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Some new non-Schurian association schemes on $2p^2$ points, p an odd prime, and related combinatorial structures

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Dedicated to the memory of Dan Archdeacon (1954–2015)

Abstract

Let p be an odd prime. We provide a construction of four non-Schurian association schemes for every prime $p \geq 5$ and two for $p = 3$. For $p > 3$ the construction is new, while for $p = 3$ it coincides with the non-Schurian schemes, obtained with the aid of a computer by A. Hanaki and I. Miyamoto. The discovered non-Schurian objects appear as algebraic mergings of the Schurian coherent configuration on $2p^2$ points and rank $6p - 2$, which corresponds to the action of the Heisenberg group of order p^3 on the set of points and lines of the classical biaffine plane. The results obtained are considered in a wider framework.

1 Introduction

This paper belongs to the area of Algebraic Graph Theory (briefly AGT). Symmetry of graphs is one of the central points of interests in AGT. A natural way to treat symmetry of a given graph Γ is to consider action of the automorphism group $\mathrm{Aut}(\Gamma)$ on diverse ingredients of Γ. Transitivity of the action of $\mathrm{Aut}(\Gamma)$ on the arc set of Γ may be approximated by combinatorial regularity: it is required that all arcs have the same value of certain local invariants. In this way concepts of a coherent configuration and its particular case of an association scheme are formulated in an axiomatic manner. Schurian association schemes naturally correspond to transitive permutation groups. Non-Schurian association schemes are, in a sense, more keen: they reflect combinatorial symmetry, which is not fully explained in group-theoretical terms.

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In this paper we describe four new infinite families of non-Schurian association schemes on $2p^2$ points, where $p > 3$ is a prime. These schemes are mergings of our starting geometrical object: a biaffine coherent configuration $\mathcal{M}$, which appears via the intransitive action of the Heisenberg group of order p^3 on two orbits of length p^2.

The main part of the current text deals with the description and justification of the existence of these non-Schurian schemes. Section 2 contains the most significant preliminaries. Section 3 introduces the classical biaffine plane with p^2 points and p^2 lines, p an odd prime, as well as the action of the Heisenberg group on the set Ω of points and lines. This action naturally implies the master coherent configuration $\mathcal{M}$ on the set Ω, having rank $6p - 2$; see Section 4. The three kinds of automorphism groups of $\mathcal{M}$ are considered in Section 6. Finally, four new infinite families of non-Schurian association schemes are defined and investigated in Section 7.

Section 5 is a stand-alone text. Here the configuration $\mathcal{M}$ is considered from scratch for $p = 3$. The presentation is accompanied by data obtained with the aid of computer packages. A number of diagrams aim to help the reader to comprehend material visually.

The last part of the paper reflects its additional expository and survey features. In Section 8 the above-mentioned distinction between combinatorial and group-theoretical regularity is discussed in a more precise context. It motivates origin of a long-standing interest to non-Schurian association schemes. In fact, with increasing value of p, the number of non-Schurian mergings of $\mathcal{M}$ also increases; in particular, part of such mergings have a relatively small constant rank—see Section 9. Section 10 has definite interdisciplinary features: here some discovered non-Schurian objects are naturally linked with a number of classical problems and objects in extremal and topological graph theory and other parts of AGT. In Section 11 we pay special tribute to the memory of Dan Archdeacon. Section 12 presents a mosaic of diverse topics related to the content of the paper, which had not been touched before in evident form.

This paper is a follower of its previous version [24] available at the arXiv. The reader will be referred to [24] several times for some concrete numerical data, extra details of presentation and even psychology of communication between the authors on the way from computer-aided experiments towards rigorous understanding and justification of the objects discovered.

There also exists a full version of the current text, containing more information regarding diverse issues, touched by us. It is available from the authors upon request.

2 Preliminaries

Below we provide a brief outline of the most significant concepts that will be used throughout the text. We refer to [20] and [36] for more detailed background.

By a *color graph* Γ we mean an ordered pair $(V, \mathcal{R})$, where V is a set of vertices and $\mathcal{R}$ a partition of $V \times V$ into binary relations. The elements of $\mathcal{R}$ are *colors*, and the number of colors is the *rank* of Γ. A *coherent configuration* is a color graph $\mathcal{W} = (\Omega, \mathcal{R})$, $\mathcal{R} = \{R_i \mid i \in I\}$, such that the following axioms are satisfied:

(i) The diagonal relation $\Delta_\Omega = \{(x, x) \mid x \in \Omega\}$ is a union of relations $\cup_{i\in I'} R_i$, for a suitable subset $I' \subseteq I$.

(ii) For each $i \in I$ there exists $i' \in I$ such that $R_i^T = R_{i'}$, where $R_i^T = \{(y, x) \mid (x, y) \in R_i\}$ is the relation transposed to R_i.

(iii) For any $i, j, k \in I$, the number $c^k_{i,j}$ of elements $z \in \Omega$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$ is a constant depending only on i, j, k, and independent on the choice of $(x, y) \in R_k$.

The numbers $c^k_{i,j}$ are called *intersection numbers*, or sometimes *structure constants* of $\mathcal{W}$.

Assume that $|\Omega| = n$, and let us put $\Omega = \{1, \ldots, n\}$. To each *basic graph* $\Gamma_i = (\Omega, R_i)$ we associate its adjacency matrix $A_i = A(\Gamma_i)$. Then the set of *basic matrices* $\{A_i \mid i \in I\}$ may be regarded as a basis of a matrix algebra $\mathcal{H}$ which contains the identity matrix, the all-ones matrix J, and is closed under transposition and Schur-Hadamard multiplication of matrices. Such an algebra is called *coherent*, and the set $\{A_i \mid i \in I\}$ is its *standard basis*. Usually, coherent configurations and coherent algebras are considered as equivalent objects.

The concepts of coherent configurations and coherent algebras were introduced by D. Higman (see e.g. [28]). Similar concepts were introduced independently by B.Ju. Weisfeiler and A.A. Leman, see [52] and also [38] for a historical discussion.

A significant source of coherent configurations appears as follows. Assume that (G, Ω) is a permutation group acting on Ω. For $(\alpha, \beta) \in \Omega^2$ the set $\{(\alpha, \beta)^g \mid g \in G\}$ is a 2-*orbit* of G, where $(\alpha, \beta)^g = (\alpha^g, \beta^g)$. More precisely, it is the 2-*orbit* of G *corresponding to* (α, β). (For a transitive permutation group (G, Ω), many authors prefer the term *orbital* for this set.)

Denoting by 2-Orb(G, Ω) the set of 2-orbits of a permutation group (G, Ω), it is easy to check that $(\Omega$, 2-Orb$(\Omega))$ is a coherent configuration. Coherent configurations that arise in this manner are called *Schurian*, otherwise we call them *non-Schurian*.

An *association scheme* $\mathcal{W} = (\Omega, \mathcal{R})$ (also called a *homogeneous coherent configuration*) is such a coherent configuration in which the diagonal relation Δ_Ω belongs to $\mathcal{R}$. Thus, Schurian association schemes are coming from transitive permutation groups.

A coherent configuration $\mathcal{W}$ is *commutative* if for all $i, j, k \in I$ we have $c^k_{ij} = c^k_{ji}$. It is *symmetric* if $R_i = R_i^T$ for all $i \in I$. It is a well known fact that a symmetric coherent configuration is also commutative, but the converse is not true in general.

To each coherent configuration $\mathcal{W}$ we may assign three groups: Aut$(\mathcal{W})$, CAut$(\mathcal{W})$ and AAut$(\mathcal{W})$. The (combinatorial) *group of automorphisms* Aut$(\mathcal{W})$ consists of the permutations $\phi : \Omega \to \Omega$ which preserve the relations, i.e. $R_i^\phi = R_i$ for all $R_i \in \mathcal{R}$. The *color automorphisms* are permitted to permute the relations from $\mathcal{R}$ and they constitute the group CAut$(\mathcal{W})$. An *algebraic automorphism* is a bijection $\psi : \mathcal{R} \to \mathcal{R}$ that satisfies $c^k_{ij} = c^{k^\psi}_{i^\psi j^\psi}$ and these form the group AAut$(\mathcal{W})$. It is easy to verify that Aut$(\mathcal{W})$ is a normal subgroup of CAut$(\mathcal{W})$, and that the quotient group CAut$(\mathcal{W})$/Aut$(\mathcal{W})$ embeds naturally into AAut$(\mathcal{W})$.

Let $\mathcal{W}$ be a coherent configuration (briefly CC) and $\mathcal{H}$ the corresponding coherent algebra. If $\mathcal{H}'$ is a coherent subalgebra of $\mathcal{H}$, then the CC $\mathcal{W}'$ corresponding to $\mathcal{H}'$ is called *merging (or fusion)* of CC $\mathcal{W}$. There exists a natural Galois correspondence between mergings of $\mathcal{W}$ and their automorphism groups, see [20].

For each group K of algebraic automorphisms of $\mathcal{W} = (\Omega, \mathcal{R})$ one can define an algebraic merging of $\mathcal{R}$ in the following way. Denote by $\mathcal{R}/K$ the set of orbits of K on $\mathcal{R}$. To each $O \in \mathcal{R}/K$ define O^+ as the union of all relations from O. Then the set of relations $\{O^+ \mid O \in \mathcal{R}/K\}$ forms a CC on Ω. We will call it an *algebraic merging* of $\mathcal{R}$ with respect to K. Note that if $K \leq \mathrm{CAut}(\mathcal{W})/\mathrm{Aut}(\mathcal{W})$ and $\mathcal{W}$ is Schurian, then the resulting merging is Schurian as well.

It is clear from the definitions that an association scheme (briefly AS) $\mathcal{W}$ is Schurian if and only if its rank coincides with the rank of its group of automorphisms $\mathrm{Aut}(\mathcal{W})$.

A few graph-theoretical symbols: K_n, $\mathcal{E}_n$, and $\mathcal{C}_n$ denote the complete graph, empty graph and directed cycle, respectively, all having n vertices. Clearly, $\mathcal{C}_n$ is unique (up to isomorphism). The symbol $m \circ \Gamma$ denotes the disjoint union of m copies of the graph Γ. For many concepts, exploited below, we refer the reader to book [14]. A classical operation of composition of graphs Γ_1 and Γ_2 is also called *wreath product*, denoted by $\Gamma_1 \,\mathrm{wr}\, \Gamma_2$. We are also using the term wreath product for permutation groups. Note that here the so-called orthodox notation $G_1 \wr G_2$ is used for the wreath product of permutation groups (G_1, Ω_1) and (G_2, Ω_2), see again [20].

A number of other used concepts, such as 2-closure of a permutation group, Weisfeiler-Leman stabilization of a color graph, etc., will be discussed in the text in an ad hoc manner later on.

We refer to Section 12.6 for information regarding the used computer packages.

3 Heisenberg group and biaffine plane

Our starting geometric concept is a *biaffine plane* of order n. It is obtained from an affine plane of order n by removal of one class of parallel lines (let us call this removed class *vertical lines*).

Let p be an odd prime, and let $\mathbb{Z}_p$ be the cyclic group of order p. Throughout this text, the set of nonzero elements in $\mathbb{Z}_p$ will be denoted by $\mathbb{Z}_p^*$. Take two copies $\mathcal{P}$ and $\mathcal{L}$ of $\mathbb{Z}_p \times \mathbb{Z}_p$. The first copy $\mathcal{P}$ is none other than the point set of the classical (Desarguesian) affine plane of order p. Each element $P \in \mathcal{P}$ may be identified uniquely with a pair of coordinates of the form $P = [x, y]$. Thus, we refer to the elements of $\mathcal{P}$ as *points*. Let $\mathcal{L}$ be the set of *"non-vertical" lines* in the affine plane, i.e. $\ell \in \mathcal{L}$ if and only if the equation for ℓ may be expressed as $y = k \cdot x + q$ for some $k, q \in \mathbb{Z}_p$. Each line ℓ is determined uniquely by a pair $\ell = (k, q)$. In order to distinguish points and lines we will use square brackets for points and parentheses for lines. We denote the resulting geometry by $\mathcal{B}_p$ (see Figure 1 depicting $\mathcal{B}_3$).

Given a point $P = [x, y]$ and a line $\ell = (k, q)$ of the biaffine plane, we define a *quasidistance* $d : (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P}) \to \mathbb{Z}_p$ by the formulas: $d(P, \ell) = k \cdot x + q - y$ and $d(\ell, P) = y - q - k \cdot x$. Note that d does not define a metric. It is just a vague analogue.

The automorphism group of the classical Desarguesian projective plane of order

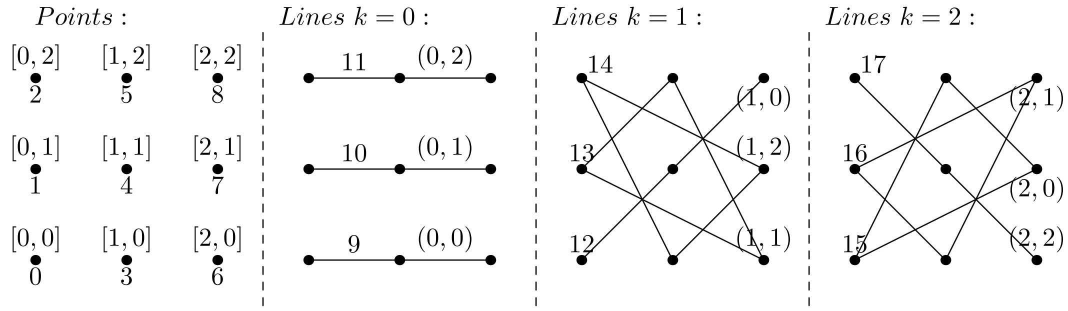

Figure 1: The objects of the biaffine plane $\mathcal{B}_3$.

p acts transitively on the set of projective points. The structure of this group is well-known, see e.g. [33]. Taking this into account, we obtain that the order of the automorphism group $\mathrm{AGL}(2,p) \cong \mathbb{Z}_p^2 \rtimes \mathrm{GL}_2(p)$ of the classical affine plane of order p is $p^2(p^2-1)(p^2-p)$. Now, in turn, the group $\mathrm{AGL}(2,p)$ acts transitively on the set of parallel classes, which has cardinality $p+1$. Finally, we obtain that the group $\mathrm{Aut}(\mathcal{B}_p)$ is of order $p^2(p-1)(p^2-p) = p^3(p-1)^2$.

In what follows, we will be interested in the Sylow p-subgroup of $\mathrm{Aut}(\mathcal{B}_p)$, which clearly has order p^3 and acts transitively on the point set $\mathcal{P}$ of $\mathcal{B}_p$. It turns out that this group may be introduced from two relatively independent points of view.

Let us now consider an action of the group $H = (\mathbb{Z}_p)^2 \rtimes \mathbb{Z}_p$ on the set $\Omega = \mathcal{P} \cup \mathcal{L}$, most conveniently described in terms of generators. To each pair $(a,b) \in \mathbb{Z}_p^2$ we associate a *translation* t_{ab} acting naturally on $\mathcal{P}$ as $[x,y] \mapsto [x+a, y+b]$, while the induced action on $\mathcal{L}$ is $(k,q) \mapsto (k, b+q-ak)$. Of course, the set of all translations forms an Abelian group of order p^2 under composition of permutations and is isomorphic to $\mathbb{Z}_p^2$. Further, let $\varphi : \mathcal{P} \to \mathcal{P}$ be defined by $\varphi : [x,y] \mapsto [x, y-x]$. The corresponding permutation on lines is $(k,q) \mapsto (k-1,q)$. Clearly, φ is a permutation of order p and it is immediate that $\varphi \cdot t_{a,b-a} = t_{a,b} \cdot \varphi$. Our group H above is generated by all translations together with the permutation φ. Note that all elements of H may be expressed in the form $\varphi^u \cdot t_{a,b}$, where a, b, u are suitable elements of $\mathbb{Z}_p$. Moreover, to distinct triples of a, b, u there correspond distinct elements of H. In other words, $H = \langle t_{a,b}, \varphi \rangle$ and $|H| = p^3$. Now set $h_{a,b,u} := \varphi^u \cdot t_{a,b}$. Then multiplication in H is given by $h_{a,b,u} \cdot h_{c,d,v} = h_{a+c, b+d-av, u+v}$.

Observe that the action of H is intransitive on Ω with two orbits $\mathcal{P}$ and $\mathcal{L}$.

Proposition 3.1. *Let p be an odd prime. Then*

(1) the rank of (H,Ω) is $6p-2$;
(2) the 2-orbits of (H,Ω) may be divided into six different types of classes A_i, B_i, C_i, D_i, E_i and F_i, which are characterized by suitable relations between coordinates of objects in Ω.

