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Note: An inequality for the line-size sum in a finite linear space

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Abstract

An inequality for finite linear spaces in relation with clique partitions of the complete graph K_n is given.

1 The inequality

For notions on finite linear spaces and clique partitions of graphs we refer the reader to [1] and [2] respectively.

Let $(\mathcal{P}, \mathcal{L})$ be a (non-degenerate) finite linear space with v points and b lines. For every point p let r_p denote the number of lines containing p (the *degree* of p) and for every line ℓ let k_ℓ denote the size of ℓ. By an *m-point* we mean a point of degree m.

Recently in [2] a lower bound for the sum of line sizes of a finite linear space has been obtained.

Theorem 1.1. *Let $(\mathcal{P}, \mathcal{L})$ be a non-degenerate finite linear space with v points, then we have*

$$\sum_{\ell \in \mathcal{L}} k_\ell \geq 3v - 3 \tag{1}$$

and equality holds if and only if $(\mathcal{P}, \mathcal{L})$ is a near-pencil.

As a consequence of Theorem 1.1 we obtain the following result.

Proposition 1.2. *Let $\mathcal{C}$ be a clique partition of the complete graph K_n whose cliques are of size at most $n-1$. Then $\sum_{C \in \mathcal{C}} |C| \geq 3n - 3$.*

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In this note, a generalization of Theorem 1.1 is given.

Theorem 1.3. *Let $(\mathcal{P}, \mathcal{L})$ be a non-trivial finite linear space on v points. Let $m \geq 2$ denote the minimum point degree. Then*

$$\sum_{\ell \in \mathcal{L}} k_\ell \geq (v - m + 1)(m + 1).$$

The equality holds if and only if $m = 2$ and $(\mathcal{P}, \mathcal{L})$ is a near-pencil.

2 Proof of Theorem 1.3

In this section, $(\mathcal{P}, \mathcal{L})$ is a finite linear space with minimal point degree m. As for any incidence structrure, in a finite linear space the following equality holds:

$$\sum_{p \in \mathcal{P}} r_p = \sum_{\ell \in \mathcal{L}} k_\ell. \tag{2}$$

Assume $\sum_{\ell \in \mathcal{L}} k_\ell \leq (v-m+1)(m+1)$. Let x denote the number of points of degree m. Then

$$xm + (v - x)(m + 1) \leq \sum_{p \in \mathcal{P}} r_p = \sum_{\ell \in \mathcal{L}} k_\ell \leq (v - m + 1)(m + 1)$$

and so

$$x \geq m^2 - 1 \geq 3.$$

If there are three non-collinear m-points, then the size of of each line is at most m and thus $m^2 - 1 \leq x \leq v \leq m(m-1) + 1$ and so $m = 2$, all the points have degree m and the linear space $(\mathcal{P}, \mathcal{L})$ is the near-pencil on three points and so $\sum_{\ell \in \mathcal{L}} k_\ell = (v - 1) \cdot (m + 1) = 6$.

Hence we may assume that all the m-points are collinear and that there are points of degree different from m. Let L be the line containing all the points of degree m. Thus $k_L \geq m^2 - 1$. Counting the number of points of the linear space via the lines on an m-point, and since all lines other than L must have size at most m, we have

$$v \leq k_L + (m - 1)^2 \leq 2k_L - 2(m - 1)$$

and so

$$v - m + 1 \leq 2k_L - 3(m - 1).$$

Since the points outside L have degree at least k_L, it follows that

$$k_L m + (v - k_L)k_L \leq \sum_{p \in \mathcal{P}} r_p = \sum_{\ell \in \mathcal{L}} k_\ell \leq (2k_L - 3(m - 1))(m + 1)$$

and so

$$(v - k_L)k_L \leq 2k_L + k_L m - 3(m^2 - 1),$$

$$v - k_L \leq m + 2 - \frac{3(m^2 - 1)}{k_L},$$

$$v \leq k_L + m + 1.$$

Thus

$$v - m + 1 \leq k_L + 2,$$

from which it follows, since each line has size at least 2, that

$$k_L + 2(b - 1) \leq k_L + \sum_{\ell \in \mathcal{L}\, \ell \neq L} k_\ell = \sum_{\ell \in \mathcal{L}} k_\ell \leq (k_L + 2)(m + 1),$$

and hence

$$2(b - 1) \leq k_L m + 2(m + 1).$$

Counting lines meeting, but different from, L gives $b - 1 \geq k_L(m - 1)$; thus

$$2k_L m - 2k_L \leq k_L m + 2(m + 1)$$

so

$$k_L(m - 2) \leq 2(m + 1).$$

But $k_L \geq m^2 - 1$, and therefore

$$(m - 1)(m - 2) \leq 2$$

and so either $m = 2$ or $m = 3$.

If $m = 3$ then $k_L = 8 = m^2 - 1$, $v \leq 12$ and all the points of L have degree $m = 3$. Moreover, from $v \geq k_L + m - 1$ it follows that $v \geq 10$.

If $v = 12$, on each point of L there are L and two lines of length 3 contradicting the fact that the four points outside L may give rise to at most six lines of length 3.

If $v = 11$, on each point of L there are L, one line of length 3 and one of lenght 2, contradicting the fact that in such a case $(\mathcal{P}, \mathcal{L})$ has at most three lines of length 3.

If $v = 10$, $(\mathcal{P}, \mathcal{L})$ is the union of L and a line of length 2 disjoint from L and the two points outside L have degree 9. Thus, $42 = \sum_{p \in \mathcal{P}} r_p = \sum_{\ell \in \mathcal{L}} k_\ell \leq (10-3+1) \cdot 4 = 32$, a contradiction.

Hence $m = 2$ and $(\mathcal{P}, \mathcal{L})$ is the near-pencil on $v = k_L + 1$ points and $\sum_{\ell \in \mathcal{L}} k_\ell = 3v - 3 = (v - m + 1)(m + 1)$. This completes the proof of Theorem 1.3. □

Let $(\mathcal{P}, \mathcal{L})$ be a finite linear space and let m be the minimum point degree. If $m = 3$ then $\sum_{\ell \in \mathcal{L}} k_\ell = \sum_{p \in \mathcal{P}} r_p \geq 3v > 3v - 3$ and if $m = 2$ then by Theorem 1.3 $\sum_{\ell \in \mathcal{L}} k_\ell \geq (v - 1)3$, so Theorem 1.1 and Proposition 1.2 follow from Theorem 1.3.

Let us end by observing that Theorem 2.5 of [2] can imply similar (and in some cases better) bounds than Theorem [2]. For instance, if we take the dual of the linear space (i.e. exchange the role of points and lines) and we assume that there is a point of degree $b-c$ (where b is the number of lines of the linear space and c is a constant), then Inequality (8) in [2] gives a lower bound $(2c+1)b - c/2 - 5c^2/2$. In contrast, Theorem 1.3 gives a lower bound less than $2cb-(c-1)v-c^2+1$, which is worse for large v.

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References

[1] L. M. Batten and A. Beutelspacher, *The Theory of finite linear spaces*, Cambridge University Press (1993).

[2] A. Davoodi, R. Javadi and B. Omoomi, Pairwise balanced designs and sigma clique partitions, *Discrete Math.* 339, (2016), 1450–1458.

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