

Theorem 1.2 (C. C. Lindner, C. A. Rodger [5]) *For each $i \in I(n)$, there exist two* MCT(n)*s, $(S, \mathcal{T}_1, \mathcal{P}_1)$ and $(S, \mathcal{T}_2, \mathcal{P}_2)$, with $|\mathcal{T}_1 \cap \mathcal{T}_2| = i$ and $\mathcal{P}_1 = \mathcal{P}_2$ if and only if*

1. *$i \notin \{\tau - 1, \tau - 2, \tau - 3, \tau - 5\}$, and*

2. *if $n = 5$ then $i \neq 0$.*

It will be convenient to define $J(n) = \{k \mid$ there exists a pair of MCT(n)s intersecting in exactly k triples$\}$ and $JS(n) = \{k \mid$ there exists a pair of SMCT(n)s intersecting in exactly k triples$\}$. In this paper, we extend Theorem 1.2 in two ways.

In Section 2, the restriction in Theorem 1.2 that $\mathcal{P}_1 = \mathcal{P}_2$ is removed. Then in Section 3, Theorem 1.2 is extended to consider the intersection problem for SMCT(n)s.

2 MCT(n)s with paddings that may differ

In this section we consider the intersection problem for MCT(n)s in which it is not required that both MCT(n)s have the same padding. Of course, in such a setting it is fine if the paddings happen to be the same, so in view of Theorem 1.2 it remains to decide whether or not this relaxation allows $\{\tau-1, \tau-2, \tau-3, \tau-5\}$ to be in $J(n)$. As will be shown, sometimes it does and sometimes it does not! (See Theorem 2.5.) We begin with some helpful lemmas.

Lemma 2.1 *Let* $(S, \mathcal{T}_1, \mathcal{P}_1)$ *and* $(S, \mathcal{T}_2, \mathcal{P}_2)$ *be two* MCT(n)*s. If* $\mathcal{P}_1$ *contains no path of length* 2 *then* $\tau - 1 \notin J(n)$.

Proof: Let MCT(n)s $(S, \mathcal{T}_1, \mathcal{P}_1)$ and $(S, \mathcal{T}_2, \mathcal{P}_2)$ be two MCT(n)s in which $\mathcal{T}_1\backslash\{t_1\} = \mathcal{T}_2\backslash\{t_2\}$. Each edge $e \in E(t_1)\backslash E(t_2)$ must be in some other triple in $\mathcal{T}_2$, say $t_2' \neq t_2$. But $\mathcal{T}_1\backslash\{t_1\} = \mathcal{T}_2\backslash\{t_2\}$, so e is in two triples in $\mathcal{T}_1$. Therefore $e \in \mathcal{P}_1$.

Since $\mathcal{P}_1$ contains no path of length 2 and since t_1 is isomorphic to K_3, it follows that t_1 contains at most one edge in $\mathcal{P}_1$.

Therefore t_1 has at most one edge that is not in t_2 and so, being a copy of K_3, this forces t_1 to equal t_2. □

Some small values have unusual intersection numbers, so are handled in the following lemma.

Lemma 2.2 $J(4) = \{2, 3\}$*,* $J(5) = \{1, 2, 4\}$ *and* $J(6) = \{0, 2, 3, 6\}$.

Proof: First consider $J(4)$. Let $S = \{1, 2, 3, 4\}$ and let $(S, \mathcal{T}_1, \mathcal{P}_1)$ and $(S, \mathcal{T}_2, \mathcal{P}_2)$ be the following two MCT(4)s:
$\mathcal{T}_1 = \{\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}\}$, $\mathcal{P}_1 = \{\{1, 2\}, \{2, 3\}, \{2, 4\}\}$,
$\mathcal{T}_2 = \{\{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$, $\mathcal{P}_2 = \{\{1, 4\}, \{2, 4\}, \{3, 4\}\}$.
Then $|\mathcal{T}_1 \cap \mathcal{T}_1| = 3$ and $|\mathcal{T}_1 \cap \mathcal{T}_2| = 2$. It is trivial to see that these are the only possibile intersection sizes.