Proof. Let $P_1 = [x_1, y_1]$, $P_2 = [x_2, y_2] \in \mathcal{P}$ and $\ell_1 = (k_1, q_1)$, $\ell_2 = (k_2, q_2) \in \mathcal{L}$. Then the types of classes are the following:

- $(P_1, P_2) \in A_i \iff x_1 = x_2$ and $y_2 - y_1 = i$, where $i \in \mathbb{Z}_p$ (note that A_0 is the diagonal relation on $\mathcal{P}$);
- $(P_1, P_2) \in B_i \iff x_2 - x_1 = i$, where $i \in \mathbb{Z}_p^*$;
- $(\ell_1, \ell_2) \in C_i \iff k_1 = k_2$ and $q_2 - q_1 = i$, where $i \in \mathbb{Z}_p$ (note that C_0 is the diagonal relation on $\mathcal{L}$);
- $(\ell_1, \ell_2) \in D_i \iff k_2 - k_1 = i$, where $i \in \mathbb{Z}_p^*$;
- $(P_1, \ell_1) \in E_i \iff k_1 \cdot x_1 + q_1 - y_1 = i$, where $i \in \mathbb{Z}_p$ (i.e., those point-line pairs whose quasidistance $d(P_1, \ell_1)$ is i);
- $(\ell_1, P_1) \in F_i \iff y_1 - k_1 x_1 - q_1 = i$, where $i \in \mathbb{Z}_p$ (i.e., those line-point pairs whose quasidistance $d(\ell_1, P_1)$ is i).

To complete the proof, one must verify two things: that the $6p - 2$ relations introduced above indeed form a partition of the set Ω^2, and that each such relation is a 2-orbit of (H, Ω). □

Remark 3.1. One can easily check from its definition that the permutation φ has exactly p fixed points in its action on $\mathcal{P}$, as well as p fixed points in its action on $\mathcal{L}$. Relying on the bijection between 2-orbits of a transitive permutation group and orbits (1-orbits) of the stabilizer of an arbitrary point (see e.g. [20]), the reader can easily deduce that there exists exactly $p + (p - 1)$ 2-orbits of the transitive action $(H, \mathcal{P})$, and similarly for the action $(H, \mathcal{L})$. Observing that there are p 2-orbits of type E_i and p of type F_i, we arrive at the desired amount of $6p - 2$.

Remark 3.2. Reflexive 2-orbits A_0 and C_0 are obviously symmetric, however, all the remaining 2-orbits are antisymmetric. Namely, we obtain that $A_i^T = A_{p-i}$, $B_i^T = B_{p-i}$, $C_i^T = C_{p-i}$, $D_i^T = D_{p-i}$, $E_i^T = F_{p-i}$ and $F_i^T = E_{p-i}$. Here and below, operations on subscripts are considered in $\mathbb{Z}_p$.

We now introduce a group H' from scratch and compare it later with H.

Let

$$V_1 = \mathbb{Z}_p^2 = \{(1, x_1, x_2) \mid x_1, x_2 \in \mathbb{Z}_p\},$$

$$V_2 = (\mathbb{Z}_p^2)^{dual} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ -1 \end{pmatrix} \mid x_1, x_2 \in \mathbb{Z}_p \right\}.$$

Define a natural scalar (dot) product of vectors in V_1 and V_2 by:

$$\left((1, x_1, x_2), \begin{pmatrix} y_1 \\ y_2 \\ -1 \end{pmatrix} \right) = y_1 + x_1 y_2 - x_2.$$

Let

$$H' = \left\{ g_{abc} = \begin{pmatrix} 1 & a & b + ac \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z}_p \right\}.$$

The matrix g_{abc} is invertible, and

$$g_{abc}^{-1} = \begin{pmatrix} 1 & -a & -b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}.$$

Clearly, the set H' together with the operation of matrix multiplication forms a group. The multiplication in H' is given by $g_{abu} \cdot g_{cdv} = g_{a+c,b+d-uc,u+v}$, and exactly this group is well known under the name *Heisenberg group modulo p,* see e.g. [11].

Define an action of H' on $\Omega = V_1 \cup V_2$ by: $x^g = \begin{cases} x \cdot g & \text{if } x \in V_1 \\ g^{-1} \cdot x & \text{if } x \in V_2, \end{cases}$ for all $g \in H'$.

Let us take arbitrary $x_1, x_2, y_1, y_2 \in \mathbb{Z}_p$. The matrix g_{abc}, where $a = y_1 - x_1$, $b = y_2 - x_2$, $c = 0$, sends $(1, x_1, x_2) \in V_1$ to $(1, y_1, y_2) \in V_1$, while g_{cab} sends $(x_1, x_2, -1)^T \in V_2$ to $(y_1, y_2, -1)^T \in V_2$. Thus, H' acts transitively on V_1 and also on V_2.

This action of H' preserves the scalar product: $(x^g, y^g) = (xg, g^{-1}y) = (xgg^{-1}, y) = (x, y)$.

Proposition 3.2. *The groups H and H' are isomorphic.*

Proof. We claim that $\Phi : H \to H'$, $h_{a,b,u} \mapsto g_{a,b+au,-u}$ is a group isomorphism. First, we have

$$\begin{aligned} \Phi(h_{a,b,u} \cdot h_{c,d,v}) = \Phi(h_{a+c,b+d-av,u+v}) &= g_{a+c,b+d-av+(a+c)(u+v),-(u+v)} = \\ &= g_{a,b+au,-u} \cdot g_{c,d+cv,-v} = \Phi(h_{a,b,u}) \cdot \Phi(h_{c,d,v}). \end{aligned}$$

Moreover, Φ is invertible: $\Phi^{-1}(g_{\alpha,\beta,\gamma}) = h_{\alpha,\beta+\alpha\gamma,-\gamma}$. Thus, Φ is a group isomorphism from H to H' as claimed. □

Proposition 3.2 allows us to identify the groups H and H'. This is why we will henceforth assign the notation H to both groups. However, the reader is advised to keep in mind the group H' as it appears in this section. In this fashion we identify elements from V_1 with the point set $\mathcal{P}$ and elements from the set V_2 with the line set $\mathcal{L}$ of the biaffine plane $\mathcal{B}_p$.

4 Schurian master coherent configuration $\mathcal{M}$

Let us consider again the intransitive action (H, Ω) of the Heisenberg group H of order p^3 with two orbits of length p^2. The isomorphism between groups H and H' established in Proposition 3.2 allows to reconsider description of 2-orbits and proof of Proposition 3.1. We briefly outline the necessary arguments and final formulations for the reader's benefit.

Orbits on $V_1 \times V_1$:

Let $g = g_{abc} \in H$ and $P_1 = (1, x_1, x_2)$, $P_2 = (1, y_1, y_2)$, $P_3 = (1, u_1, u_2)$, $P_4 = (1, v_1, v_2) \in V_1$. Then $(P_1, P_2)^g = (P_3, P_4)$ if and only if

$$\begin{aligned} u_1 &= x_1 + a, \\ v_1 &= y_1 + a, \\ u_2 &= c \cdot x_1 + x_2 + b + a \cdot c, \\ v_2 &= c \cdot y_1 + y_2 + b + a \cdot c. \end{aligned}$$

We can see that if (P_1, P_2) and (P_3, P_4) belong to the same orbit, then necessarily $y_1 - x_1 = v_1 - u_1$, $a = u_1 - x_1 = v_1 - y_1$, and $v_2 - u_2 = c(y_1 - x_1) + (y_2 - x_2)$.

(i) If $y_1 - x_1 = 0$, then it is necessary to have $v_2 - u_2 = y_2 - x_2$, and by choosing $c = 0$ and $b = u_2 - x_2$ we are getting a suitable g sending (P_1, P_2) to (P_3, P_4).

(ii) If $y_1 - x_1 = k \in \mathbb{Z}_p^*$, then we can choose $c = k^{-1}(v_2 - u_2 + x_2 - y_2)$, $b = u_2 - cx_1 - x_2 - ac$ and we get $(P_1, P_2)^g = (P_3, P_4)$.

Hence we obtained two types of orbits, say A_k and B_k on $V_1 \times V_1$:

- $(P_1, P_2) \in A_k$ if and only if $y_1 = x_1$ and $y_2 - x_2 = k$, where $k \in \mathbb{Z}_p$.
- $(P_1, P_2) \in B_k$ if and only if $y_1 - x_1 = k$, where $k \in \mathbb{Z}_p^*$.

Three other cases are handled similarly; for details see [24].

Altogether, we get the same six types of orbits of H on $\Omega \times \Omega$ up to the used notation of elements from Ω with 2 and 3 coordinates.

Let us denote by $\mathcal{M} = \mathcal{M}_p = (\Omega, 2-\mathrm{Orb}(H, \Omega))$ the Schurian CC on $2p^2$ vertices with two fibers of size p^2. According to Section 3, it has rank $6p-2$. In what follows, we will call $\mathcal{M}$ the *classical biaffine CC*. In the context of this paper it plays the role of the *master CC*.

We are interested in the intersection numbers of $\mathcal{M}$. In this section we display these numbers with the aid of tables. In each table the superscript is fixed, the symbol in the row indicates the first subscript, and the symbol in the column indicates the second subscript. We are using the *Kronecker's symbol* $\delta_{i,j}$ in order to shorten computations and formulas.

Proposition 4.1. *The tensor of structure constants of the biaffine CC $\mathcal{M}$ is given as follows:*

$c^{A_k}_{r_i,c_j}$	A_j	B_j	F_j
A_i	$\delta_{i+j,k}$	0	0
B_i	0	$p \cdot \delta_{i+j,0}$	0
E_i	0	0	$p \cdot \delta_{i+j,k}$

$c^{C_k}_{r_i,c_j}$	C_j	D_j	E_j
C_i	$\delta_{i+j,k}$	0	0
D_i	0	$p \cdot \delta_{i+j,0}$	0
F_i	0	0	$p \cdot \delta_{i+j,k}$

$c^{B_k}_{r_i,c_j}$	A_j	B_j	F_j
A_i	0	$\delta_{j,k}$	0
B_i	$\delta_{i,k}$	$p \cdot \delta_{i+j,k}$	0
E_i	0	0	1

$c^{D_k}_{r_i,c_j}$	C_j	D_j	E_j
C_i	0	$\delta_{j,k}$	0
D_i	$\delta_{i,k}$	$p \cdot \delta_{i+j,k}$	0
F_i	0	0	1

$c^{E_k}_{r_i,c_j}$	C_j	D_j	E_j
A_i	0	0	$\delta_{i+j,k}$
B_i	0	0	1
E_i	$p \cdot \delta_{i+j,k}$	1	0

$c^{F_k}_{r_i,c_j}$	A_j	B_j	F_j
C_i	0	0	$\delta_{i+j,k}$
D_i	0	0	1
F_i	$p \cdot \delta_{i+j,k}$	1	0

where i, j, k go through all feasible values. All structure constants not displayed here are zero.

Proof (Outline). First observe that for all $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_p^*$, we have $A_i, B_j \subseteq \mathcal{P} \times \mathcal{P}$, $C_i, D_j \subseteq \mathcal{L} \times \mathcal{L}$, $E_i \subseteq \mathcal{P} \times \mathcal{L}$ and $F_i \subseteq \mathcal{L} \times \mathcal{P}$. Thus all structure constants of the form $c^{X_k}_{Y_i,Z_j}$ are zero provided X, Y, Z satisfy any of the following:

- $X \in \{A, B, E\}$ and $Y \in \{C, D, F\}$, or $Y \in \{A, B, E\}$ and $X \in \{C, D, F\}$,
- $X \in \{A, B, F\}$ and $Z \in \{C, D, E\}$, or $Z \in \{A, B, F\}$ and $X \in \{C, D, E\}$,
- $Y \in \{A, B, F\}$ and $Z \in \{C, D, F\}$, or $Y \in \{C, D, E\}$ and $Z \in \{A, B, E\}$.

The reason is simple: in these cases the composition of relations Y_i and Z_j is either impossible, or giving a relation disjoint to X_k. This observation is crucial in order to understand that those structure constants, which in principle may be nonzero, are covered just by the six kinds of tables presented above. Simple algebraic manipulations with coordinates lead us to the following:

$$c^{A_k}_{A_i,A_j} = c^{C_k}_{C_i,C_j} = \delta_{i+j,k}, \quad c^{A_k}_{B_i,B_j} = c^{C_k}_{D_i,D_j} = p \cdot \delta_{i+j,0}, \quad c^{B_k}_{A_i,B_j} = c^{D_k}_{C_i,D_j} = \delta_{j,k},$$

$$c^{B_k}_{B_i,A_j} = c^{D_k}_{D_i,C_j} = \delta_{i,k}, \text{ and } c^{B_k}_{B_i,B_j} = c^{D_k}_{D_i,D_j} = p \cdot \delta_{i+j,k}.$$

Computation of the remaining structure constants requires a bit more sophistication. These may be determined by counting with introduced coordinates; see [24] for more details. □

5 Case $p = 3$ on 18 points from scratch

In this section we are trying to arrange a self-contained consideration restricted to the smallest case $p = 3$. This will make it possible to consider with enough details results of some computer aided computations. Also we will involve the reader with a few nice small structures. We are using the concept of WL-closure without explanation; for details see Section 8.

5.1 Computations with COCO

One of the advantages of the case $p = 3$ is that here all necessary computations can be done, using the initial version of COCO, as it was developed and presented in [19].

Let us label all elements of the set $\mathcal{P}$ lexicographically by numbers from $\{0, 1, \ldots, 8\}$, while elements of $\mathcal{L}$ by numbers $\{9, \ldots, 17\}$ (cf. Figure 1). In this notation for the Heisenberg group $H = H(3)$ we obtain three natural generators, thus $H = \langle h_1, h_2, h_3\rangle$. Here

$$\begin{aligned} h_1 &= (0,3,6)(1,4,7)(2,5,8)(12,14,13)(15,16,17),\\ h_2 &= (0,1,2)(3,4,5)(6,7,8)(9,10,11)(12,13,14)(15,16,17),\\ h_3 &= (3,4,5)(6,8,7)(9,12,15)(10,13,16)(11,14,17). \end{aligned}$$

We construct a color graph with two orbits of length 9 and of rank 16. The representatives of 2-orbits $R_0, R_1, R_2, R_{11}, R_{12}, R_{13}$ of valency 1 are $(0,0)$, $(0,1)$, $(0,2)$, $(9,9)$, $(9,10)$, $(9,11)$, respectively. The remaining ten 2-orbits all have valency 3 with representatives $(0,3)$, $(0,6)$, $(0,9)$, $(0,10)$, $(0,11)$, $(9,0)$, $(9,1)$, $(9,2)$, $(9,12)$, $(9,15)$. The latter ten 2-orbits are R_i for $i = 3,4,5,6,7,8,9,10,14,15$, respectively. Orbits R_0 and R_{11} are reflexive, while all other 2-orbits are anti-symmetric with pairing $\{R_1, R_2\}$, $\{R_3, R_4\}$, $\{R_5, R_8\}$, $\{R_6, R_{10}\}$, $\{R_7, R_9\}$, $\{R_{12}, R_{13}\}$, $\{R_{14}, R_{15}\}$.

COCO returns a list of 34 proper mergings, which are ASs. For each merging we obtain its rank and basic relations. Finally, COCO returns for each merging $\mathcal{X}$ the order of its group of automorphisms $\mathrm{Aut}(\mathcal{X})$; rank of this group; whether $\mathrm{Aut}(\mathcal{X})$ is transitive or not; subdegrees in transitive case.

Traditional COCO does not distinguish isomorphic ASs. Thus, for this job we were using COCO II, though in this case the result could be obtained by hand.

#	Merging	rank	group order	gr.rank	valencies	Schurian	size
0	starting	16	27	16		Yes	1
1	(0,11)(1,13)(2,12)(3,14)(4,15)(5,8)(6,10)(7,9)	8	54	8	$1^3, 3^5$	Yes	2
2	(0,11)(1,12)(2,13)(3,14)(4,15)(5,8)(6,9)(7,10)	8	27	16	$1^3, 3^5$	No	6
3	(0,11)(1,12)(2,13)(5,8)(3,4,14,15)(6,9)(7,10)	7	108	7	$1^3, 3^3, 6$	Yes	3
4	(0,11)(1,13)(2,12)(5,8)(3,4,14,15)(6,10)(7,9)	7	108	7	$1^3, 3^3, 6$	Yes	1
5	(0,11)(1,12)(2,13)(3,14)(4,15)(5,6,7,8,9,10)	6	13122	6	$1^3, 3^2, 9$	Yes	4
6	(0,11)(1,2,12,13)(3,14)(4,15)(5,8)(6,7,9,10)	6	54	8	$1, 2, 3^3, 6$	No	6
7	(0,11)(1,2,12,13)(7,9)(3,4,14,15)(5,6,8,10)	5	216	5	$1, 2, 3, 6^2$	Yes	3
8	(0,11)(1,2,12,13)(3,14)(4,15)(5,6,7,8,9,10)	5	839808	5	$1, 2, 3^2, 9$	Yes	2
9	(0,11)(1,12)(3,4,14,15)(2,13)(5,6,7,8,9,10)	5	52488	5	$1^3, 6, 9$	Yes	2
10	(1,2,12,13)(3,4,14,15)(0,11)(5,6,7,8,9,10)	4	3359232	4	$1, 2, 6, 9$	Yes	1
11	(0,11)(1,12)(2,13)(3,4,5,6,7,8,9,10,14,15)	4	524880	4	$1^3, 15$	Yes	2
12	(0,11)(1,2,12,13)(3,4,5,6,7,8,9,10,14,15)	3	33592320	3	$1, 2, 15$	Yes	1
13	(0,11)(5,6,7,8,9,10)(1,2,3,4,12,13,14,15)	3	263363788800	3	$1, 8, 9$	Yes	1

Table 1: AS mergings for $p = 3$.

