AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **67(3)** (2017), Pages 508–514

Inequalities for two systems of subspaces with prescribed intersections

Gábor Hegedüs

Antal Bejczy Center for Intelligent Robotics
Kiscelli utca 82, Budapest, H-1032
Hungary
hegedus.gabor@nik.uni-obuda.hu

Abstract

Let W denote a linear space over a fixed field $\mathbb{F}$. We define the notions of weak ISP-system and weak (u,v)-system $\mathcal{S} = \{(U_i, V_i) : 1 \leq i \leq m\}$ of subspaces of W. We give upper bounds for the size of weak ISP-systems and weak (u,v)-systems.

1 Introduction

First we recall the notion of q-binomial coefficients.

The *q-binomial coefficient* $\begin{bmatrix} n \\ m \end{bmatrix}_q$ is a q-analog for the binomial coefficient, also called a Gaussian coefficient or a Gaussian polynomial. The q-binomial coefficient is given by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{[n]_q!}{[n-m]_q! \cdot [m]_q!} \tag{1}$$

for $n, m \in \mathbb{N}$, where $[n]_q!$ is the q-factorial (see [2], p. 26)

$$[n]_q! := (1+q) \cdot (1+q+q^2) \cdots (1+q+q^2+\ldots+q^{n-1}).$$

Clearly we have $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$. If we substitute $q = 1$ into (1), then this substitution reduces this definition to that of binomial coefficients.

Bollobás, in [1], proved the following two remarkable results in extremal combinatorics.

Theorem 1.1 *Let $A_1, \ldots A_m$ and $B_1, \ldots B_m$ be finite sets satisfying the conditions*

(i) $A_i \cap B_i = \emptyset$ for each $1 \leq i \leq m$;

(ii) $A_i \cap B_j \neq \emptyset$ for each $i \neq j$ $(1 \leq i, j \leq m)$.

Then

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$

Theorem 1.2 *Let $A_1, \ldots A_m$ be r-element sets and $B_1, \ldots B_m$ be s-element sets such that*

(i) $A_i \cap B_i = \emptyset$ for each $1 \leq i \leq m$;

(ii) $A_i \cap B_j \neq \emptyset$ for each $i \neq j$ $(1 \leq i, j \leq m)$.

Then

$$m \leq \binom{r+s}{s}.$$

Tuza proved the following two versions of Bollobás' Theorem.

Theorem 1.3 *Let p be an arbitrary real number, $0 < p < 1$ and $t := 1 - p$. Let $A_1, \ldots A_m$ and $B_1, \ldots B_m$ be finite sets satisfying the conditions*

(i) $A_i \cap B_i = \emptyset$ for each $1 \leq i \leq m$;

(ii) $A_i \cap B_j \neq \emptyset$ or $A_j \cap B_i \neq \emptyset$ for $i \neq j$ $(1 \leq i, j \leq m)$.

Then

$$\sum_{i=1}^{m} p^{|A_i|} t^{|B_i|} \leq 1.$$

Theorem 1.4 *Let $A_1, \ldots A_m$ be r-element sets and $B_1, \ldots B_m$ be s-element sets satisfying the conditions*

(i) $A_i \cap B_i = \emptyset$ for each $1 \leq i \leq m$;

(ii) $A_i \cap B_j \neq \emptyset$ or $A_j \cap B_i \neq \emptyset$ for $i \neq j$ $(1 \leq i, j \leq m)$.

Then

$$m \leq \frac{(r+s)^{r+s}}{r^r s^s}.$$

Tuza, in [4], raised the following question: Let a, b be fixed positive integers. Determine the largest integer $m := m(a,b)$ such that there exists a system $\mathcal{S} = \{(A_i, B_i) : 1 \leq i \leq m\}$ of $m(a,b)$ pairs of sets satisfying the conditions:

(i) $A_1, \ldots A_m$ are a-element sets and $B_1, \ldots B_m$ are b-element sets;

(ii) $A_i \cap B_i = \emptyset$ for each $1 \leq i \leq m$;

(iii) $A_i \cap B_j \neq \emptyset$ or $A_j \cap B_i \neq \emptyset$ for $i \neq j$ $(1 \leq i, j \leq m)$.