Next is the case $n = 5$. Let $S = \{1, 2, 3, 4, 5\}$ and consider the three following MCT(5)s:
$\mathcal{T}_1 = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{3, 4, 5\}\}$, $\mathcal{P}_1 = \{\{1, 2\}, \{1, 2\}\}$,
$\mathcal{T}_2 = \{\{1, 2, 3\}, \{2, 3, 4\}, \{2, 3, 5\}, \{1, 4, 5\}\}$, $\mathcal{P}_2 = \{\{2, 3\}, \{2, 3\}\}$, and
$\mathcal{T}_3 = \{\{1, 3, 4\}, \{2, 3, 4\}, \{3, 4, 5\}, \{1, 2, 5\}\}$, $\mathcal{P}_3 = \{\{3, 4\}, \{3, 4\}\}$ Then $|\mathcal{T}_1 \cap \mathcal{T}_1| = 4$, $|\mathcal{T}_1 \cap \mathcal{T}_2| = 1$ and $|\mathcal{T}_1 \cap \mathcal{T}_3| = 2$. It is easy to see that these are the only possibile intersection sizes since once the padding is chosen the MCT(5) is completely determined.

Finally consider $J(6)$. In view of Theorem 1.2, the only intersection numbers remaining to be considered are those in $\{1, 3, 4, 5\}$. Let $S = \{0, 1, 2, 3, 4, 5\}$ and define the following MCT(6)s:

$\mathcal{T}_1 = \{\{0, 1, 2\}, \{0, 1, 3\}, \{0, 4, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}\}$, so
$\mathcal{P}_1 = \{\{0, 1\}, \{2, 3\}, \{4, 5\}\}$, and
$\mathcal{T}_2 = \{\{0, 1, 2\}, \{0, 2, 4\}, \{0, 3, 5\}, \{1, 3, 4\}, \{1, 4, 5\}, \{2, 3, 5\}\}$, so
$\mathcal{P}_2 = \{\{0, 2\}, \{1, 4\}, \{3, 5\}\}$.

Then $|\mathcal{T}_1 \cap \mathcal{T}_2| = 3$. By Lemma 2.1, we know $5 \notin J(6)$. A straightforward search shows that $1, 4 \notin J(6)$. □

We now focus on the case where $n \equiv 5 \pmod 6$, beginning with the next lemma which is a vital ingredient in showing $\tau - 5 \in J(n)$.

Lemma 2.3 *Let H be the graph on 7 vertices formed by a cycle of length 3 and a cycle of length 5 sharing exactly one vertex. There exist two coverings of $K_7 - E(H)$, each having a doubled edge as its padding, with no triple occurring in both coverings.*

Proof: The result follows by defining

$\mathcal{T}_1 = \{\{0, 1, 3\}, \{0, 1, 6\}, \{0, 4, 5\}, \{2, 3, 4\}, \{2, 5, 6\}\}$ and
$\mathcal{T}_2 = \{\{0, 3, 4\}, \{0, 5, 6\}, \{1, 2, 3\}, \{1, 2, 6\}, \{2, 4, 5\}\}$ with
$H = (0, 1, 2) \cup (1, 4, 6, 3, 5)$, so $\mathcal{P}_1 = \{\{0, 1\}, \{0, 1\}\}$ and $\mathcal{P}_2 = \{\{1, 2\}, \{1, 2\}\}$. □

Lemma 2.4 *Let $n \equiv 5 \pmod 6$. Then $J(n) = I(n) \backslash \{\tau - 1\}$.*

Proof: We consider four cases in turn.

Case 1: $\tau - 1 \notin J(n)$ by Lemma 2.1 since the padding is a doubled edge.

Case 2: $\tau - 2 \in J(n)$. Let (S, T, L) be a maximum packing of K_n with triples with leave the 4-cycle (a, b, c, d) (see [6], for example). Form two MCT(n)s $(S, \mathcal{T}_1, \mathcal{P}_1)$ and $(S, \mathcal{T}_2, \mathcal{P}_2)$ by defining $\mathcal{T}_1 = T \cup \{\{a, b, c\}, \{a, c, d\}\}$ and $\mathcal{T}_2 = T \cup \{\{a, b, d\}, \{b, c, d\}\}$. Then $\mathcal{P}_1 = \{\{a, c\}, \{a, c\}\}$ and $\mathcal{P}_2 = \{\{b, d\}, \{b, d\}\}$, and clearly $|\mathcal{T}_1 \cap \mathcal{T}_2| = \tau - 2$ as required.