Table 1 shows 13 isomorphism classes of ASs mergings; their rank; order of automorphism group; rank of the group; valencies; size of the isomorphism class. Two of the classes present non-Schurian ASs (in this case the rank of group is larger than the rank of AS). Note that the starting CC $\mathcal{M}$, which is counted in Table 1 as class #0, as well as the trivial AS of rank 2 are not included in the list of mergings obtained with the aid of COCO. We denote the mergings as they appear in the output from

COCO. For example $(1, 2)$ displays merging of relations R_1 and R_2.

Below we will explain (interpret) the significant part of data obtained by COCO.

5.2 Basic graphs of $\mathcal{M}$

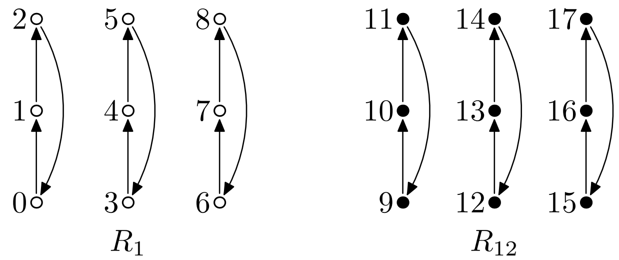

Figure 2: The digraphs $(R_1, \mathcal{P})$ and $(R_{12}, \mathcal{L})$.

There are two isomorphic graphs of valency 1: $(R_1, \mathcal{P})$ and $(R_{12}, \mathcal{L})$, depicted on Figure 2. Another two isomorphic basic graphs are transposed to them: $R_2 = R_1^T$ and $R_{13} = R_{12}^T$. There are two isomorphic graphs of valency 3: $(R_3, \mathcal{P})$ and $(R_{14}, \mathcal{L})$, depicted below. Here the triple arrow in Figure 3 substitutes 9 directed arcs from one 3-element subset of vertices to another such subset. Similarly, here $R_4 = R_3^T$ and $R_{15} = R_{14}^T$.

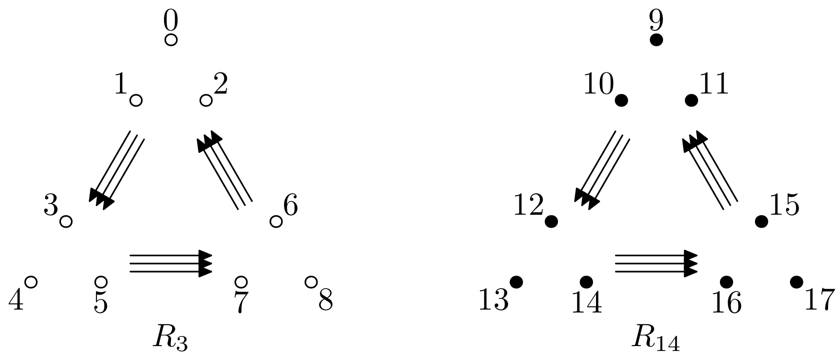

Figure 3: The digraphs $(R_3, \mathcal{P})$ and $(R_{14}, \mathcal{L})$.

All six bipartite graphs between sets $\mathcal{P}$ and $\mathcal{L}$ graphs are isomorphic. In each graph vertices from one part have the common out-valency equal to 3, while vertices in the other part have common in-valency, also equal to 3. First, we start from the diagram of graph (R_5, Ω), where directed Hamiltonian cycle (for weakly connected digraph) is visible, see Figure 4. The opposite graph $R_8 = R_5^T$ is also clear from this diagram. We depict one more diagram of (R_5, Ω) simultaneously, see Figure 5. In the same style as Figure 5 we depict diagram of (R_6, Ω) on Figure 6. We are not presenting the remaining diagram of R_7, because $R_7 = (\mathcal{P} \times \mathcal{L}) \setminus (R_5 \cup R_6)$.

Clearly, $\mathrm{Aut}(R_1, \mathcal{P}) \cong S_3 \wr \mathbb{Z}_3$ is a group of order $6 \cdot 3^3 = 162$. $\mathrm{Aut}(R_3, \mathcal{P}) \cong \mathbb{Z}_3 \wr S_3$ is a group of order $3 \cdot 6^3 = 648$. The group $Y = \mathrm{Aut}(R_5, \Omega)$ contains the subgroup

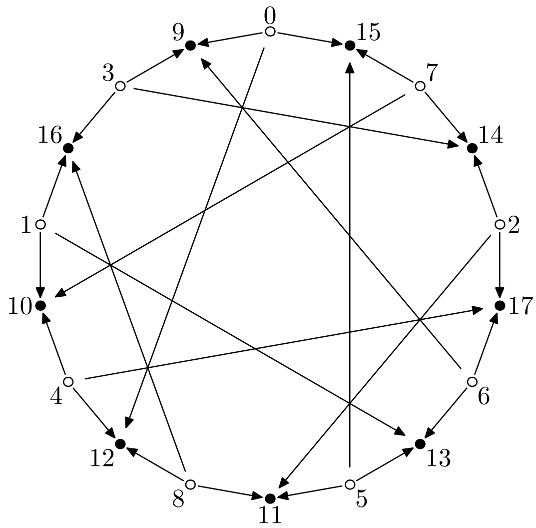

Figure 4: Digraph (R_5, Ω).

H and is transitive on $\mathcal{P}$. Due to the permutation h_3 we claim that the point-wise stabiliser $Y_{0,1,2}$ has order at least 3. Let us find involutions in Y_0. Claiming that all the neighbours of 0 remain on the place, we obtain

$$t_1 = (1, 2)(3, 6)(4, 8)(5, 7)(10, 11)(13, 14)(16, 17).$$

By allowing transposition of 9 and 15 we obtain

$$t_2 = (1, 2)(3, 5)(6, 7)(9, 15)(10, 17)(11, 16)(13, 14).$$

The permutation $t_3 = t_1t_2 = t_2t_1$ is also an involution, thus the index of $Y_{0,1,2}$ is at least 4 as a subgroup of Y_0, and therefore the order of Y is at least $27 \cdot 4 = 108$. In fact, $|Y| = 108$.

5.3 Discussion of the list of mergings: eight evident cases

We start from eight more or less evident Schurian cases with rank in range between 3 to 6 and relatively large automorphism group. The main information is organized in Table 2. For each AS we consider, we provide its number according to Table 1. We extract one concrete representative of ASs from the corresponding isomorphism class of ASs. In case #9 we obtain classical rank 5 DRG, generated by the incidence graph of $\mathcal{B}_3$. In the other 7 cases, the group appearing is explained in terms of wreath products of suitable cyclic and symmetric groups.

5.4 Three more Schurian ASs of ranks 7 and 8 through the scope of Pappus graph

Our starting object now is the incidence graph of $\mathcal{B}_3$, which is well-known under the name *Pappus graph* P. The pictorial presentation of P is still very helpful. We

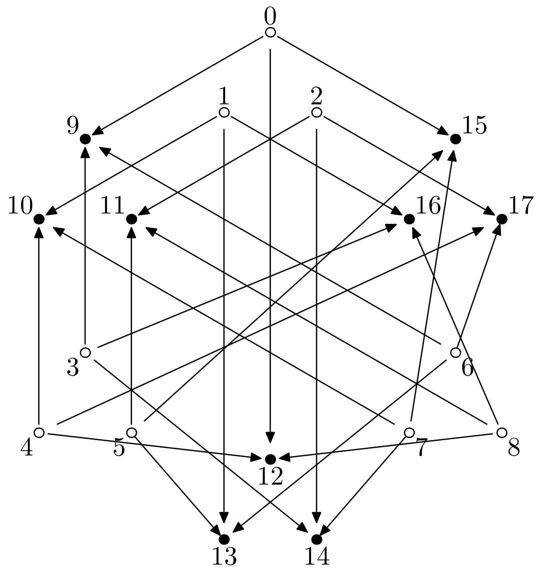

Figure 5: Digraph (R_5, Ω).

suggest the reader looks at the diagram of P as it is depicted in Figures 4 and 5, where arrows on edges are omitted.

The following significant features of P are visible: this is a connected, bipartite cubic graph of girth 6 and diameter 4. We are also aware that P is vertex- and arc-transitive. Moreover, it is antipodal with 6 blocks of size 3. A naive way to express this is to consider the cyclic group $\langle h_2 \rangle$ of order 3, which preserves each of the antipodal blocks. Considering these blocks as metavertices, we observe a quotient graph $P/\mathcal{E}_3$, which is isomorphic to $K_{3,3}$. This visual observation can be translated into algebraic language. Namely, it turns out that $\langle h_2 \rangle \cong \mathbb{Z}_3$ is a normal subgroup of $\mathrm{Aut}(P)$. Thus, we get that $\mathrm{Aut}(P) \cong \mathbb{Z}_3.\mathrm{Aut}(K_{3,3}) = \mathbb{Z}_3.(S_2 \wr S_3)$. The presentation so obtained is a non-split extension. The group $\mathrm{Aut}(K_{3,3})$ of order 72 has three subgroups of index 2. They are in bijective correspondence with subgroups of index 2 in $\mathrm{Aut}(P)$, also presented in the form $\mathbb{Z}_3.K$, where K is a suitable group of order 36. Two of these normal subgroups of order 108 are now of special interest to us. In principle, they can be constructed by hand, giving the groups $\langle h_1, h_2, h_3, t_4, t_5 \rangle$ and $\langle h_1, h_2, h_3, t_5, t_6 \rangle$, where

$$\begin{aligned} t_4 &= (0,9)(1,11)(2,10)(3,15)(4,17)(5,16)(6,12)(7,14)(8,13), \\ t_5 &= (3,6)(4,7)(5,8)(12,15)(13,16)(14,17), \\ t_6 &= (0,9)(1,10)(2,11)(3,5,6,12)(4,16,7,13)(5,17,8,14). \end{aligned}$$

Both presented groups are transitive of rank 7 with subdegrees 1^3, 3^3, 6. Their centralizer algebras provide two Schurian ASs (#4 and #5).

Note the essential difference in combinatorial properties of the detected schemes. In one case we are getting three symmetric bipartite graphs between $\mathcal{P}$ and $\mathcal{L}$. In another case one of the graphs (the copy of P, considered here) is symmetric, while two other pairs are antisymmetric. Finally, we consider the (second) group of order

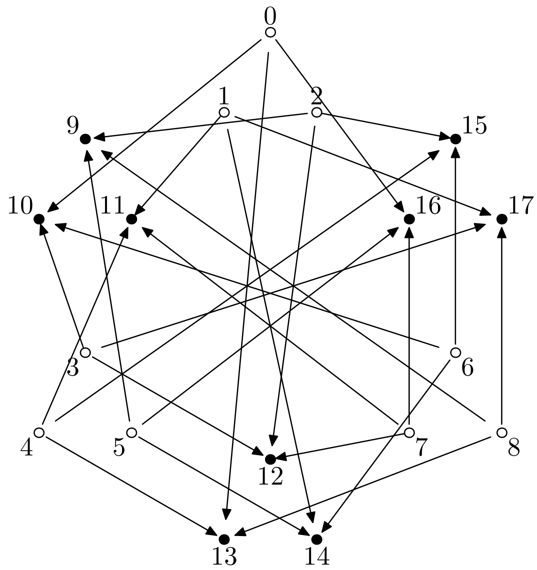

Figure 6: Digraph (R_6, Ω).

108, which has three symmetric 2-orbits between $\mathcal{P}$ and $\mathcal{L}$. It turns out that it has two subgroups of order 54. Each of these subgroups has centralizer algebra of rank 8 with subdegrees 1^3, 3^5. These two centralizer algebras form one isomorphism class of Schurian ASs, which we denote by #1. The only symmetric non-reflexive relations in it are the 2-orbits between $\mathcal{P}$ and $\mathcal{L}$.

This rank 8 group of order 54 appears as $\langle h_1, h_2, h_3, t_7 \rangle$, where

$$t_7 = (0,9)(1,11)(2,10)(3,12)(4,14)(5,13)(6,15)(7,17)(8,16).$$

Remark 5.1. The fact that the group $\langle h_2 \rangle$ appears to be the (unique) normal subgroup of order 3 in $\mathrm{Aut}(P)$ is quite crucial in our consideration. The reason is that the antipodal system of blocks corresponds to the vertical lines removed from $AG(2,3)$, while $\langle h_2 \rangle$ stabilizes this system.

5.5 Heisenberg graph on 18 vertices

Let us consider a representative $\mathcal{X}_2$ of the class of non-Schurian ASs, numbered above by #2. (Note that the labeling corresponds to the order, in which mergings are listed by COCO.)

Here the union of relations $R_5 \cup R_8$ provides the edge set of our copy of the Pappus graph, and all other relations are antisymmetric. There are six ASs in this isomorphism class. Analysis of the lattice of the mergings shows that $\mathcal{X}_2 = \mathrm{W}(R_3 \cup R_{14}, R_6 \cup R_9)$. In fact, it is easy to check that the union $R_3 \cup R_{14} \cup R_6 \cup R_9$ still generates $\mathcal{X}_2$. (We use the WL-closure, see Section 8.)

Let us now describe the group $G = \mathrm{Aut}(\mathcal{X}_2) = \mathrm{Aut}(R_3 \cup R_{14}) \cap \mathrm{Aut}(R_6 \cup R_9)$. Also $G \leq \mathrm{Aut}(P)$. Thus $\mathbb{Z}_3 = \langle h_2 \rangle$ is a normal subgroup of G. Recall that $H \leq G$. Because G is also a subgroup of $\mathrm{Aut}(R_1 \cup R_{12})$, the orbits of $\mathbb{Z}_3$ form a partition of

#	Generators	Formula	Group	Order of group
5	(1,12) (3,14)	$6 \circ \mathcal{C}_3$ $2 \circ (\mathcal{C}_3 \,\mathrm{wr}\, \mathcal{E}_3)$	$S_2 \wr (\mathbb{Z}_3 \wr \mathbb{Z}_3)$	$2 \cdot (3^4)^2$
7	(7,9)	$\mathrm{Inc}(\mathcal{B}_3)$	$((E_9.\mathbb{Z}_2).S_3).\mathbb{Z}_2$	216
8	(1,2,12,13) (3,14)	$6 \circ K_3$ $2 \circ (\mathcal{C}_3 \,\mathrm{wr}\, \mathcal{E}_3)$	$S_2 \wr (\mathbb{Z}_3 \wr S_3)$	$2 \cdot (3 \cdot 6^3)^2$
9	(3,4,14,15) (1,12)	$2 \circ (K_3 \,\mathrm{wr}\, \mathcal{E}_3)$ $6 \circ \mathcal{C}_3$	$S_2 \wr (S_3 \wr \mathbb{Z}_3)$	$2 \cdot (6 \cdot 3^3)^2$
10	(1,2,12,13) (3,4,14,15)	$6 \circ K_3$ $2 \circ (K_3 \,\mathrm{wr}\, \mathcal{E}_3)$	$S_2 \wr (S_3 \wr S_3)$	$2 \cdot (6 \cdot 6^3)^2$
11	(1,12)	$6 \circ \mathcal{C}_3$	$S_6 \wr \mathbb{Z}_3$	$6! \cdot 3^6$
12	(1,2,12,13)	$6 \circ K_3$	$S_6 \wr S_3$	$6! \cdot 6^6$
13	(5,6,7,8,9,10)	$2 \circ K_9$	$S_2 \wr S_9$	$2 \cdot (9!)^2$

Table 2: Explanations of some AS mergings for $p = 3$.