Tuza proved the following property of the numbers $m(a,b)$ in [4].

Proposition 1.5 $m(a,1) = 2a+1$ *for each* $a \geq 1$. *For every* $a, b \geq 1$,

$$m(a,b) \geq m(a,b-1) + m(a-1,b).$$

Proposition 1.5 gives a lower bound for $m(a,b)$ near to $2\binom{a+b}{a}$ for every a and b.

Lovász, in [3], used tensor product methods to prove the following skew version of Bollobás' Theorem for subspaces.

Theorem 1.6 *Let* $\mathbb{F}$ *be an arbitrary field. Let* $U_1, \dots U_m$ *be* r*-dimensional and* $V_1, \dots V_m$ *be* s*-dimensional subspaces of a linear space* W *over the field* $\mathbb{F}$. *Assume that*

(i) $U_i \cap V_i = \{0\}$ *for each* $1 \leq i \leq m$;

(ii) $U_i \cap V_j \neq \{0\}$ *whenever* $i < j$ $(1 \leq i, j \leq m)$.

Then

$$m \leq \binom{r+s}{r}.$$

In this paper our main aim is to give a subspace version of Theorems 1.3 and 1.4.

The following definitions were motivated by Theorems 1.4 and 1.6.

Definition. Let $\mathbb{F}$ be a fixed field. We say that a system $\mathcal{S} = \{(U_i, V_i) : 1 \leq i \leq m\}$ is a *weak* ISP*-system of subspaces* of an n-dimensional linear space W over the field $\mathbb{F}$, if $\mathcal{S}$ satisfies the following conditions:

(i) $U_i \cap V_i = \{0\}$ for each $1 \leq i \leq m$;

(ii) $U_i \cap V_j \neq \{0\}$ or $U_j \cap V_i \neq \{0\}$ for $i \neq j$ $(1 \leq i, j \leq m)$.

Definition. Let $\mathbb{F}$ be a fixed field. We say that a system $\mathcal{S} = \{(U_i, V_i) : 1 \leq i \leq m\}$ of subspaces of a linear space W over the field $\mathbb{F}$ is a *weak* (u,v)*-system*, if $\mathcal{S}$ satisfies the conditions

(i) $\mathcal{S}$ is a weak ISP-system;

(ii) $\dim(U_i) = u$ and $\dim(V_i) = v$ for each $1 \leq i \leq m$.

Our main results are upper bounds for the size of weak ISP-systems and weak (u,v)-systems.

Theorem 1.7 *Let $\mathcal{S} = \{(U_i, V_i) : 1 \le i \le m\}$ be a weak ISP-system of subspaces of a linear space W over the finite field $\mathbb{F}_q$. Let $u_i := \dim(U_i)$ and $v_i := \dim(V_i)$ for each $1 \le i \le m$. Let $0 \le j \le n$ be an arbitrary, but fixed integer. Then we have*

$$\sum_{i=1}^{m} \frac{\begin{bmatrix} n-v_i-u_i \\ j-u_i \end{bmatrix}_q q^{(j-u_i)v_i}}{\begin{bmatrix} n \\ j \end{bmatrix}_q} \le 1.$$

Theorem 1.8 *Let $\mathcal{S} = \{(U_i, V_i) : 1 \le i \le m\}$ be a weak (u, v)-system of subspaces of an n-dimensional linear space W over the finite field $\mathbb{F}_q$. Then*

$$m \le \left(\frac{q}{q-1}\right)^n q^{uv}.$$

2 Proofs of our main results

In the proof of our main results we use the following bounds for the q-binomial coefficients.

Lemma 2.1 *Let $0 \le j \le n$ be natural numbers. Then*

$$\begin{bmatrix} n \\ j \end{bmatrix}_q \le \left(\frac{q}{q-1}\right)^n q^{j(n-j)}.$$

Proof. This follows immediately from the inequalities

$$q^{\binom{n}{2}} \le [n]_q! \le \left(\frac{q}{q-1}\right)^n q^{\binom{n}{2}}.$$

□

In the proof of Theorem 1.7 we also use the following simple lemma (see Lemma 2.2 in [5]).