Case 3: $\tau - 3 \in J(n)$. Let (S, B) be a PBD($6n$ +5), $n \geq 1$, with one block $f = \{a, b, c, d, e\}$ of size 5 and all other blocks of size 3 (see [6], for example). Form two MCT(n)s $(S, \mathcal{T}_1, \mathcal{P}_1)$ and $(S, \mathcal{T}_2, \mathcal{P}_2)$ by defining $\mathcal{T}_1 = (B \backslash \{f\}) \cup \{\{a, b, c\}, \{a, b, d\}, \{a, b, e\}, \{c, d, e\}\}$ and $\mathcal{T}_2 = (B \backslash \{f\}) \cup \{\{a, b, c\}, \{b, c, d\}, \{b, c, e\}, \{a, d, e\}\}$. Then $\mathcal{P}_1 = \{\{a, b\}, \{a, b\}\}$ and $\mathcal{P}_2 = \{\{b, c\}, \{b, c\}\}$, and clearly $|\mathcal{T}_1 \cap \mathcal{T}_2| = \tau - 3$ as required.

Case 4: $\tau - 5 \in J(n)$. If $n = 5$ then $\tau = 4$, so in this case we can assume that $n \geq 11$. We begin by considering the two smallest cases, $n = 11$ and 17. In each case the MCT(n)s are constructed by starting with the MCT(7)s $(\{0, 1, \ldots, 6\}, \mathcal{T}_1, \mathcal{P}_1)$ and $(\{0, 1, \ldots, 6\}, \mathcal{T}_2, \mathcal{P}_2)$ defined in Lemma 2.3. If $n = 11$ then to each of $\mathcal{T}_1$ and $\mathcal{T}_2$ add the triples in:

$\{\{1,4,7\},\{4,6,10\},\{3,6,8\},\{3,5,9\},\{1,5,10\},\{0,1,8\},\{0,2,7\},\{1,2,9\},$
$\{5,7,8\},\{6,7,9\},\{3,7,10\},\{4,8,9\},\{2,8,10\},\{0,9,10\}\}.$

If $n = 17$ then use Lemma 2.3 to form the two sets of triples $\mathcal{T}_1$ and $\mathcal{T}_2$, except first rename the vertices so that $H = (0,1,2,3,4) \cup (0,5,6)$. Then the result follows by defining two MCT(17)s on the vertex set $S = \{0,1,\dots,6\} \cup (\{0,1,2,3,4\} \times \{0,1\})$ by adding to each of $\mathcal{T}_1$ and $\mathcal{T}_2$ the triples in:
$\{\{i, i+1, (2+i, 0)\}, \{2+i, (i,0), (i+1,0)\}, \{i, (i,0), (i,1)\}, \{(i,0), (i+2,0), (i+1,1)\}, \{i+4, (i,1), (i+3,1)\}, \{i+3, (i,1), (i+1,1)\}, \{5, (i,0), (i+2,1)\}, \{6, (i,0), (i+3,1)\} \mid 0 \le i \le 4\} \cup \{\{0,5,6\}\}$, reducing all sums modulo 5.

Now let $n = 6x+5$ with $x \ge 3$. Then there exists a GDD (V, B) with x groups of size 6 and all blocks of size 3 (for example, see [6]). Form the two required MCT(n)s, on the vertex set $V \cup S$ of order n as follows. On the 11 vertices formed by the 6 vertices in one group and the 5 vertices in S, place the triples in the two MCT(11)s just formed; so these intersect in all but 5 triples. On the 11 points formed by each other group together with S, place the triples of $K_{11} \backslash K_5$ (see, for example, "The Quasigroup with Holes Construction" in [6]), the hole of size 5 being the vertices in S. □

Theorem 2.5 *$J(4) = \{2,3\}$, $J(5) = \{1,2,4\}$, $J(6) = \{0,2,3,6\}$, and if $n \ge 7$ then*

1. *$J(n) = I(n) \setminus \{\tau - 1\}$ when $n \equiv 0$ or $5 \pmod 6$, and*
2. *$J(n) = I(n)$ when $n \equiv 2$ or $4 \pmod 6$.*

PROOF: In view of Theorem 1.2 it remains to decide whether or not this relaxation where the paddings can be different allows the integers $\{\tau-1, \tau-2, \tau-3, \tau-5\}$ to be in $J(n)$ in the cases where $n \equiv 0, 2, 4$ or $5 \pmod 6$. We consider various cases in turn. The cases where $n \le 6$ are settled in Lemma 2.2, so assume that $n \ge 8$.