Ω, which is preserved by G. Therefore $G = K \cdot Q$, where K is the stabilizer of the latter partition, Q a suitable subgroup of G. Taking into account the above analysis of $\mathrm{Aut}(P)$, we obtain that $K = \mathbb{Z}_3 = \langle h_2 \rangle$. Looking at the diagram of $R_3 \cup R_{14}$, we conclude that $Q \leq \mathrm{Aut}(2 \circ \mathcal{C}_3) = S_2 \wr \mathbb{Z}_3$, that is Q has order at most 18. On the other hand, $G \geq H$, thus Q has order at least 9. It remains to figure out whether G contains an involution, or not.

We need to depict the directed regular graph $(R_6 \cup R_9, \Omega)$ of valency 3. In principle, it can be considered as gluing two graphs, one of which visible in Figure 6. Instead of such a presentation we prefer to depict a special decomposition of this graph, which is given in Figure 7. Its quotient with respect to the partition $6 \circ \mathcal{E}_3$, consisting of 6 cocliques of order 3, is $K_{3,3}$, (cf. Remark 5.1). Moreover, to each of the nine undirected edges of $K_{3,3}$ a corresponding (induced) subgraph of $(R_6 \cup R_9, \Omega)$ is a directed cycle $\mathcal{C}_6$. A system of such 9 directed cycles is presented in Figure 7 and it is identified with the graph itself. The hexagons are labeled by letters from a to i.

Proposition 5.1. *The group $G = \mathrm{Aut}(\mathcal{X}_2)$ does not contain any involutions.*

Proof. Suppose that $|G| = 54$ and $y \in G$ is an involution. If G is intransitive, then y has a fixed point, say, 0. This implies that one of the hexagons a, b, c remains in place. Any involution preserving a directed hexagon acts semi-regularly on its vertices, a contradiction. Thus, the group G acts transitively on Ω. Again, we conclude that y preserves one of the hexagons, say a. Thus, the action of y on the vertex set of a is $(0, 9)(1, 10)(2, 11)$. Considering this action on the remaining elements of Ω and on hexagons, we obtain $y = (0, 9)(1, 10)(2, 11)(3, 12)(4, 13)(5, 14)(6, 15)(7, 16)(8, 17)$, while $\tilde{y} = (a)(b, d)(c, g)(e)(f, h)(i)$. Here $\tilde{y}$ is the induced action of y on the hexagons. Trying to get the induced action of $\tilde{y}$ on hexagon e, we have to claim that it reverses arc $(3, 12)$ in e. Again a contradiction. □

Corollary 5.2. *The following hold:*

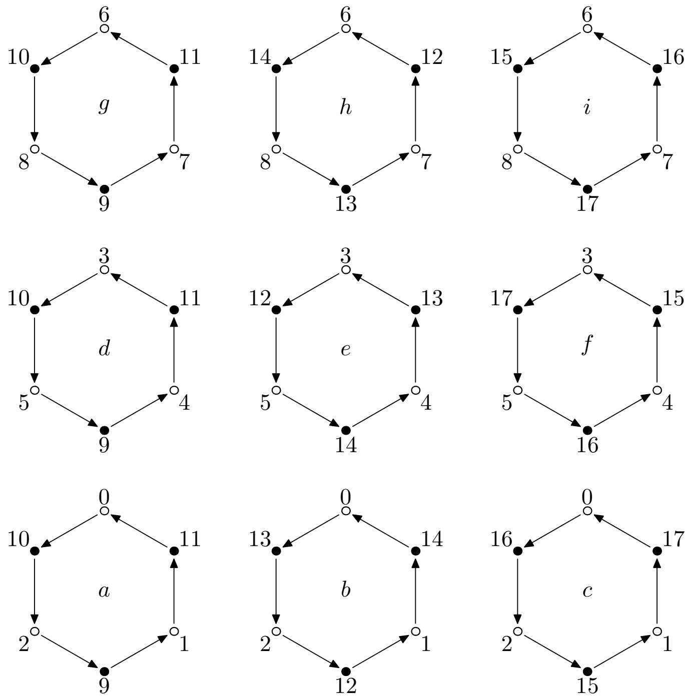

Figure 7: Nine directed hexagons, partitioning the arcs of $(R_6 \cup R_9, \Omega)$.

(a) $G = \mathrm{Aut}(\mathcal{X}_2) = H$ *is a group of order* 27.
(b) G *acts intransitively on* Ω.
(c) $\mathcal{X}_2$ *is a non-Schurian AS.*

Proof. Parts (a) and (b) follow immediately. To prove (c) without the use of a computer, the reader has to conclude that $\mathcal{X}_2 = \mathrm{W}(R_3 \cup R_{14}, R_6 \cup R_9)$. □

Remark 5.2. We will call the directed color graph $\mathcal{H} = ((R_3 \cup R_{14}, R_6 \cup R_9), \Omega)$ *the Heisenberg graph on* 18 *points.* As was explained, it can be also regarded as a usual directed graph. Its advantage is that it generates a non-Schurian AS#2, serving as a combinatorial object, whose full symmetry is described by the Heisenberg group of order 27.

5.6 Non-Schurian AS#6

To consider the remaining case of non-Schurian AS#6, we first take another glance at the Schurian AS#1. Looking at the lattice of mergings of $\mathcal{M}$ we conclude that

$$AS\#1 = \mathrm{W}(R_1 \cup R_{13}, R_3 \cup R_{14}, R_6 \cup R_{10}).$$

Now we present a merging scheme $\mathcal{X}_6$, a representative of class #6, as

$$\{\{R_0, R_{11}\}, \{R_1, R_2, R_{12}, R_{13}\}, \{R_3, R_{14}\}, \{R_4, R_{15}\}, \{R_5, R_8\}, \{R_6, R_7, R_9, R_{10}\}\}.$$

Clearly, $\mathcal{X}_6$ is also a merging of the above copy of AS#1. Therefore $\mathrm{Aut}(\mathcal{X}_6)$ contains a group of order 54. Once more, we consider WL-stabilisation and obtain that

$$\mathcal{X}_6 = \mathrm{W}(R_3 \cup R_{14}, R_6 \cup R_7 \cup R_9 \cup R_{10}).$$

Figures 3 and 7 provide visualization of both the necessary graphs. Indeed, the relation $\psi = R_6 \cup R_7 \cup R_9 \cup R_{10}$ is decomposed into 9 undirected hexagons on Figure 7 (just disregard the direction of arcs on the hexagons). This allows us to check once more that $\mathrm{Aut}(\mathcal{X}_6) \geq \langle H, t_7 \rangle$. Indeed, t_7 preserves relation $R_3 \cup R_{14}$. On the other hand, $\bar{t}_7 = (\bar{a})(\bar{b}, \bar{d})(\bar{c}, \bar{g})(\bar{e})(\bar{f}, \bar{h})(\bar{i})$. Here $\bar{x}$ is the symmetrization of hexagon x, and $\bar{t}_7$ is the induced action of t_7 on such hexagons.

Proposition 5.3. *Let $\mathcal{X}_6 = \mathrm{W}(R_3 \cup R_{14}, R_6 \cup R_7 \cup R_9 \cup R_{10})$. Then*

(a) the group $G = \mathrm{Aut}(\mathcal{X}_6)$ has order 54;
(b) the WL-closure $\mathcal{X}_6$ is an AS of rank 6, which is obtained via merging of $\mathcal{M}$:

$$\{\{R_0, R_{11}\}, \{R_1, R_2, R_{12}, R_{13}\}, \{R_3, R_{14}\}, \{R_4, R_{15}\}, \{R_5, R_8\}, \{R_6, R_7, R_9, R_{10}\}\};$$

(c) $\mathcal{X}_6$ is a non-Schurian AS.

Proof. We already know that G acts transitively on Ω. Let us consider stabilizer G_0 of element $0 \in \Omega$ in G. Assume that $R_6 \cup R_7 \cup R_9 \cup R_{10}$ is a 2-orbit of (G, Ω). We take $(0, 10) \in R_6$ and $(0, 11) \in R_7$. Then there exists a permutation $\varphi \in G_0$ such that $10^\varphi = 11$. Because $0^\varphi = 0$ and φ preserves $R_3 \cup R_{14}$, we obtain that φ preserves $\Omega_1 = \{3, 4, 5\}$, $\Omega_2 = \{6, 7, 8\}$ and $\Omega_3 = \{1, 2\}$. Because G is a subgroup of $\mathrm{Aut}(R_5 \cup R_8)$, we obtain that φ also preserves $\Omega_5 = \{9, 12, 15\}$ and $\Omega_6 = \{10, 11, 16, 17, 14, 13\}$. Looking at the symmetrization of diagram on Figure 7, we step by step finish the construction of φ,

$$\varphi = (0)(1, 2)(3)(4, 5)(6)(7, 8)(9)(10, 11)(12)(13, 14)(15)(16, 17).$$

However, the resulting permutation does not preserve $\bar{e}$, nor any of $\bar{f}$, $\bar{h}$ or $\bar{i}$. This contradiction proves (c).

To prove (a) we have to check that the first group of order 54 coincides with $\mathrm{Aut}(\mathcal{X}_6)$. Note that now, in addition, $\mathrm{Aut}(\mathcal{X}_6)$ also preserves the relation $R_6 \cup R_{10}$.

The proof of (b) (as was mentioned above) is again a trivial exercise in the use of WL-stabilization. □

6 Automorphism groups related to $\mathcal{M}$

In this section we describe three main groups attributed to the master CC $\mathcal{M}$.

Lemma 6.1. *For the graphs (on vertex set $\mathcal{P}$) defined by relations A_i (for each $i \in \{1, \ldots, p-1\}$) the following hold:*

(a) They are isomorphic to $p \circ \mathcal{C}_p$, that is to disjoint union of p copies of $\mathcal{C}_p$.
(b) Their automorphism group is of order $p! \cdot p^p$ and isomorphic to $S_p \wr \mathbb{Z}_p$.

Similar results hold for the graphs defined by relations C_i on the vertex set $\mathcal{L}$. □

Lemma 6.2. *For the graphs (on vertex set $\mathcal{P}$) defined by relations B_i the following hold:*

(a) They are isomorphic to $\mathcal{C}_p$ wr $\mathcal{E}_p$.
(b) Their group of automorphisms is of order $p \cdot (p!)^p$ and isomorphic to $\mathbb{Z}_p \wr S_p$.

Similar results hold for graphs D_i with vertex set $\mathcal{L}$. □

Lemma 6.3. *The automorphism group of the CC $\mathcal{M}$ restricted to the vertex set $\mathcal{P}$ is isomorphic to the group $\mathbb{Z}_p \wr \mathbb{Z}_p$ of order p^{p+1}. Similar result holds after the restriction onto the vertex set $\mathcal{L}$.*

Proof. It is convenient to regard elements from $\mathcal{P}$ as points of a grid with p horizontal lines and p vertical lines. Restriction of $\mathcal{M}$ to each line coincides with (thin) AS $2-\mathrm{Orb}(\mathbb{Z}_p, \mathbb{Z}_p)$. Clearly, its automorphism group is $\mathbb{Z}_p$. Restriction to the vertical lines has similar properties. It is easy to check that the wreath product $\mathbb{Z}_p \wr \mathbb{Z}_p$ preserves all basic elements of $\mathcal{M}$ on $\mathcal{P}$. This group is a Sylow p-subgroup of the intersection of $S_p \wr \mathbb{Z}_p$ with $\mathbb{Z}_p \wr S_p$. The 2-orbits of $\mathbb{Z}_p \wr \mathbb{Z}_p$ have valencies 1 and p and coincide with the graphs defined by A_i and B_i. Thus, the resulting group is nothing else but the 2-closure $(\mathbb{Z}_p \wr \mathbb{Z}_p)^{(2)}$. According to [55], the latter group is again a p-group, which has order p^{p+1} due to the above-mentioned arguments.

The proof works similarly for the restriction on $\mathcal{L}$. □

Theorem 6.4. *The combinatorial automorphism group* $\mathrm{Aut}(\mathcal{M})$ *coincides with H.*

Proof. The Schurian CC $\mathcal{M}$ is formed by the 2-orbits of (H, Ω). Therefore $\mathrm{Aut}(\mathcal{M}) \geq H$. The group $\mathrm{Aut}(\mathcal{M})$ is an intransitive group of degree $2p^2$ with two orbits of length p^2. Therefore it is a subgroup of the direct product of two groups of degree p^2, namely, of the automorphism groups of the restriction of $\mathcal{M}$ on the sets $\mathcal{P}$ and $\mathcal{L}$. According to Lemma 6.3, we obtain that $|\mathrm{Aut}(\mathcal{M})| \leq p^{2(p+1)}$.

On the other hand, the group $\mathrm{Aut}(\mathcal{M})$ preserves each of the basic graphs corresponding to relations E_i and F_i. Any such graph is a (directed) incidence graph of the classical biaffine plane $\mathcal{B}_p$. As it was demonstrated in Section 3, $|\mathrm{Aut}(\mathcal{B}_p)| = p^3(p-1)^2$. This, together with the above inequality for $|\mathrm{Aut}(\mathcal{M})|$, implies that $|\mathrm{Aut}(\mathcal{M})| \leq p^3$. Since $|H| = p^3$, we are done. □

Remark 6.1. A more straightforward proof will be presented in the next section.

Now, we will describe the group $\mathrm{AAut}(\mathcal{M})$. Let us start with the following permutations on the set of relations of the CC $\mathcal{M}$. (We assume that ω is a primitive element of $\mathbb{Z}_p^*$.)

$$\begin{aligned}
g_1 &= (A_0, C_0)(E_0, F_0) \prod_{i=1}^{p-1} (A_i, C_i)(B_i, D_i)(E_i, F_i),\\
g_2 &= (E_0, E_{p-1}, E_{p-2}, \ldots, E_2, E_1)(F_0, F_1, F_2, \ldots, F_{p-1}),\\
g_3 &= (A_1, A_\omega, A_{\omega^2}, \ldots, A_{\omega^{p-2}})(C_1, C_\omega, \ldots, C_{\omega^{p-2}})(E_1, E_\omega, \ldots, E_{\omega^{p-2}})(F_1, F_\omega, \ldots, F_{\omega^{p-2}}),\\
g_4 &= (B_1, B_\omega, B_{\omega^2}, \ldots, B_{\omega^{p-2}}),\\
g_5 &= (D_1, D_\omega, D_{\omega^2}, \ldots, D_{\omega^{p-2}}).
\end{aligned}$$

The permutation g_1 is an involution corresponding to the duality between points and lines in $\mathcal{B}_p$ (it interchanges their roles); g_2 is of order p, while the remaining permutations are of order $p-1$.

Theorem 6.5. *The group* $\mathrm{AAut}(\mathcal{M})$ *is of order* $2p(p-1)^3$ *and*

$$\mathrm{AAut}(\mathcal{M}) \cong \langle g_1, g_2, g_3, g_4, g_5\rangle \cong (\mathbb{Z}_2 \wr \mathbb{Z}_{p-1}) \times \mathrm{AGL}(1,p).$$

Proof. First, we have to check that each of the permutations $g_1, \dots, g_5$ preserves the tensor of structure constants. Thus, we have that $\langle g_1, g_2, g_3, g_4, g_5\rangle \leq \mathrm{AAut}(\mathcal{M})$. In order to see that $\mathrm{AAut}(\mathcal{M}) \cong \langle g_1, g_2, g_3, g_4, g_5\rangle$ it is sufficient to compute and compare the orders of $\langle g_1, g_2, g_3, g_4, g_5\rangle$ and $\mathrm{AAut}(\mathcal{M})$. The order of $\mathrm{AAut}(\mathcal{M})$ may be determined by repeated application of the orbit-stabilizer lemma. A crucial issue on this way is to check that if the basic relations E_0, B_1, D_1 and A_1 are fixed under the action of an algebraic automorphism ψ, then all other basic graphs are preserved, as well. In other words, ψ is the identity element of $\mathrm{AAut}(\mathcal{M})$.

Clearly, $\{g_2, g_3\}$ is a standard set of generators for the affine linear group over the finite field $\mathbb{F}_p$ of order p, that is, $\langle g_2, g_3\rangle \cong \mathrm{AGL}(1,p)$. Also it is easy to see that $\langle g_4, g_5\rangle \cong \mathbb{Z}_{p-1}^2$ and $\langle g_1, g_4, g_5\rangle \cong \mathbb{Z}_{p-1}^2 \rtimes_\phi \mathbb{Z}_2$, where ϕ interchanges g_4 with g_5. A more careful inspection of the semidirect product $\mathbb{Z}_{p-1}^2 \rtimes_\phi \mathbb{Z}_2$ shows that it is the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}_{p-1}$. By routine computation one can further show that $\langle g_1, g_4, g_5\rangle \trianglelefteq \langle g_1, g_2, g_3, g_4, g_5\rangle$, $\langle g_2, g_3\rangle \trianglelefteq \langle g_1, g_2, g_3, g_4, g_5\rangle$, and $\langle g_1, g_4, g_5\rangle \cap \langle g_2, g_3\rangle = \{e\}$. Thus $\langle g_1, g_4, g_5\rangle \times \langle g_2, g_3\rangle \cong \langle g_1, g_2, g_3, g_4, g_5\rangle$, which confirms that $\langle g_1, g_2, g_3, g_4, g_5\rangle \cong (\mathbb{Z}_2 \wr \mathbb{Z}_{p-1}) \times \mathrm{AGL}(1,p)$. Finally, $|\langle g_1, g_2, g_3, g_4, g_5\rangle| = 2p(p-1)^3$. ☐

At this stage we are in position to describe the group $\mathrm{CAut}(\mathcal{M})$.