Lemma 2.2 *Let V denote the n-dimensional vector space over the finite field $\mathbb{F}_q$ and fix an $(n-d)$-dimensional subspace K of V, where $0 \le d \le n$. Let U_1 be a fixed ℓ_1-subspace of V such that $U_1 \cap K = \{0\}$. Let $u(n, d; \ell_1, \ell_2)$ denote the number of ℓ_2-subspaces U_2 of V satisfying $U_2 \cap K = \{0\}$ and $U_1 \subseteq U_2$. Then*

$$u(n, d; \ell_1, \ell_2) = \frac{\begin{bmatrix} d \\ \ell_2 \end{bmatrix}_q \begin{bmatrix} \ell_2 \\ \ell_1 \end{bmatrix}_q q^{(\ell_2-\ell_1)(n-d)}}{\begin{bmatrix} d \\ \ell_1 \end{bmatrix}_q}.$$

□

Proof of Theorem 1.7:

Let $1 \le i \le m$, $0 \le j \le n$ be fixed integers. Let $\mathcal{F}(i,j)$ denote the following subset of subspaces of W:

$$\mathcal{F}(i,j) := \{U \le W : \dim(U) = j, U_i \subseteq U, V_i \cap U = \{0\}\}.$$

Then it follows immediately from Lemma 2.2 that

$$|\mathcal{F}(i,j)| = \frac{\begin{bmatrix} n-v_i \\ j \end{bmatrix}_q \begin{bmatrix} j \\ u_i \end{bmatrix}_q q^{(j-u_i)v_i}}{\begin{bmatrix} n-v_i \\ u_i \end{bmatrix}_q}.$$

for each $0 \le j \le n$.

Lemma 2.3 *Let $0 \le j \le n$ be fixed. Let $1 \le i_1 < i_2 \le m$ be two indices. Then*

$$\mathcal{F}(i_1, j) \cap \mathcal{F}(i_2, j) = \emptyset.$$

Proof. We can prove this statement by an indirect argument. Suppose that there exist two indices $1 \le i_1 < i_2 \le m$ such that $\mathcal{F}(i_1,j) \cap \mathcal{F}(i_2,j) \ne \emptyset$. Let $U \in \mathcal{F}(i_1,j) \cap \mathcal{F}(i_2,j)$ be an arbitrary, but fixed subspace. Then $U_{i_1} \subseteq U$ and $V_{i_1} \cap U = \{0\}$. Similarly $U_{i_2} \subseteq U$ and $V_{i_2} \cap U = \{0\}$. Hence we find that

$$U_{i_1} \cap V_{i_2} = \{0\}$$

and

$$U_{i_2} \cap V_{i_1} = \{0\},$$

which gives a contradiction, because $\mathcal{S} = \{(U_i, V_i) : 1 \le i \le m\}$ is a weak (u, v)-system of subspaces of the linear space W. □

In the following, let $0 \le j \le n$ be a fixed integer. It follows from Lemma 2.3 that

$$\sum_{i=1}^m |\mathcal{F}(i,j)| = |\bigcup_{i=1}^m \mathcal{F}(i,j)| \le \begin{bmatrix} n \\ j \end{bmatrix}_q,$$

because $\mathcal{F}(i,j) \subseteq \{U \le W : \dim(U) = j\}$. Hence

$$\sum_{i=1}^m \frac{\begin{bmatrix} n-v_i \\ j \end{bmatrix}_q \begin{bmatrix} j \\ u_i \end{bmatrix}_q q^{(j-u_i)v_i}}{\begin{bmatrix} n-v_i \\ u_i \end{bmatrix}_q} \le \begin{bmatrix} n \\ j \end{bmatrix}_q. \qquad (2)$$