Case 1: $n \equiv 0 \pmod 6$. The fact that $\tau - 1 \notin J(n)$ follows from Lemma 2.1 since the padding is a 1-factor.

Let $n = 6x$ for some $x \ge 2$, let $S = \{0,1,\dots,6x-4\}$ and let $S_1 = \{0,1,\dots,6x-1\}$. Let (S,T) be a partial STS$(n-3)$ in which the leave is the 2-factor consisting of the cycles $(0,1,\dots,6x-7)$ and $(6x-6, 6x-5, 6x-4)$ (see "The $6n+5$ Construction" in [6], for example). Form an MCT(n) $(S_1, \mathcal{T}_1, \mathcal{P}_1)$ from T by adding: the set $\mathcal{T}_1'$ of triples in an MCT(6) on the vertex set $\{6x-6, 6x-5, \dots, 6x-1\}$; the triples in $\{\{6x-3+i, 2j+i, 2j+1+i\} \mid 0 \le i \le 1, 0 \le j \le 3x-4\}$, reducing the sum modulo $6x-6$; and the triples in $\{\{6x-1, 2j, 2j+1\} \mid 0 \le j \le 3x-4\}$. Form another MCT($n$) $(S_1, \mathcal{T}_2, \mathcal{P}_2)$ by defining $\mathcal{T}_2 = (\mathcal{T}_1 \cup \{\{6x-1, 0, 2\}, \{6x-1, 1, 3\}\} \setminus \{\{6x-1, 0, 1\}, \{6x-1, 2, 3\}\}$. Finally, form third MCT(n) $(S_1, \mathcal{T}_3, \mathcal{P}_3)$ by defining $\mathcal{T}_3 = (\mathcal{T}_1 \cup \mathcal{T}_3') \setminus \mathcal{T}_1'$, where $\mathcal{T}_3'$ is the set of triples in an MCT(6) on the vertex set $\{6x-6, 6x-5, \dots, 6x-1\}$ for which $|\mathcal{T}_1' \cap \mathcal{T}_3'| = 3$ (see Lemma 2.2). Then $|\mathcal{T}_1 \cap \mathcal{T}_2| = \tau - 2$, $|\mathcal{T}_1 \cap \mathcal{T}_3| = \tau - 3$ and $|\mathcal{T}_2 \cap \mathcal{T}_3| = \tau - 5$.

Case 2: $n \equiv 2$ or $4 \pmod 6$. Let $(\{1,2,\dots,n-1\}, T)$ be an STS$(n-1)$. Append a vertex n to form an MCT(n) with padding:

$P_1 = \{\{1, n\}, \{1, 2\}, \{1, 3\}\} \cup \{\{2x, 2x + 1\} \mid 2 \le x \le (n - 2)/2\}$
by defining $T_1 = T \cup \{\{1, 2, n\}, \{1, 3, n\}\} \cup \{\{2x, 2x + 1, n\} \mid 2 \le x \le (n - 2)/2\}$. Similarly, define four more MCT(n)s by defining:

1. $T_2 = (T_1 \cup \{\{2, 3, n\}\}) \setminus \{\{1, 3, n\}\}$ which has padding $P_2 = (P_1 \cup \{\{2, n\}, \{2, 3\}\}) \setminus \{\{1, n\}, \{1, 3\}\}$,
2. $T_3 = (T_1 \cup \{\{4, 6, n\}, \{5, 7, n\}\}) \setminus \{\{4, 5, n\}, \{6, 7, n\}\}$ which has padding $P_3 = (P_1 \cup \{\{4, 6\}, \{5, 7\}\}) \setminus \{\{4, 5\}, \{6, 7\}\}$,
3. $T_4 = (T_2 \cup \{\{4, 6, n\}, \{5, 7, n\}\}) \setminus \{\{4, 5, n\}, \{6, 7, n\}\}$ which has padding $P_4 = (P_2 \cup \{\{4, 6\}, \{5, 7\}\}) \setminus \{\{4, 5\}, \{6, 7\}\}$, and
4. if $n \ge 14$ then $T_5 = (T_4 \cup \{\{8, 10, n\}, \{9, 11, n\}\}) \setminus \{\{8, 9, n\}, \{10, 11, n\}\}$ which has padding $P_4 = (P_2 \cup \{\{8, 10\}, \{9, 11\}\}) \setminus \{\{8, 9\}, \{10, 11\}\}$.