Proposition 6.6. *Consider the following permutations* h_1, h_2, h_3, h_4 *acting on the set of basic relations of* $\mathcal{M}$*, where*

$$\begin{aligned}
h_1 &= \prod_{i=1}^{(p-1)/2} (B_i, D_i, B_{p-i}, D_{p-i}) \cdot \prod_{i=0}^{p-1} (A_i, C_i)(E_i, F_i),\\
h_2 &= (A_1, A_\omega, A_{\omega^2}, \dots, A_{\omega^{p-2}})(C_1, C_\omega, \dots, C_{\omega^{p-2}})(D_1, D_\omega, D_{\omega^2}, \dots, D_{\omega^{p-2}}) \cdot\\
&\cdot\ (E_1, E_\omega, \dots, E_{\omega^{p-2}})(F_1, F_\omega, \dots, F_{\omega^{p-2}}),\\
h_3 &= (B_1, B_\omega, B_{\omega^2}, \dots, B_{\omega^{p-2}})(D_1, D_{\omega^{p-2}}, D_{\omega^{p-3}}, \dots, D_{\omega^2}, D_\omega),\\
h_4 &= (E_0, E_{p-1}, E_{p-2}, \dots, E_2, E_1)(F_0, F_1, F_2, \dots, F_{p-1}).
\end{aligned}$$

The following hold:

(a) $L = \langle h_1, h_2, h_3, h_4\rangle \cong \mathrm{Di}_{(p-1)/2} \cdot \mathrm{AGL}(1,p)$*;*
(b) $|L| = 2p(p-1)^2$*;*
(c) $\langle h_2, h_4\rangle \cong \mathrm{AGL}(1,p)$*;*
(d) $\langle h_1, h_3\rangle \cong \mathrm{Di}_{(p-1)/2}$*;*

where Di_x *is the* dicyclic group *of order* $4x$*, see e.g. [47]. The formula in* (a) *means factorization of* L *as a product of two disjoint groups.*

Proof. By definition, h_1 is of order 4, h_4 is of order p, while h_2 and h_3 are of order $p-1$. The proof of (c) is evident, since we have a standard set of generators for $\mathrm{AGL}(1,p)$. One can also easily check that $h_3^{p-1}=1$, $h_1^2=h_3^{(p-1)/2}$ and $h_3h_1h_3=h_1$ in order to show (d).

Now, we show that the intersection of the groups $L_1=\langle h_1,h_3\rangle$ and $L_2=\langle h_2,h_4\rangle$ is trivial. By contradiction, suppose that a non-identity element $g\in L$ belongs to $L_1\cap L_2$. For $i\neq 0$ we have orbits $A_i^{L_1}=\{A_i,C_i\}$ and $A_i^{L_2}=\{A_1,\ldots,A_{p-1}\}$, therefore g fixes all relations of type A_i. The same argument works for all relations of type C_i, E_i and F_i. Since relations A_i, C_i and D_i behave similarly in the group L_2, we can also say that all the relations D_i are fixed by g whenever A_i and C_i are fixed. Further, each relation B_i is fixed in L_2, therefore g also fixes these relations. Thus, g fixes all the relations, a contradiction. Hence $|L|\geq 2p(p-1)^2$.

Finally, by routine calculations one obtains $h_2h_1=(h_1h_3)h_2$, $h_2h_3=h_3h_2$, $h_4h_1=h_1h_4$, $h_4h_3=h_3h_4$. This implies that each element of L can be represented as a product of an element from $\mathrm{Di}_{(p-1)/2}$ and an element from $\mathrm{AGL}(1,p)$. Therefore $|L|\leq 2p(p-1)^2$, completing the proof of (a) and (b). □

Remark 6.2. It follows from the previous proof that $h_3=h_1^{-1}h_2h_1h_2^{-1}$, and so $L\cong\langle h_1,h_2,h_4\rangle$.

Proposition 6.7. *The group* L*, introduced in Proposition 6.6, is a subgroup of index* $p-1$ *in the group* $\mathrm{AAut}(\mathcal{M})$*.*

Proof. Observe that $h_4=g_2$, $h_2=g_3g_5$ and $h_1=g_5^{-1}g_1g_4^{-1}$. Compare the orders of L and $\mathrm{AAut}(\mathcal{M})$. □

We are still looking for a more suitable, in a sense canonical description of the group L.

Proposition 6.8. *The group* L *can be presented in the form of a split extension of the dicyclic group of order* $2p(p-1)$ *with the cyclic group of order* $p-1$*, i.e.* $L\cong\mathrm{Di}_{p(p-1)/2}\rtimes\mathbb{Z}_{p-1}$*.*

Proof. Let us define $\mathbb{Z}_{p-1}=\langle h_2\rangle$. The subgroup, which will serve as $\mathrm{Di}_{p(p-1)/2}$, we define as $\langle h_1,h_3h_4\rangle$, since $(h_3h_4)^{p(p-1)}=1$, $(h_3h_4)^{p(p-1)/2}=h_1^2$ and $(h_3h_4)h_1(h_3h_4)=h_1$. It remains to check that $h_2^{-1}h_1h_2=h_3h_1=(h_3h_4)^p\cdot h_1\in\mathrm{Di}_{p(p-1)/2}$, and that $h_2^{-1}(h_3h_4)h_2=(h_3h_4)^{k-p(k-1)}\in\langle h_1,h_3h_4\rangle$, where k is the smallest integer, for which h_4^{k-1} sends F_1 to F_ω. □

Proposition 6.9. *The group* $\mathrm{CAut}(\mathcal{M})$ *contains the subgroup* $H\rtimes L$*, where* L *is the above group* $\mathrm{Di}_{p(p-1)/2}\rtimes\mathbb{Z}_{p-1}$ *of order* $2p(p-1)^2$*.*

Proof. Recall that $L=\langle h_1,h_2,h_4\rangle$. Thus for each permutation h_i, $i\in\{1,2,4\}$ we need to find a permutation f_i, acting on the set Ω of cardinality $2p^2$, which belongs to the normalizer of H in $S(\Omega)$ and induces permutation h_i on the set of basic relations of $\mathcal{M}$.

Define $f_1 : \Omega \to \Omega$ by $[x, y] \mapsto (x, y)$ and $(k, q) \mapsto [-k, q]$. This permutation induces h_1 on the relations of $\mathcal{M}$. The permutation f_2 defined by $[x, y] \mapsto [x, \omega y]$ on points, and $(k, q) \mapsto (k\omega, q\omega)$ on lines (where ω is a primitive element of $\mathrm{GF}(p)$) induces h_2. Finally, f_4 defined by $[x, y] \mapsto [x, y + 1]$ on points and fixing all lines in $\mathcal{L}$ induces h_4. □

Theorem 6.10. *The group* $\mathrm{CAut}(\mathcal{M})$ *is isomorphic to* $H \rtimes L = H \rtimes (\mathrm{Di}_{p(p-1)/2} \rtimes \mathbb{Z}_{p-1})$ *and has order* $2p^4(p-1)^2$.

Proof. It follows from Proposition 6.9 that $H \rtimes L \leq \mathrm{CAut}(\mathcal{M})$, where L is presented in Proposition 6.8. By the proof of Theorem 6.5, the relations A_1, B_1, D_1, E_0 form a base of $\mathrm{AAut}(\mathcal{M})$. In the context of the current proof this fact can be reformulated as follows: any permutation from $\mathrm{CAut}(\mathcal{M})$, which is an automorphism of the relations A_1, B_1, D_1, E_0, belongs to the group $\mathrm{Aut}(\mathcal{M}) = H$. Taking into account that $\mathrm{Aut}(E_0) \cong \mathrm{Aut}(\mathcal{B}_p)$ has order $p^3(p-1)^2$, while this group acts transitively on the relations A_i and B_i with two orbits of length $p - 1$, we conclude, using the orbit-stabilizer lemma, that $\mathrm{Aut}(A_1) \cap \mathrm{Aut}(B_1) \cap \mathrm{Aut}(D_1) \cap \mathrm{Aut}(E_0)$ is of order p^3. □

A significant consequence of the results presented in this section is that the group $\mathrm{CAut}(\mathcal{M})/\mathrm{Aut}(\mathcal{M}) \cong L$ has index $p - 1$ in the group $\mathrm{AAut}(\mathcal{M})$. In other words, for all odd primes p, the group $\mathrm{AAut}(\mathcal{M})$ contains proper algebraic automorphisms, that is, those that are not induced by suitable elements from $\mathrm{CAut}(\mathcal{M})$.

7 Four new infinite families of non-Schurian ASs

Now we wish to introduce four families of color graphs with vertex set Ω. The proof of the fact that we are getting fusion schemes will be given in two independent ways: a combinatorial one, via the use of the tensor $\mathcal{T}$ of structure constants of $\mathcal{M}$; and an algebraic one, via interpreting all detected color graphs as algebraic mergings.

7.1 Introducing new color graphs

Let us consider the following subsets of $\Omega \times \Omega$:

- $R_0 = A_0 \cup C_0$,
- $S_i = A_i \cup C_i$, where $i \in \mathbb{Z}_p^*$,
- $T_i = B_i \cup D_i$, where $i \in \mathbb{Z}_p^*$,
- $U_i = E_i \cup F_i$, where $i \in \mathbb{Z}_p$. Note that the relation U_0 coincides with the set of (undirected) flags in the *canonical* copy of the biaffine plane $\mathcal{B}_p$.

Further, let $S_i^* = S_i \cup S_{p-i}$, $T_i^* = T_i \cup T_{p-i}$ and $U_i^* = U_i \cup U_{p-i}$ be the respective symmetrizations of the relations S_i, T_i and U_i, canonically denoted for each $i \in \{1, \ldots, (p-1)/2\}$. Finally, let $S = S_1 \cup \ldots \cup S_{p-1}$ and $U = U_1 \cup \ldots \cup U_{p-1}$. Observe that U is the set of antiflags in $\mathcal{B}_p$.

It is straightforward to check that $\{R_0, S_1, \ldots, S_{p-1}, T_1, \ldots, T_{p-1}, U_0, U_1, \ldots, U_{p-1}\}$ forms a partition of $\Omega \times \Omega$.

It remains to define the requested color graphs $\mathcal{M}_i$ with vertex set Ω, where $i \in \{1, \ldots, 4\}$.

Color graph 1. Denote by $\mathcal{M}_1$ the color graph with colors given by the sets R_0, $S_1, \ldots, S_{p-1}$, $T_1, \ldots, T_{p-1}$, U_0, $U_1, \ldots, U_{p-1}$.

Color graph 2. Denote by $\mathcal{M}_2$ the color graph with colors given by R_0, S_1^*, $S_2^*, \ldots, S_{(p-1)/2}^*$, T_1, $T_2, \ldots, T_{p-1}$, U_0, U_1^*, $U_2^*, \ldots, U_{(p-1)/2}^*$.

Color graph 3. Denote by $\mathcal{M}_3$ the color graph with colors given by R_0, S, T_1, $T_2, \ldots, T_{p-1}$, U_0, U.

Color graph 4. Finally, denote by $\mathcal{M}_4$ the color graph with colors given by R_0, S, T_1^*, $T_2^*, \ldots, T_{(p-1)/2}^*$, U_0, U.

Note that for $p = 3$ the color graphs $\mathcal{M}_2$ and $\mathcal{M}_3$ coincide.

These four color graphs play a central role in this paper.

7.2 Proving the existence of ASs

We show that the color graphs $\mathcal{M}_1$–$\mathcal{M}_4$ defined in the previous section are ASs. It is sufficient to show the existence of intersection numbers (structure constants) because all other axioms of an AS are trivially satisfied. For convenience, we invoke the following notational simplification for the indices of intersection numbers. We write si, ti, ui in place of S_i, T_i, U_i, respectively. For example, $c_{si,tj}^{uk}$ indicates the number of elements $z \in \Omega$ such that $(x, z) \in S_i$ and $(z, y) \in T_j$ for any $(x, y) \in U_k$. A subscripted or superscripted zero always indicate the relation R_0, while any index i not accompanied by a specified symbol indicates any feasible relation. For the sake of brevity, we indicate only those intersection numbers in which starred relations (such as U_k^*) do occur in the superscript. In each case we are providing only one argumentation (usually for points), because the dual consideration (for lines) is similar. We make frequent use of *Kronecker's symbol* $\delta_{i,j}$ in order to shorten computations and formulas.

To make the enumeration easier the following observations are helpful:

Observation 7.1. *For all values of integers $i \in \{1, \ldots, p-1\}$ and $j \in \{1, \ldots, (p-1)/2\}$ we have $R_0, S, S_i, T_i, S_j^*, T_j^* \subseteq (\mathcal{P} \times \mathcal{P}) \cup (\mathcal{L} \times \mathcal{L})$, and $U, U_0, U_i, U_j^* \subseteq (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$. Thus, the intersection numbers of type $c_{si,sj}^{uk}, c_{si,tj}^{uk}, c_{ti,sj}^{uk}, c_{ti,tj}^{uk}, c_{ui,uj}^{uk}, c_{si,uj}^{sk}$, $c_{si,uj}^{tk}, c_{ti,uj}^{sk}, c_{ti,uj}^{tk}, c_{ui,sj}^{sk}, c_{ui,sj}^{tk}, c_{ui,tj}^{sk}, c_{ui,tj}^{tk}$ are zero for all choices of i, j, k.*

Remark 7.1. Below, the symbol $*$ indicates *composition* of relations. In a coherent algebra this operation corresponds to a product of corresponding adjacency matrices. In a CC the result is usually a *multirelation*, that is, a set of relations together with their multiplicities.

Observation 7.2. *For the compositions of relations S_i, S_j, T_i, T_j we have $S_i * T_j = T_j$, $T_i * S_j = T_i$ and if $i + j \neq 0$, then $S_i * S_j = S_{i+j}$ and $T_i * T_j = T_{i+j}$. As a consequence we obtain the following: $c_{si,tj}^{sk} = c_{tj,si}^{sk} = c_{si,sj}^{tk} = 0$, and for $i + j \neq 0$: $c_{ti,tj}^{sk} = 0$, $c_{si,sj}^{sk} = \delta_{i+j,k}$.*

Observation 7.3. *For each color X we have $R_0 * X = X * R_0 = X$, and for any $Y \neq X^T$ we obtain $X * Y \neq R_0 \neq Y * X$. Thus for all i and j such that $i \neq j$ we have $c^j_{0,i} = c^j_{i,0} = 0$, and when $i \neq j'$ we have $c^0_{i,j} = 0$.*

Observation 7.4. *Let $P_1, P_2 \in \mathcal{P}$ be two collinear points in $\mathcal{B}_p$, and let $L_1 = \{\ell \in \mathcal{L} \mid P_1 \in \ell\}$. Then for all $\ell_i, \ell_j \in L_1$ we have $d(P_2, \ell_i) = d(P_2, \ell_j)$ if and only if $i = j$, where $d(P, l)$ is the previously defined quasidistance.*

All these intersection numbers appear in Appendix 1 to [24]. They were derived by geometrical arguments, usually by considering points and lines at a given quasidistance from two objects.

Theorem 7.5. *The following hold:*

(a) $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4$ *are ASs.*
(b) The combinatorial groups of automorphisms of $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4$ *contain a subgroup isomorphic to* H.
(c) $\mathcal{M}_2$ *is a merging of* $\mathcal{M}_1$, $\mathcal{M}_3$ *is a merging of* $\mathcal{M}_2$, *and* $\mathcal{M}_4$ *is a merging of* $\mathcal{M}_3$.
(d) $\mathrm{rk}(\mathcal{M}_1) = 3p - 1$, $\mathrm{rk}(\mathcal{M}_2) = 2p$, $\mathrm{rk}(\mathcal{M}_3) = p + 3$, $\mathrm{rk}(\mathcal{M}_4) = (p + 7)/2$.