But it is easy to verify that

$$\frac{\begin{bmatrix} n-v_i \\ j \end{bmatrix}_q \begin{bmatrix} j \\ u_i \end{bmatrix}_q}{\begin{bmatrix} n-v_i \\ u_i \end{bmatrix}_q} = \begin{bmatrix} n - v_i - u_i \\ j - u_i \end{bmatrix}_q,$$

and hence it follows from inequality (2) that

$$\sum_{i=1}^{m} \begin{bmatrix} n-v_i-u_i \\ j-u_i \end{bmatrix}_q q^{(j-u_i)v_i} \le \begin{bmatrix} n \\ j \end{bmatrix}_q,$$

which was to be proved. □

Proof of Theorem 1.8: If $\mathcal{S} = \{(U_i, V_i) : 1 \le i \le m\}$ is a weak (u, v)-system of subspaces of the linear space W, then $u_i = \dim(U_i) = u$ and $v_i = \dim(V_i) = v$ for each $1 \le i \le m$. It follows from Theorem 1.7 that

$$\sum_{i=1}^{m} \frac{\begin{bmatrix} n-u-v \\ j-u \end{bmatrix}_q q^{(j-u)v}}{\begin{bmatrix} n \\ j \end{bmatrix}_q} \le 1$$

for each $1 \le j \le n$. Let $j := n - v$. This choice implies that

$$\sum_{i=1}^{m} \frac{q^{(n-v-u)v}}{\begin{bmatrix} n \\ v \end{bmatrix}_q} \le 1.$$

It follows from Lemma 2.1 that

$$\sum_{i=1}^{m} \frac{q^{(n-v-u)v}}{\left(\frac{q}{q-1}\right)^n q^{v(n-v)}} \le 1.$$

But then

$$m \frac{q^{-uv}}{\left(\frac{q}{q-1}\right)^n} \le 1,$$

which was to be proved. □

3 Concluding remarks

We can raise the following natural question: Let u, v be fixed positive integers. Let $\mathbb{F}$ be a fixed field. Determine the largest integer $t := t(u, v)$ such that there exists a weak (u, v)-system $\mathcal{S} = \{(U_i, V_i) : 1 \le i \le t\}$ of $t(u, v)$ pairs of subspaces of an n-dimensional linear space W over the field $\mathbb{F}$.

If $\mathbb{F}$ is the finite field $\mathbb{F}_q$, then we proved in Theorem 1.8 that

$$t(u, v) \le \left(\frac{q}{q-1}\right)^n q^{uv}.$$

On the other hand, it is easy to verify the lower bound $m(u, v) \le t(u, v)$. Namely, let $\{e_1, \ldots, e_n\}$ denote a fixed basis of the n-dimensional linear space W over $\mathbb{F}$. By the definition of the number $m(u, v)$ there exists a system $\mathcal{S} = \{(A_i, B_i) : 1 \le i \le m(u, v)\}$ of $m(u, v)$ pairs of sets satisfying the conditions:

(i) $A_1, \dots A_m$ are u-element sets and $B_1, \dots B_m$ are v-element sets;

(ii) $A_i \cap B_i = \emptyset$ for each $1 \le i \le m$;

(iii) $A_i \cap B_j \ne \emptyset$ or $A_j \cap B_i \ne \emptyset$ for $i \ne j$ $(1 \le i, j \le m)$.

Define the generated subspaces $U_i := \langle \{e_k : \ k \in A_i\} \rangle$ and $V_i := \langle \{e_l : \ l \in B_i\} \rangle$ for each $1 \le i \le m(u, v)$.

Then it is easy to verify that the system $\mathcal{S} = \{(U_i, V_i) : \ 1 \le i \le m(u, v)\}$ of $m(u, v)$ pairs of subspaces is a weak (u, v)-system.

References

[1] B. Bollobás, On generalized graphs, *Acta Math. Hung.* **16** (3) (1965), 447–452.

[2] W. Koepf, *Hypergeometric Summation: An Algorithmic Approach to Summation and Special Function Identities*, Vieweg, 1998.

[3] L. Lovász, Flats in matroids and geometric graphs, in: *Combinatorial surveys*, Proc. 6th British Combin. Conf., Egham 1977, Acad. Press, London 1977, 45–86.

[4] Z. Tuza, Application of Set-Pair Method in Extremal Hypergraph Theory, in: "Extremal problems for Finite Sets", *Bolyai Soc. Math. Studies* **3**, János Bolyai Math. Soc., Budapest, 1994, 479–514.

[5] K. Wang and Z. Li, Lattices associated with vector spaces over a finite field, *Lin. Algebra Appl.* **429** (2) (2008), 439–446.

(Received 7 Oct 2016; revised 5 Jan 2017)