Then $|T_1 \cap T_2| = \tau - 1, |T_1 \cap T_3| = \tau - 2, |T_1 \cap T_4| = \tau - 3$ and $|T_1 \cap T_5| = \tau - 5$. If $n = 8$ then let $(\{1, 2, \ldots, 7\}, T_0')$ be a STS(7) with $|T_0' \cap T| = 3$ and define $T_0 = T_0' \cup \{\{1, 2, 8\}, \{2, 3, 8\}, \{4, 5, 8\}, \{6, 7, 8\}\}$; then $|T_0 \cap T_1| = \tau - 5$. If $n = 10$ then let $T_0 = T \cup \{\{1, 4, 10\}, \{2, 3, 10\}, \{5, 6, 10\}, \{7, 8, 10\}, \{7, 9, 10\}\}$; then $|T_0 \cap T_1| = \tau - 5$.

Case 3: $n \equiv 5 \pmod 6$. This follows from Lemma 2.4. □

3 Simple paddings

In this section we settle the intersection problem for two minimum coverings in which the paddings are required to be simple, but can be different. Since the only case where the padding is not simple in Theorem [5] is when $n \equiv 5 \pmod 6$, the padding being a doubled edge, we now focus on that case where we require the padding to be a 5-cycle. We begin with some small ingredients needed for the general result.

Lemma 3.1 $JS(5) = \{0, 2, 3, 5\}$.

Proof: Let $S = \{0, 1, 2, 3, 4\}$. For $1 \le i \le 4$ let $(S, \mathcal{T}_i, \mathcal{P}_i)$be the MCT(5) defined by:
$\mathcal{T}_1 = \{\{0, 1, 2\}, \{0, 1, 3\}, \{0, 3, 4\}, \{1, 2, 4\}, \{2, 3, 4\}\}$, $\mathcal{P}_1$ is the 5-cycle $(0, 1, 2, 4, 3)$,
$\mathcal{T}_2 = \{\{0, 1, 4\}, \{0, 2, 3\}, \{0, 2, 4\}, \{1, 2, 3\}, \{1, 3, 4\}\}$, $\mathcal{P}_2 = (0, 2, 3, 1, 4)$,
$\mathcal{T}_3 = \{\{0, 1, 3\}, \{0, 1, 4\}, \{0, 2, 4\}, \{1, 2, 3\}, \{2, 3, 4\}\}$, $\mathcal{P}_3 = (0, 1, 3, 2, 4)$, and
$\mathcal{T}_4 = \{\{0, 1, 2\}, \{0, 1, 4\}, \{0, 3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}\}$, $\mathcal{P}_4 = (0, 1, 2, 3, 4)$.
Then $|\mathcal{T}_1 \cap \mathcal{T}_2| = 0$, $|\mathcal{T}_1 \cap \mathcal{T}_3| = 2$, $|\mathcal{T}_1 \cap \mathcal{T}_4| = 3$ and $|\mathcal{T}_1 \cap \mathcal{T}_1| = 5$. A computer search shows that these are the only intersection numbers in $JS(5)$. □

Lemma 3.2 $JS(11) = I(11)$.

PROOF: Let $X = \{0, 1, 2, 3, 4\}$. For $1 \le i \le 4$ let $(X, \mathcal{T}_i, \mathcal{P}_i)$ be the MCT(5) defined in Lemma 3.1. For $1 \le i \le 7$ let $\alpha_i : X \mapsto X$ be a bijection. Let $\mathcal{F} = \{F_0, F_1, F_2, F_3, F_4\}$ be a 1-factorization of K_6 on the vertex set $\{5, 6, 7, 8, 9, 10\}$. Let $S = \{0, 1, \dots, 10\}$.

For $1 \le i \le 4$, form the MCT(11) $(S, C_i, \mathcal{P}_i)$ by defining $C_i = \mathcal{T}_i \cup \{\{j, x, y\} \mid j \in X, \{x, y\} \in F_{\alpha_i(j)}\}$.