Proof. Parts (a) and (b) have already been proven. Proofs of (c) and (d) easily follow from the definition of $\mathcal{M}_i$, where $i \in \{1, \dots, 4\}$. □

As we have seen in the previous section, the algebraic group $\mathrm{AAut}(\mathcal{M})$ is of order $2p(p-1)^3$. It is easy to check that this group has four orbits of length 2, $2p-2$, $2p-2$ and $2p$, respectively, on the set of basic relations of $\mathcal{M}$. These orbits are:

$$\{A_0, C_0\}, \{A_1, \dots, A_{p-1}, C_1, \dots, C_{p-1}\}, \{B_1, \dots, B_{p-1}, D_1, \dots, D_{p-1}\},$$
$$\{E_0, E_1, \dots, E_{p-1}, F_0, F_1, \dots, F_{p-1}\}.$$

Now we provide an alternative proof of the fact that the color graphs $\mathcal{M}_i$ are ASs, where $i \in \{1, \dots, 4\}$. This proof is simple and elegant.

Theorem 7.6. *The color graphs $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ and $\mathcal{M}_4$ appear as algebraic mergings of $\mathcal{M}$ and are ASs.*

Proof. Let $q = (p-1)/2$. Then the algebraic mergings corresponding to the subgroups $K_1 = \langle g_1 \rangle$, $K_2 = \langle g_1, g_3^q \rangle$, $K_3 = \langle g_1, g_3 \rangle$ and $K_4 = \langle g_1, g_3, g_4^q, g_5^q \rangle$ of $\mathrm{AAut}(\mathcal{M})$ lead to $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ and $\mathcal{M}_4$, respectively. This implies that each of these color graphs is a CC. Note that permutation g_1 merges together A_0 and C_0 into R_0. Therefore $\mathcal{M}_1$ becomes an AS. Taking into account that the groups K_2, K_3 and K_4 are overgroups of K_1, the remaining color graphs are also ASs. □

Proposition 7.7. *The following holds: $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ are commutative, but non-symmetric; $\mathcal{M}_4$ is symmetric and hence commutative.* □

Remark 7.2. A careful inspection of the introduced subgroups K_i of $\mathrm{AAut}(\mathcal{M})$, where $i \in \{1, \dots, 4\}$, shows that for $p \geq 5$ none of them is a subgroup of the group $K = \mathrm{CAut}(\mathcal{M})/\mathrm{Aut}(\mathcal{M})$. Therefore, according to general properties of algebraic mergings, the obtained ASs might be, in principle, non-Schurian. We now prove rigorously that exactly this is the case.

7.3 Combinatorial automorphism groups of constructed ASs

In this section we will focus on the combinatorial groups of automorphisms of the constructed schemes. Recall that this group consists of all permutations $\phi : \Omega \to \Omega$ that preserve relations, i.e. $R_i^\phi = R_i$ for all $R_i \in \mathcal{R}$.

Theorem 7.8. *Let* $\mathrm{Aut}(\mathcal{M}_1)$, $\mathrm{Aut}(\mathcal{M}_2)$, $\mathrm{Aut}(\mathcal{M}_3)$ *and* $\mathrm{Aut}(\mathcal{M}_4)$ *be the combinatorial groups of automorphisms of* $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ *and* $\mathcal{M}_4$, *respectively. Then the following hold:*

(a) $\mathrm{Aut}(\mathcal{M}_1) \le \mathrm{Aut}(\mathcal{M}_2) = \mathrm{Aut}(\mathcal{M}_3) \le \mathrm{Aut}(\mathcal{M}_4)$,
(b) $|\mathrm{Aut}(\mathcal{M}_1)| = p^3$,
(c) $|\mathrm{Aut}(\mathcal{M}_2)| = 2p^3$,
(d) $|\mathrm{Aut}(\mathcal{M}_3)| = 2p^3$,
(e) $|\mathrm{Aut}(\mathcal{M}_4)| = 8p^3$.

Proof. It is clear that the previously defined permutations t_{ab} and φ (in definition of group H) are elements of each automorphism group $\mathrm{Aut}(\mathcal{M}_i)$, that is, H is a subgroup of $\mathrm{Aut}(\mathcal{M}_i)$ for each $1 \le i \le 4$.

(a) The chain of inequalities $\mathrm{Aut}(\mathcal{M}_1) \le \mathrm{Aut}(\mathcal{M}_2) \le \mathrm{Aut}(\mathcal{M}_3) \le \mathrm{Aut}(\mathcal{M}_4)$ follows directly from Theorem 7.5, simply by applying Galois correspondence to the lattice of CCs and that of their corresponding automorphism groups. The equality $\mathrm{Aut}(\mathcal{M}_2) = \mathrm{Aut}(\mathcal{M}_3)$ follows by inspection of the group orders, to be accomplished in parts (c) and (d).

Below, we consider separately the claims (b) through (e). In each proof, G denotes the group $\mathrm{Aut}(\mathcal{M}_i)$, while $G_{[0,0],(0,0)}$ will denote the stabilizer in G of both the point $[0,0]$ and the line $(0,0)$. The proof of (b) is given with enough details.

(b) We apply the orbit-stabilizer lemma to prove that $|\mathrm{Aut}(\mathcal{M}_1)| \le p^3$. We already know that H is a subgroup of $\mathrm{Aut}(\mathcal{M}_1)$, hence the result follows. (Moreover, this shows that $\mathrm{Aut}(\mathcal{M}_1) \cong H \cong \mathbb{Z}_p^2 \rtimes \mathbb{Z}_p$.) For the sake of brevity, let us denote $G = \mathrm{Aut}(\mathcal{M}_1)$.

First, we claim that there is no automorphism that sends a point to a line. Toward a contradiction, suppose that $\alpha \in G$ sends the point P_1 to the line $r = (k,q)$, $k,q \in \mathbb{Z}_p$. Without loss of generality we may assume $P_1 = [0,0]$, because G acts transitively on $\mathcal{P}$. In such a case, we have $\mathcal{P}^\alpha = \mathcal{L}$ and $\mathcal{L}^\alpha = \mathcal{P}$, because of the relations S_i. Consider now the line $l = (0,0)$ and its image $l_1^\alpha = (u,v)$, $u,v \in \mathbb{Z}_p$. Clearly $(P_1,l_1) \in U_0$, whence $v = k \cdot u + q$. Now consider the point $[1,0]$. Since $([0,0],[1,0]) \in T_1$ and $([1,0],(0,0)) \in U_0$, it follows that $[1,0]^\alpha = (k+1, q-u)$. Similarly, $(1,0)^\alpha = [u+1, q+k(u+1)]$. However $([1,0],(1,0)) \in U_1$, and therefore $([1,0],(1,0))^\alpha \in U_1$. But $([1,0],(1,0))^\alpha = ((k+1,q-u),[u+1,q+k(u+1)]) \in U_{-1}$, since $q+k(u+1)-(k+1)(u+1)-q+u = -1$, a contradiction for any odd prime p. This proves that $\mathcal{P}^G = \mathcal{P}$ and $\mathcal{L}^G = \mathcal{L}$, as claimed.

Since G is transitive on the points, $|[0,0]^G| = |\mathcal{P}| = p^2$. Let $G_{[0]} := G_{[0,0]}$ be the stabilizer of the point $[0,0]$ in G. The points $[0,1],[0,2],\ldots,[0,p-1]$ are fixed by $G_{[0]}$, because they form unique pairs together with $[0,0]$ in the relations $S_1,\ldots,S_{p-1}$, respectively. As the line $(0,0)$ contains the point $[0,0]$, there are at most p distinct images of $(0,0)$ under the action of $G_{[0]}$. However, it is easy to check that $(0,0)^{\varphi^i} = (-i,0)$ for $i \in \{0,1,\ldots,p-1\}$, which proves that $|(0,0)^{G_{[0]}}| = p$.

Now let $G_0 = G_{[0,0],(0,0)}$ be the stabilizer in G of both $(0,0)$ and $[0,0]$. Then G_0 fixes all lines $(0,i)$ parallel to $(0,0)$, because $(0,i)$ forms a unique pair with $(0,0)$ in S_i. If we now consider an arbitrary point $[x,y]$ with $x \neq 0$, then its image under G_0 must be contained in the line $(0,y)$. Moreover, $([0,y],[x,y]) \in T_x$ and for any $\pi \in G_0$ it follows that $([0,y],[x,y])^\pi = ([0,y],[t,y]) \in T_x$ for some $t \in \mathbb{Z}_p$. This establishes that $t = x$, and hence the point $[x,y]$ is fixed under G_0. Thus G_0 fixes all points and therefore all lines as well. By the orbit-stabilizer lemma, $|G| = p^2 \cdot p \cdot 1 = p^3$ and so $G \cong H$.

(c) By routine inspection, the mapping π defined by $[x,y] \mapsto (x,-y-2x)$, $(x,y) \mapsto [x+2,-y]$ is an automorphism of $\mathcal{M}_2$. From this it follows that $G = \mathrm{Aut}(\mathcal{M}_2)$ is transitive on $\Omega = \mathcal{P} \cup \mathcal{L}$, whence $|[0,0]^G| = 2p^2$. Similarly, as in part (b), we may again show that the line $(0,0)$ has p distinct images under the action of the stabilizer $G_{[0]}$ of $[0,0]$. Again, considering the stabilizer G_0 of $[0,0]$ and $(0,0)$ we conclude (see [24]) that $|G| = 2p^3$.

(d) It is clear that the permutation π appearing in part (c) is also an automorphism of $\mathcal{M}_3$, i.e. $\pi \in \mathrm{Aut}(\mathcal{M}_3) = G$. Moreover, the initial steps of part (c) again establish $|[0,0]^G| = 2p^2$. We consider once more the stabilizer G_0 of $(0,0)$ and $[0,0]$. Again, we obtain $|G| = 2p^3$ as desired. (In fact, we confirmed that $\mathrm{Aut}(\mathcal{M}_2) \cong \mathrm{Aut}(\mathcal{M}_3) \cong \langle t_{01}, t_{10}, \varphi, \pi\rangle$.)

(e) Here we set $G = \mathrm{Aut}(\mathcal{M}_4)$, and consider permutations α and β defined by $[x,y]^\alpha = [x,-y]$, $(k,q)^\alpha = (-k,-q)$ and $[x,y]^\beta = [-x,-y]$, $(k,q)^\beta = (k,-q)$. Similarly to parts (b) and (c), one can verify that $G \cong \langle t_{10}, t_{01}, \varphi, \pi, \alpha, \beta\rangle$, whence $|G| = 8p^3$. (Comparing α with f_1, f_2, f_4, inducing proper color automorphisms of $\mathcal{M}$, one can notice that $\alpha = f_2^{(p-1)/2}$.)

□

Remark 7.3. We are now ready to share with the reader a possibility for the alternative proof of the claim that $\mathrm{Aut}(\mathcal{M}) = H$, which was mentioned in the previous section. It immediately follows from (b) of Theorem 7.8, since $\mathcal{M}_1$ is a merging of $\mathcal{M}$ and due to the Galois correspondence, we obtain $H = \mathrm{Aut}(\mathcal{M}_1) \geq \mathrm{Aut}(\mathcal{M}) \geq H$ implying $\mathrm{Aut}(\mathcal{M}) = H$. We believe that the proof of this equality, proved earlier by different methods, is of independent interest, demonstrating different methodology, not appealing to the use of the orbit-stabilizer theorem.

7.4 Main corollary

Corollary 7.9. *For $p > 3$ the $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ and $\mathcal{M}_4$ are pairwise distinct non-Schurian ASs.*

Proof. Recall that the rank of $H = \mathbb{Z}_p^2 \rtimes \mathbb{Z}_p$ on $V = \mathcal{P} \cup \mathcal{L}$ is $6p-2$. Thus $\mathrm{Aut}(\mathcal{M}_1)$ is also of rank $6p-2$, while the rank of $\mathcal{M}_1$ is $3p-1$. This proves that $\mathcal{M}_1$ is non-Schurian for $p \geq 3$.

For the ASs $\mathcal{M}_2$ and $\mathcal{M}_3$, we consider the permutation $\pi \in \mathrm{Aut}(\mathcal{M}_2) = \mathrm{Aut}(\mathcal{M}_3)$ introduced in part (c) of Theorem 7.8. As the result of π, we obtain the following 2-orbits: $A_i \cup C_{-i}$, $B_j \cup D_j$ and $E_i \cup F_{-i}$, for $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_p^*$. This proves that $\mathrm{Aut}(\mathcal{M}_2) = \mathrm{Aut}(\mathcal{M}_3)$ is of rank $3p-1$. As the ranks of $\mathcal{M}_2$ and $\mathcal{M}_3$ are $2p$ and $p+3$ respectively, we conclude that $\mathcal{M}_2$ and $\mathcal{M}_3$ are non-Schurian for $p \geq 3$. However, one can check that $\mathcal{M}_2$ and $\mathcal{M}_3$ coincide when $p = 3$.

Finally, with the aid of permutations α and β introduced in part (e) of Theorem 7.8, it is easy to see that the 2-orbits of $\mathrm{Aut}(\mathcal{M}_4)$ are $A_i \cup A_{-i} \cup C_i \cup C_{-i}$, $B_j \cup B_{-j} \cup D_j \cup D_{-j}$ and $E_i \cup E_{-i} \cup F_i \cup F_{-i}$, for $i \in \{0, 1, \ldots, (p-1)/2\}$ and $j \in \{1, \ldots, (p-1)/2\}$. Thus the rank of $\mathrm{Aut}(\mathcal{M}_4)$ is equal to $(3p+1)/2$. As the rank of $\mathcal{M}_4$ is $(p+7)/2$, we conclude that $\mathcal{M}_4$ is non-Schurian for $p > 3$. □

Remark 7.4. For $p = 3$ we obtain only two non-Schurian ASs: $\mathcal{M}_1$ and $\mathcal{M}_2$. Indeed, the ASs $\mathcal{M}_2$ and $\mathcal{M}_3$ coincide, while $\mathcal{M}_4$ is Schurian, since $(3 \cdot 3+1)/2 = (3+7)/2 = 5$. These two non-Schurian ASs are of order $2 \cdot 3^2 = 18$ and according to the catalogue of small ASs of Hanaki and Miyamoto [26], we cover all non-Schurian ASs of order 18.

8 Combinatorial vs. group-theoretical regularity

This section is devoted to a discussion of the concept of symmetry in graph theory. Suppose that Γ is a graph, possibly directed, undirected, colored. Usually, diverse requirements to the symmetry of Γ are formulated in the terms of transitivity of action of the group $G = \mathrm{Aut}(\Gamma)$ on ingredients of Γ of a concrete kind, like vertices, edges, arcs, etc. Such requirements are of global nature and the related investigation depends on the use of tools from group theory. In many cases the knowledge of consequences of the Classification of Finite Simple Groups turns out to be very helpful, see e.g. [15]. In this fashion classification of highly symmetrical graphs may be regarded as a problem in applied group theory. In AGT there is a long-standing tradition to approximate group-theoretical symmetry by other requirements, formulated in terms of combinatorial or spectral regularity. Combinatorial regularity is expressed in local terms. Most simple and well-known example is a regular graph: each vertex has the same amount of neighbours.

CCs serve as a combinatorial approximation of transitivity of the action of group $G = \mathrm{Aut}(\Gamma)$ on arcs of each monochromatic graph of a colored graph Γ. One of the most significant concepts in CCs is the *coherent closure* $W(\Gamma)$ of a given graph Γ: this is the smallest CC, which contains Γ as a union of its basic relations. The definition of $W(\Gamma)$ is correct, see [37]. Usually, $W(\Gamma)$ is called *Weisfeiler-Leman closure*, paying tribute to the seminal paper [53] written by Weisfeiler and Leman, see also [52]. In fact, they described a polynomial-time algorithm for the computation of $W(\Gamma)$. Usually, it is formulated and fulfilled in the terms of matrices, while $W(\Gamma)$ is regarded as a coherent algebra, equivalent to the corresponding CC. We refer to [7] for one of the existing practical program implementations of the WL-stabilization, the modern name for the efficient procedure, introduced in [53]. A small example of the use of WL-stabilization for purely pedagogical purposes is presented in our text [34] (Section 6.4, Example 6.31).

For a few decades the smallest known non-Schurian CC was a rank 3 AS on 15 points, formed by a *doubly regular tournament* $\mathcal{T}_{15}$ and its complement, see [46]. It turns out that the group $\mathrm{Aut}(\mathcal{T}_{15})$ is of order 21, thus it can not act transitively on the vertex set of $\mathcal{T}_{15}$, neither on the set of its directed arcs, having cardinality 105. The fact that this AS is non-Schurian may be justified on a local level with the aid of 5-vertex invariants of arcs of $\mathcal{T}_{15}$, see [39]. Note that the use of 4-vertex invariants is not enough for the same purposes.

There are 16 non-Schurian ASs on 16 points. We briefly discuss one of them.

Example 8.1. The Shrikhande graph Sh appears as the complement to the Latin square graph over the group $\mathbb{Z}_4$. It follows from the general theory of such graphs, that $|\mathrm{Aut}(Sh)| = 192$, see e.g. [27]. Sh is also a strongly regular graph (briefly SRG). In other words, the reflexive relation R_0, Sh and $\overline{Sh}$ together are basic graphs of a rank 3 AS. It is clear that this AS is non-Schurian, because the vertex stabilizer of order 12 can not act transitively on the set of nine non-neighbours of the considered vertex in graph $\overline{Sh}$. Note that this proof of the non-Schurian property was obtained with the aid of information of global nature, related to properties of the group G.

Good news are that once more we may justify the desired property, relying only on local invariants of Sh, this time considering its 4-vertex subconfigurations, see [38].

Following the outlined methodology, we are trying to find the smallest value of κ, for which the non-Schurian property of the detected schemes may be justified (using just local invariants of size κ). Some computer aided experiments are arranged (jointly with M. Ziv-Av). It is already clear that $\kappa \geq 5$. The results will be reported elsewhere.

A simple bipartite regular graph Γ is *semisymmetric*, if $\mathrm{Aut}(\Gamma)$ acts transitively on the edge set, but intransitively on the vertex set. Systematic consideration of semisymmetric graphs goes back to the seminal paper [22] by Folkman. He proved that there are no semisymmetric graphs on $2p^2$ vertices, when p is a prime. In other words, global symmetry on the edges of a regular bipartite graph on $2p^2$ vertices implies global symmetry on the vertices. The existence of the AS $\mathcal{M}_1$ demonstrates that this is not valid for color graphs.

As it was discussed, all detected non-Schurian ASs demonstrate local symmetry, which is not fully supported by global symmetry.

9 Some ASs of small rank appearing as mergings of $\mathcal{M}$

Recall that the main result of this paper is the discovery and investigation of four infinite families of non-Schurian ASs, which appear as algebraic mergings of the master CC $\mathcal{M}$. For each of these four families rank of schemes grows linearly with increasing p. From the earliest inception of AGT, special attention has been paid to ASs of small rank. The smallest possible (non-trivial) rank is 3. Especially interesting are primitive rank 3 schemes, which are generated by SRGs in the symmetric case,

p	AS	Schur	NonSch	NonComm	NonSym
$p = 3$	13	11	2	2	9
$p = 5$	29	23	6	6	17
$p = 7$	51	33	18	8	38

Table 3: Numbers of mergings of $\mathcal{M}$.

p	AS	Schur	NonSch	NonComm	NonSym
$p = 3$	3	2	1	0	2
$p = 5$	8	5	3	0	4
$p = 7$	15	6	9	0	11
$p = 11$	15	6	9	0	11
$p = 13$	35	15	20	0	19

Table 4: Numbers of algebraic mergings of $\mathcal{M}$.

and doubly regular tournaments in the non-symmetric case. One may attempt to prove that in our case such primitive object can not appear for $p > 5$. Similarly, it is possible to justify that there exist only predictable (Schurian) rank 4 mergings with large imprimitive automorphism groups, that is, objects that do not carry any surprises. For merging schemes of rank at least 5 the picture is quite different.

9.1 Attempts of systematical enumeration of all mergings

At initial stage of this project we were enumerating all the ASs that arise as mergings of $\mathcal{M}$. For $p = 3$, using traditional COCO we immediately enumerate 34 proper mergings, which form 13 isomorphism classes, see Section 5. For $p = 5$ COCO does not finish computations in time measurable by a few days. This is why we used COCO II (version by Ziv-Av), which enumerates all AS mergings up to combinatorial isomorphism. A similar task may be fulfilled for $p = 7$, but for $p = 11$ it appears to be practically impossible. A summary of results is included in Table 3.

At the next stage of systematic enumeration we were hunting just for those ASs (up to isomorphism) that arise as algebraic mergings of $\mathcal{M}$. Here it was possible to get complete results for values of p up to 13. A summary of these results is given in Table 4.

A legend to these tables is as follows: *AS* = # of ASs, *Schur* = # of Schurian ASs, *NonSch* = # of non-Schurian ASs, *NonComm* and *NonSym* = # of non-commutative and non-symmetric ASs, respectively.

Let us switch from discussion of computational efforts to their theoretical generalizations.

9.2 Coherent closure of basic graphs of $\mathcal{M}$

Recall that, up to isomorphism, there are three kinds of basic graphs of the master CC $\mathcal{M}$. Let us consider each of them separately. In this context the vertex set of graphs coincides with the set Ω. We can identify the basic graphs with corresponding basic relations.

Proposition 9.1. *The following hold for the basic graphs* (A_i, Ω) *and* (C_i, Ω)*:*

(a) The rank of the WL-closure of the graphs is equal to $p + 5$*.*
(b) The WL-closure of the graphs is Schurian.

Proof. We prove the results for the graph (A_i, Ω), as similar results for (C_i, Ω) can be proved analogously. By Lemma 6.1 we have $\mathrm{Aut}(A_i, \Omega) \cong (S_p \wr \mathbb{Z}_p) \times S_{p^2}$. This group has two orbits: $\mathcal{P}$ and $\mathcal{L}$. There are $p+1$ 2-orbits on the set $\mathcal{P}$, two 2-orbits on $\mathcal{L}$, and two 2-orbits between $\mathcal{P}$ and $\mathcal{L}$. Altogether we obtain that $\mathrm{rk}(\mathrm{Aut}(A_i, \Omega)) = p + 5$.

Now, we have to take into account that the directed cycle $\mathcal{C}_p$ with p vertices is naturally generating its powers in the WL-procedure. This implies (a) and (b). □

Proposition 9.2. *The following hold for the basic graphs* (B_i, Ω) *and* (D_i, Ω)*:*

(a) The rank of the WL-closure of the graphs is equal to $p + 5$*.*
(b) The WL-closure of the graphs is Schurian.

Proof. We prove the results for the graph (B_i, Ω), as similar results for (D_i, Ω) can be proved analogously. This time by Lemma 6.2 we have $\mathrm{Aut}(B_i, \Omega) \cong (\mathbb{Z}_p \wr S_p) \times S_{p^2}$. Further reasonings are similar as in the proof of the previous proposition. □

Proposition 9.3. *The following hold for the basic graphs* (E_i, Ω) *and* (F_i, Ω)*:*

(a) The rank of the WL-closure of the graphs is equal to 10.
(b) The WL-closure of the graphs is Schurian.

Proof. There are 3 basic relations on the vertex set $\mathcal{P}$ of the biaffine plane: reflexivity, collinearity of points, and their non-collinearity. Dually, we have 3 basic relations on the set $\mathcal{L}$. There are two relations from $\mathcal{P}$ to $\mathcal{L}$ (incidence and non-incidence), as well as from $\mathcal{L}$ to $\mathcal{P}$. This proves (a). To prove (b), let us interpret the group $\mathrm{Aut}(\mathcal{B}_p)$ as $\mathbb{F}_p^2 \rtimes \mathrm{UGL}(2, p)$, where $\mathbb{F}_p^2$ is the 2-dimensional vector space over the field $\mathbb{F}_p$, and $\mathrm{UGL}(2, p)$ is the group of upper triangle matrices with non-zero determinant over $\mathbb{F}_p$. Clearly, the group $\mathrm{Aut}(\mathcal{B}_p)$ has the desired order $(p-1)^2 \cdot p^3$. Now we have to describe 5 orbits of the stabilizer of vector $(0, 0)$ on $\mathcal{P}$, having lengths 1, $p - 1$, $p(p - 1)$, p and $p(p - 1)$. Similarly, there are 5 orbits of this stabilizer on the set $\mathcal{L}$. □

9.3 Classical biaffine plane as a distance-regular graph

Definition of a distance-regular graph is usually given in more sophisticated terms. Advantage of our definition, provided below, is the use of the of WL-closure.

A regular connected undirected graph Γ of valency k and diameter d is called a *distance-regular graph* (briefly DRG), if $\mathrm{rk}(\mathrm{W}(\Gamma)) = d$. Similar definition can be given in case, when Γ is a directed graph without undirected edges. Trivial examples of DRGs are given by undirected cycles. We refer the interested reader to [14] for reasonably full theory of DRGs.

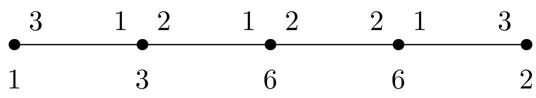

Figure 8: The intersection diagram of the Pappus graph.

Example 9.1. The Pappus graph P is a DRG on 18 vertices of valency 3 and diameter 4. Significant properties of P appear on the *intersection diagram*, see Figure 8.

Usually, instead of an intersection diagram of a DRG Γ, its *intersection array* $i(\Gamma)$ is used. In our case we obtain $i(P) = \{3, 2, 2, 1; 1, 1, 2, 3\}$.

One more essential property of P is that it is antipodal with naturally defined quotient graph $\bar{\Gamma}$. It turns out that the quotient graph is also a DRG. In our situation it has diameter 2 and is isomorphic to the complete bipartite graph $K_{3,3}$.

Note that P is also *distance-transitive* (briefly DTG). One of the possible definitions of this property is to claim that this is a DRG whose WL-closure is Schurian.

We are now ready to formulate the crucial property of biaffine planes in the framework of the theory of DTGs. Recall that $(E_i \cup F_i, \Omega)$ is the undirected incidence graph of the classical biaffine plane $\mathcal{B}_p$. Let us denote it by $\mathrm{Inc}(\mathcal{B}_p)$.

Theorem 9.4. *Let p be a prime. For the incidence graph $\mathrm{Inc}(\mathcal{B}_p)$ of the biaffine plane $\mathcal{B}_p$ the following hold:*

(a) It is a DRG of valency p and diameter 4 with intersection array

$$i(\mathrm{Inc}(\mathcal{B}_p)) = \{p, p-1, p-1, 1; 1, 1, p-1, p\}.$$

(b) It is a DTG with automorphism group of order $2(p-1)^2 \cdot p^3$.
(c) It is bipartite and antipodal.
(d) Its spectrum is:

$$\{p^1, \sqrt{p}^{(p-1)p}, 0^{2p-2}, (-\sqrt{p})^{(p-1)p}, (-p)^1\}.$$

Proof. This graph appears as a particular case considered on p.425 in [14], here $m = k = p$, $\mu = 1$. A proof of parts (a)-(c) was imitated in Example 9.1 for $p = 3$. To prove (d) one has to take into account that the WL-closure of $\mathrm{Inc}(\mathcal{B}_p)$ is a symmetric rank 5 AS and to consult [9] for general techniques of description of spectra of ASs. □

An interesting task is to classify all DTGs with intersection array as in Theorem 9.4. It was considered in a number of publications, including [16] and [31]. Finally, a complete classification was obtained using twisted fields of Albert type.

9.4 An auxiliary Schurian rank 8 AS and its automorphism group

We are now in position to present a few infinite families of non-Schurian ASs of rank 5 and 6, arising by mergings of $\mathcal{M}$.

The subgroup K of $\mathrm{AAut}(\mathcal{M})$ was considered, where $K = \langle g_1, g_3, g_4^2, g_5^2 \rangle$. (Here we are following the notation from the previous sections.) In the next step we applied K on $\mathcal{M}$ for all odd primes up to $p = 19$. In each case we constructed the corresponding algebraic merging $\mathcal{N}_6$ and investigated its properties. As a result, we concluded with the following observation.

Observation 9.5. *For all odd primes p there exists a non-Schurian rank 6 algebraic merging $\mathcal{N}_6$ of the master CC $\mathcal{M}$. The group $\mathrm{Aut}(\mathcal{N}_6)$ is a transitive rank 8 group of order $\frac{1}{2}(p-1)^2p^3$. The group $\mathrm{CAut}(\mathcal{N}_6)$ has order $(p-1)^2p^3$. The group $\mathrm{AAut}(\mathcal{N}_6)$ has order 2 and thus coincides with the group $\mathrm{CAut}(\mathcal{N}_6)/\mathrm{Aut}(\mathcal{N}_6)$. The scheme $\mathcal{N}_6$ is commutative. It is non-symmetric when $p \equiv 3 \pmod 4$ and symmetric when $p \equiv 1 \pmod 4$.*

Naturally, we became interested in the group $\mathrm{Aut}(\mathcal{N}_6)$ and its centralizer algebra, a Schurian AS of rank 8, which plays here a role of an auxiliary structure. Let us denote it by $\mathcal{N}_8$. Again, it is non-symmetric when $p \equiv 3 \pmod 4$ and symmetric when $p \equiv 1 \pmod 4$. They are all commutative. In what follows we concentrate on $\mathcal{N}_8$ and its group $\mathrm{Aut}(\mathcal{N}_8) = \mathrm{Aut}(\mathcal{N}_6)$.

- Basic relations of $\mathcal{N}_8$ have valencies 1, $(p-1)/2$, $(p-1)/2$, $(p^2-p)/2$, $(p^2-p)/2$, p, $(p^2-p)/2$, $(p^2-p)/2$. We label them R_i for $i \in \{0, 1, \ldots, 7\}$.
- The graphs $(R_1, \mathcal{P})$ and $(R_2, \mathcal{P})$ are isomorphic to $p \circ \mathcal{T}_p$, where $\mathcal{T}_p$ is a classical (Paley) DRT on p points, when $p \equiv 3 \pmod 4$ and to $p \circ \mathrm{Pal}_p$, where Pal_p is Paley SRG on p points, when $p \equiv 1 \pmod 4$.
- The graphs $(R_3, \mathcal{P})$ and $(R_4, \mathcal{P})$ are wreath products $\mathcal{T}_p \operatorname{wr} E_p$ and $\mathrm{Pal}_p \operatorname{wr} E_p$, respectively (depending on the value $p \mod 4$).
- Restrictions of the (disconnected) graphs (R_i, Ω) onto $\mathcal{L}$ are described similarly, that is $(R_i, \mathcal{L})$, for $i \in \{1, \ldots, 4\}$.
- The graph (R_5, Ω) is isomorphic to $\mathrm{Inc}(\mathcal{B}_p)$.
- Finally, the graphs (R_6, Ω) and (R_7, Ω) appear as unions of $(p-1)/2$ another copies of $\mathrm{Inc}(\mathcal{B}_p)$, which are labeled with the aid of non-zero squares and non-squares in the finite field $\mathbb{F}_p$, provided that labeling of R_5 corresponds to 0 in $\mathbb{F}_p$.

9.5 All mergings of $\mathcal{N}_8$ in case $p \equiv 3 \pmod 4$

It turns out that the lattice of all mergings of $\mathcal{N}_8$ is slightly richer in the case when $p \equiv 3 \pmod 4$. This is why we start by considering this case.

Proposition 9.6. *Let $p \equiv 3 \pmod 4$, $p > 3$.*

(a) There exist exactly 14 proper mergings of $\mathcal{N}_8$. Their ranks vary from 3 to 7.
(b) There are 5 non-Schurian mergings and 9 Schurian mergings.
(c) There are 3 non-Schurian mergings of rank 6 and 2 of rank 5.

(d) *Non-Schurian rank* 6 *mergings correspond to the partitions*

$$\{\{R_0\},\{R_1,R_2\},\{R_3\},\{R_4\},\{R_5,R_6\},\{R_7\}\},$$

$$\{\{R_0\},\{R_1,R_2\},\{R_3\},\{R_4\},\{R_5,R_7\},\{R_6\}\},$$

$$\{\{R_0\},\{R_1,R_2\},\{R_3\},\{R_4\},\{R_5\},\{R_6,R_7\}\}.$$

(e) *There are two non-Schurian rank* 5 *mergings. They correspond to the partitions*

$$\{\{R_0\},\{R_1,R_2\},\{R_3,R_4\},\{R_5,R_6\},\{R_7\}\},$$

$$\{\{R_0\},\{R_1,R_2\},\{R_3,R_4\},\{R_5,R_7\},\{R_6\}\}.$$

They are isomorphic and their group have rank 7 *with order twice larger than* $|\mathrm{Aut}(\mathcal{N}_8)|$.

Proof. A rigorous proof is based on literal extrapolation of the results obtained for small values of p. This is possible due to the essence of the described basic graphs of $\mathcal{N}_8$. □

Remark 9.1. In fact, we detected two infinite families of non-Schurian rank 6 ASs. Let us denote them (up to isomorphisms) by $\mathcal{N}_{6,1}$ and $\mathcal{N}_{6,2}$. It is clear that valencies for basic relations of $\mathcal{N}_{6,1}$ are 1, $p-1$, $(p^2-p)/2$, $(p^2-p)/2$, $(p^2-p)/2$, $(p^2+p)/2$, while for $\mathcal{N}_{6,2}$ they are 1, $p-1$, $(p^2-p)/2$, $(p^2-p)/2$, p, p^2-p. It turns out that $\mathcal{N}_6=\mathcal{N}_{6,2}$.