Three more MCT(11)s are needed. For $5 \le i \le 7$, form $(S, C_i, \mathcal{P}_i)$ by defining $\mathcal{T}_5 = \{\{0, 5, 3\}, \{0, 1, 4\}, \{0, 2, 4\}, \{1, 2, 3\}, \{2, 3, 4\}\}$ and $C_5 = \mathcal{T}_5 \cup \{\{j, x, y\} \mid j \in X, \{x, y\} \in F_{\alpha_5(j)}\}$, so $\mathcal{P}_5 = (0, 5, 3, 2, 4)$, defining $\mathcal{T}_6 = \{\{0, 5, 2\}, \{0, 1, 3\}, \{0, 3, 4\}, \{1, 2, 4\}, \{2, 3, 4\}\}$ and $C_6 = \mathcal{T}_6 \cup \{\{j, x, y\} \mid j \in X, \{x, y\} \in F_{\alpha_6(j)}\}$, so $\mathcal{P}_6 = (0, 5, 2, 4, 3)$, and finally, letting $C_7 = \mathcal{T}_1 \cup \{\{j, x, y\} \mid j \in X, \{x, y\} \in F_{\alpha_7(j)}\}$, so $\mathcal{P}_7 = \mathcal{P}_1$. Notice that $|\mathcal{T}_1 \cap \mathcal{T}_5| = 1$ and $|\mathcal{T}_1 \cap \mathcal{T}_6| = 4$.

For $1 \le i < k \le 6$, let $D(i, k) = |\{j \mid \alpha_i(j) = \alpha_k(j)\}|$; notice that $D(i, k)$ can be chosen to take on any value in $\{0, 1, 2, 3, 5\}$. Then, since each 1-factor contains 3 edges, $|C_1 \cap C_2| = 3D(1, 2)$, $|C_1 \cap C_3| = 2 + 3D(1, 3)$, $|C_1 \cap C_4| = 3 + 3D(1, 4)$, $|C_1 \cap C_5| = 1 + 3D(1, 5)$, $|C_1 \cap C_6| = 4 + 3D(1, 6)$, and $|C_1 \cap C_7| = 5 + 3D(1, 7)$. By suitably choosing the bijections, it now follows that $JS(11) = I(11) = \{0, 1, \dots, 20\}$. □

Lemma 3.3 $JS(17) = I(17)$.

PROOF: This proof closely follows that of Lemma 3.2, so some details are omitted.

We begin with the following factorization $\mathcal{F}_1 = \{F_1, F_2, F_3, F_4, F_5, T_1\}$ of K_{12}:
$F_1 = \{\{0, 4\}, \{1, 8\}, \{5, 9\}, \{2, 6\}, \{3, 10\}, \{7, 11\}\}$,
$F_2 = \{\{0, 5\}, \{1, 9\}, \{4, 8\}, \{2, 7\}, \{3, 11\}, \{6, 10\}\}$,
$F_3 = \{\{0, 6\}, \{1, 7\}, \{2, 8\}, \{3, 9\}, \{4, 10\}, \{5, 11\}\}$,
$F_4 = \{\{0, 7\}, \{1, 6\}, \{2, 9\}, \{3, 8\}, \{4, 11\}, \{5, 10\}\}$,
$F_5 = \{\{0, 8\}, \{1, 5\}, \{2, 10\}, \{3, 7\}, \{4, 9\}, \{6, 11\}\}$, and
$T_1 = \{\{i, i + 1, i + 3\} \mid i = 0, 1, \dots, 11\}$, reducing the sums modulo 12.

Let $T_2 = \{\{i, i + 2, i + 3\} \mid i = 0, 1, \dots, 11\}$. Then $\mathcal{F}_2 = (\mathcal{F}_1 \cup T_2) \backslash T_1$, reducing the sums modulo 12, is a second factorization of K_{12}. Notice that $|T_1 \cap T_2| = 0$.

Let $X = \{12, 13, \dots, 16\}$. Let $T' = \{\mathcal{T}'_i \mid 1 \le i \le 7\}$ where $\mathcal{T}'_i$ is formed from $\mathcal{T}_i$ in Lemma 3.2 by renaming vertex j with $j + 12$ except that 5 is renamed with 0.

Each integer in $I(17) = \{0, 1, \dots, 47\}$ can be written as $a + 12b + 6c$, where $a \in \{0, 1, \dots, 5\}, b \in \{0, 1\}$ and $c \in \{0, 1, 2, 3, 5\}$. Two MCT(17)s with intersection $a + 12b + 6c$ can be constructed by the union of the following choices: each MCT(17) chooses one of the sets in T' in such a way that they intersect in a triples; each MCT(17) chooses triples in either T_1 or T_2, chosen to be the same if $b = 1$ and different if $b = 0$; and each chooses one of the sets $\{\{j, x, y\} \mid j \in X, \{x, y\} \in F_{\alpha_1(j)}\}$ and $\{\{j, x, y\} \mid j \in X, \{x, y\} \in F_{\alpha_2(j)}\}$, where $\alpha_1(j)$ and $\alpha_2(j)$ are bijections from $\{1, 2, \dots, 5\}$ to itself, chosen so that $D(1, 2) = c$.

So it follows that $JS(17) = I(17) = \{0, 1, \dots, 47\}$. □

With Lemmas 3.1, 3.2 and 3.3 in hand we can now proceed to the main construction for $n \equiv 5 \pmod 6$, proving the following result.

Theorem 3.4 *Let* $n \equiv 5 \pmod 6$. *Then* $JS(5) = \{0, 2, 3, 5\}$ *and* $JS(n) = I(n)$ *for all* $n \geq 11$.

Proof: If $n \leq 17$ then the result follows from Lemmas 3.1, 3.2 and 3.3. So we can assume that $n \geq 23$. Let $n = 6m + 5 = 5 + 3(2m)$; so we are assuming that $2m \geq 6$. Hence there exists a GDD $G = (Y, \mathcal{G}, \mathcal{B})$ with $|Y| = 2m$, blocks of size 3, and all groups of size 2 except possibly for one of size 4 (easily constructed by deleting a point either from a STS or from a PBD with 1 block of size 5, the rest of size 3). It is also worth noting that $\tau = 6m^2 + 9m + 5$

First note that each integer in $A = \{0, 3, 6, 9, \ldots, 9s\} \setminus \{9s-3\}$, where $s = |\mathcal{B}| \geq 4$ can be written as the sum of s integers, say $z_1, z_2, \ldots, z_s$, each being in $\{0, 3, 9\}$. Then, if $\mathcal{G}$ contains $\epsilon \in \{0, 1\}$ groups of size 4, it follows that each integer in $I(n) = \{0, 1, \ldots, 6m^2+9m+5\}$ can be written as $x+y+z$ where $x \in JS(11+6\epsilon), y \in \{15i \mid 0 \leq i \leq m-1-\epsilon\}$, and $z \in A$. (To see this, it helps to note that the largest values x, y and z can take are $\tau(11+6\epsilon) = 20+27\epsilon$, $15(m-1-\epsilon)$ and $9s = 9((m(2m-2)-4\epsilon)/3)$ respectively, the sum of these three values being $6m^2 + 9m + 5$.)

Let X be a set of 5 vertices. Two MCT(n)s on the vertex set $S = X \cup (Y \times \{1, 2, 3\})$ with intersection $x+y+z$ can be constructed by the union of the following choices:

(i) Each MCT(n) chooses the triples in an MCT(17) or an MCT(11) if G does or does not contain a group of size 4 respectively, chosen so that they intersect in x triples;

(ii) for each remaining group in $\mathcal{G}$, say $\{a, b\}$, each MCT(n) takes the 15 triples in a K_3-decomposition of $K_{11} - K_5$ on the vertex set $S = X \cup (\{a, b\} \times \{1, 2, 3\})$ with hole X, chosen so that they are 15 identical triples for $y/15$ of the groups and are disjoint otherwise; and

(iii) for each block $b_i = \{a, b, c\} \in \mathcal{B} = \{b_1, b_2, \ldots, b_s\}$, each MCT($n$) chooses the blocks in a K_3-decomposition of the complete tripartite graph $K_{3,3,3}$ (with parts $\{a\} \times \{1, 2, 3\}$, $\{b\} \times \{1, 2, 3\}$, and $\{c\} \times \{1, 2, 3\}$), chosen so that they agree in z_i triples.

Note that the paddings are the 5-cycles defined by the MCTs defined in (i).

Then it follows that $JS(n) = I(n)$. □

Acknowledgments

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