The valencies for the rank 5 merging $\mathcal{N}_5$ are 1, $p-1$, p^2-p, $(p^2-p)/2$, $(p^2+p)/2$.

Remark 9.2. By analyzing the lattice of the mergings obtained we easily conclude that $\mathcal{N}_{6,1}=\mathrm{W}(R_3,R_5\cup R_6)$, $\mathcal{N}_{6,2}=\mathrm{W}(R_3,R_6\cup R_7)$, and $\mathcal{N}_5=\mathrm{W}(R_5\cup R_6)$. Thus both $\mathcal{N}_{6,1}$ and $\mathcal{N}_{6,2}$ appear as the WL-closure of two basic graphs, while for $\mathcal{N}_5$ just one graph is sufficient.

9.6 All mergings of $\mathcal{N}_8$ in case $p\equiv 1 \pmod 4$

As was mentioned, this case is a bit simpler.

Proposition 9.7. *Let* $p\equiv 1 \pmod 4$. *Then the following hold:*

(a) *There exist exactly* 10 *proper mergings of* $\mathcal{N}_8$. *Ranks of mergings vary from* 3 *to* 7.
(b) *There is only one non-Schurian merging, it has rank* 6, *and* 9 *Schurian mergings.*
(c) *Non-Schurian rank* 6 *merging* $\mathcal{N}_6$ *corresponds to partition*

$$\{\{R_0\},\{R_1,R_2\},\{R_3\},\{R_4\},\{R_5\},\{R_6,R_7\}\}.$$

(d) $\mathrm{Aut}(\mathcal{N}_6)=\mathrm{Aut}(\mathcal{N}_8)$.
(e) *Basic graphs of* $\mathcal{N}_6$ *have valencies* 1, $p-1$, $(p^2-p)/2$, $(p^2-p)/2$, p, p^2-p.

Proof. For parts (a)–(d), the proof is similar to one in the previous proposition. □

9.7 Towards one more family of non-Schurian rank 5 mergings

It is becoming clear that the four infinite families of ASs $\mathcal{M}_1$–$\mathcal{M}_4$, considered in our text, are just a tip of the iceberg of non-Schurian mergings of the master CC $\mathcal{M}$. The amount of non-Schurian mergings grows with increasing value of p, while rank of such schemes varies from 5 to $3p-1$. Other infinite families of low constant rank non-Schurian ASs exist, as well. Below, we describe methodology that allowed to detect existence of one such family and discuss its significant features.

Let us start from $p = 13$. The list of 35 algebraic mergings contains a rank 11 Schurian, commutative, non-symmetric AS $\mathcal{N}_{11}$. $\mathrm{Aut}(\mathcal{N}_{11})$ is a transitive group of order $158,184 = 72 \cdot 13^3$. $\mathrm{AAut}(\mathcal{N}_{11}) \cong \mathrm{CAut}(\mathcal{N}_{11})/\mathrm{Aut}(\mathcal{N}_{11}) \cong \mathbb{Z}_4$. The subdegrees of $\mathrm{Aut}(\mathcal{N}_{11})$ are 1^1, 3^4, 13^1, 39^4, 156^1. The 2-orbits of valency 3 are antisymmetric. Relations of valency 13 and 39 correspond to symmetric bipartite graphs, in particular (R_6, Ω) is the incidence graph of $\mathcal{B}_{13}$.

Clearly, the group $\mathrm{Aut}(\mathcal{N}_{11})$ is a subgroup of $\mathrm{Aut}(\mathrm{Inc}(\mathcal{B}_{13}))$. A more careful analysis shows that $\mathrm{Aut}(\mathcal{N}_{11}) = (\mathbb{Z}_{13}^2 : K) \cdot \mathbb{Z}_2$, where K is a subgroup of $\mathrm{UGL}(2, 13)$ of index 4. More precisely, $K = \mathrm{USL}(2, 13) \cdot \mathbb{Z}_3$, where $\mathrm{USL}(2, 13)$ consists of upper triangular matrices, whose determinant is equal to 1. COCO detected 13 mergings of $\mathcal{N}_{11}$, among them four combinatorially isomorphic non-Schurian rank 5 mergings $\mathcal{N}_{5,2}$ with valencies $1, 12, 52, 117, 156$. Such a merging appears as a WL-closure of a bipartite graph Γ of valency 52. This graph Γ is a union of 4 copies of $\mathrm{Inc}(\mathcal{B}_{13})$. It is significant to mention that Γ is not arc-transitive. The group $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\mathcal{N}_{11}) = \mathrm{Aut}(\mathcal{N}_{5,2})$ splits edges of Γ into two 2-orbits of valency 13 and 39. However, in the context of the generated AS, these edges are not distinguishable.

We also computed the spectrum of the basic graphs for $\mathcal{N}_{5,2}$. In particular,

$$\Lambda(\Gamma) = \{\pm 52^1, 0^{24}, \pm\sqrt{39}^{156}\}.$$

The considered example appears as the first member of an infinite family of rank 5 non-Schurian ASs $\mathcal{N}_{5,2}$ with the automorphism group of rank 11 and order $p^3 \cdot (p-1)^2/2$. Such an AS $\mathcal{N}_{5,2}$ exists for each prime $p = 12k + 1$, in particular, for $p = 13, 37, \ldots$. It is generated by a bipartite graph Γ of valency $p(p + 3)/4$, which is not edge-transitive. Its diameter is 3 for $p = 13$.

10 Links to known combinatorial structures

It turns out that our master CC $\mathcal{M}$, as well as the diverse ASs arising as its mergings, are essentially related to a number of known combinatorial structures. These structures are holders of other kinds of nice symmetry, expressible in terms of extremal and spectral graph theory. We will consider not only sporadic examples, but also some infinite families.

As usual, the smallest case $p = 3$ attracts special attention. We just recall once more that the graph defined by U_0 for $p = 3$ is the famous Pappus graph. Recall that an SRG Γ is called *primitive*, if both Γ and its complement $\bar{\Gamma}$ are connected. Using COCO, we confirmed existence of such a rank 3 primitive merging just for $p = 5$ (in comparison with a few other investigated values of p). It is generated by the famous Hoffman-Singleton graph, which is SRG with parameters $(50, 7, 0, 1)$, and can be described in terms of $\mathcal{M}$. More precisely, to obtain this graph we have to take the union of relations A_1, A_4, C_1, C_4, E_0 and F_0. This construction seems to be one of the motivations for McKay, Miller and Širáň in their discovery of an infinite family of graphs on $2q^2$ vertices, where q is a prime power. The original description was given in [43] in terms of voltage assignments. Here we consider only the graphs with $2p^2$ vertices, p an odd prime. We are using notation $\mathcal{H}(p)$ for the graphs considered. The original description was simplified a few times by different authors, see references in [48]. For our purposes the best one is that given by Hafner (see [25]).

McKay-Miller-Širáň graphs may be obtained as a suitable union of relations of $\mathcal{M}$. Specifically:

$$\mathcal{H}(p) = E_0 \cup F_0 \cup \bigcup_{i \in X} A_i \cup \bigcup_{j \in X'} C_j.$$

In particular, $\mathcal{H}(5)$ is the Hoffman-Singleton graph [30], which was discussed above. The McKay-Miller-Širáň graphs are currently the best known solutions to the degree/diameter problem for diameter 2 and valency $(3q-1)/2$, where q is a prime power.

The description provided of the graph $\mathcal{H}(p)$ stresses promising potential of the methodology presented in the current paper for the purposes of further clever hunting for extremal graphs. Indeed, to get a graph $\mathcal{H}(p)$ we start from $\mathrm{Inc}(\mathcal{B}_p)$ and simply add to it some very natural mergings of restrictions of $\mathcal{M}$ on $\mathcal{P}$ and $\mathcal{L}$. Clearly there is a lot of freedom how to generalize graphs $\mathcal{H}(p)$, provided correct vision of their structure is used in the role of the starting polygon. At this stage we simply refer to [2], [3] and [8]. We warn the reader that the terminology used in these texts strongly differs from the one in the current paper. On the other hand, we pay attention to the short note [4], written in a friendly style, with nicely depicted graph $\mathcal{H}(3)$.

The incidence graphs of biaffine planes are regarded as a particular case of a rich family of so-called *Wenger graphs* [54]. In a recent paper [17] the spectra of these graphs are derived, and it also contains many references concerning Wenger graphs.

Knowledge of the spectrum of a graph Γ provides general possibility to judge symmetry of Γ from one more point of view. Usually, graphs with a few distinct eigenvalues are regarded as of most interest. For example, graphs with three distinct eigenvalues serve as a spectral analogue of a concept of an SRG, see, e.g. [44]. In this context it is clear that graphs appearing in the master CC $\mathcal{M}$ are of a special interest. For families $\mathcal{M}_1 - \mathcal{M}_4$ of non-Schurian ASs, as well as for mergings of small rank, the number of different eigenvalues of basic graphs remains small.

One more interesting class of regular graphs is also defined in terms of spectra. Recall that a connected simple regular graph Γ of valency k is called *Ramanujan* if and only if for all its eigenvalues λ with $|\lambda| \neq k$ we have $|\lambda| \leq 2\sqrt{k-1}$. We refer to a nice book [50], where the concept of Ramanujan graph is considered, in particular, jointly with the representation theory of Heisenberg groups. The incidence graphs of $\mathcal{B}_p$ provide one of the simplest and well-known classes of Ramanujan graphs.

Example 10.1. We consider again the rank 5 merging of $\mathcal{M}$ presented in Section 9 for $p = 13$. Its basic graphs have valencies 1, 12, 52, 117 and 156. The spectrum of the basic graph Γ_2 of valency 52 is $\{52^1, \sqrt{39}^{156}, 0^{24}, -\sqrt{39}^{156}, -52^1\}$. Surely, $\sqrt{39} < 2\sqrt{51}$. Moreover, the "spectral gap" $\lambda_1 - \lambda_2$ is quite large here. Thus, Γ_2 is indeed a Ramanujan graph on 338 vertices.

Remark 10.1. It is well-known that the most interesting Ramanujan graphs have small valency in comparison with the number of vertices. Thus the provided graph does not carry any great surprises. Nevertheless, the way how it was considered might be of some interest for experts.

There is one more fairly well known graph on 18 vertices that may be easily found inside our master CC on 18 points. This graph belongs to the family of *directed strongly regular graphs* (briefly DSRGs), a natural generalization of strongly regular (undirected) graphs to the case of mixed graphs. This concept was introduced by A. Duval in his seminal paper [18]. We will use the established notation (n, k, t, λ, μ) for its parameter set. The discussed graphs are regular digraphs of valency k, and satisfy $AJ = JA = kJ$ and $A^2 = t \cdot I + \lambda \cdot A + \mu(J - I - A)$, where I is the identity matrix. Clearly, we always have $0 \le t \le k$. If $t = k$ then we are getting the usual SRGs, while the case $t = 0$ corresponds to doubly regular tournaments. For $0 < t < k$ the wording *genuine DSRG* was suggested. The main ingredients of the theory of DSRGs were developed in [18] by Duval. In particular, he mentioned the existence of a DSRG with the parameters $(18, 4, 3, 0, 1)$, and constructed an infinite family of such graphs with parameters $(k^2 + k, k, 1, 0, 1)$, where $k \ge 2$. It seems that Duval was unaware of the fact that his particular graph on 18 vertices had already been discovered 10 years earlier by J. Bosák (see e.g. [12, 13]), who was looking for so-called *mixed Moore graphs*. In modern terms, these are DSRGs with $\lambda = 0$ and $\mu = 1$, which appear as a natural generalization of classical (undirected) Moore graphs. For this class of mixed graphs, Bosák developed a theory quite similar to the more general one established by Duval for all DSRGs later on.

For a long time Bosák's results remained undetected by the majority of experts in AGT. Fortunately, the authors of [45] rediscovered Bosák's publications and breathed new life into them. The results of Bosák were definitely not known to the authors of [21] and [35]. They, in fact, independently duplicated some of Bosák's constructions in terms of 2-designs, paying special attention to the case $(18, 4, 3, 0, 1)$. Nowadays we refer to the graph with these parameters as the *Bosák graph* B_{18}. This graph is one of the main heroes in [35]; see Example 7.2 of that paper. The depiction of B_{18} provided there vividly shows that it has $K_{3,3}$ as a quotient graph. Also it was shown in [35] that $G = \mathrm{Aut}(B_{18})$ has order 108, and an explicit set of generators for G was given. (Incidentally, the original depiction of B_{18} given by Bosák in [13] carries very much the same flavour as the one given in [35].) The graph B_{18} also attracted attention of Jørgensen [32] who gave a description of B_{18} as a Cayley graph over a suitable (non-Abelian) group of order 18. This led the current authors to prepare a draft (2013) in which B_{18} is considered in the framework of our master configuration $\mathcal{M}$ on 18 points. In particular, we showed that the coherent closure of B_{18} is a certain rank 7 Schurian AS which arises as a merging of $\mathcal{M}$. The arc set of B_{18} is described as a union of relations of $\mathcal{M}$, specifically A_1, C_2, E_0 and F_0. The undirected part of B_{18} is isomorphic to the Pappus graph.

11 Regarding mathematics of Dan Archdeacon

Our paper is dedicated to the memory of Dan Archdeacon. We feel the scope of his mathematical interests, style of presentation, and extraordinary personal features visible from his texts, to be very relevant to our own presentation.

D.A. started from classical combinatorics (Latin squares); however, he very quickly switched to graph theory, having a lucky possibility to participate in the establishment of research targets, development of new techniques and posing diverse attractive open problems in the area of topological graph theory (briefly TGT). In a sense, TGT was maturing together with him. His PhD thesis obtained reputation quite quickly as a ground-breaking and highly cited text in TGT.

In a number of papers D.A. was investigating embeddings of bipartite graphs. Later on this research scope was extended, with each new step relying more and more seriously on the use of voltage assignments. The concept of a voltage assignment is of a purely algebraic nature, presenting a significant ingredient of techniques from group theory, a demand of which naturally grew out of TGT.

The automorphism group G of a graph (map) Γ reflects symmetry of Γ. For many nice graphs a suitable subgroup of G may be reflected in an ad hoc picture, creating a special feeling of beauty for the reader. The paper [5] with self-explanatory title contains a number of such nice diagrams.

We continue with [6]; Theorem 2.2 in it deals with a family of maps M_n on $2n^2$ vertices of valency $2n$. It is shown that $\mathrm{Aut}(M_n)$ is a group of order $8n^3$, satisfying certain defining presentations. Special attention is paid to the maps $R_{10.13}$, $n = 3$ and $R_{76.20}$, $n = 5$. (The labeling refers to the famous census of regular maps by Marston Conder, 2009.)

For $n = p$, p an odd prime, the graphs considered coincide with the ones treated in our paper; see also [24]. We believe that many other graphs, touched upon in this article, will be targeted in TGT in future.

12 Concluding discussion

One of the goals of this paper was to arrange an arena suitable for interdisciplinary exchange of ideas at the intersection of graph theory, geometry, group theory and computer algebra. Below, we collect additional information which was not included to the main body of the text. To save space we have not includeid all the related references.

12.1 Biaffine plane

The authors learned the term "biaffine plane" from the paper [56] of P. Wild who used it in a few of his publications as well as in his Ph.D. thesis (University of London, 1980). The term was used earlier by G. Pickert, one of the classic experts of modern finite geometries. A more extensive bibliographical search resulted in the discovery of a paper written by A. Bennett [10] and published in 1925. This text does not contain any references. Thus, we ask experts in history of finite geometries to determine the earliest origins of this terminology.

Of course, the incidence graph of a biaffine plane is a certain antipodal DRG of diameter 4.

12.2 Heisenberg group

The name *Heisenberg group*, for a class of groups, goes back to *Werner Heisenberg*, one of the creators of quantum mechanics. He was definitely among the pioneers who were using the language of (non-commutative) matrix groups in physics. Real history of related efforts is not so straightforward, and became more clear recently, see e.g. a paper by Ian Aitchison et al. (2004). Original subjects of interest of physicist were Heisenberg groups over continuous fields, e.g. $\mathbb{R}$. As was mentioned, the texts [11, 50] played a significant role to pay attention of the physicists to the finite case. Another independent source of activity is [1]. Here the term Heisenberg group is not used, though actually it lives inside of the extended Clifford group.

These and other pioneering papers literally opened the box of Pandora. A handful of keywords for related modern research activities are: coherent states, discrete space-time, equivalent tight frames, mutually unbiased bases, quantum nets. Fortunately, a number of experts in physics realized natural links with AGT. We refer explicitly just to [40] and [51].

12.3 Wielandt's influence

Helmut Wielandt (1910–2001) was a student of Issai Schur. Wielandt is definitely one of the creators of modern finite permutation group theory, with a relatively short but influental textbook (1964), a pearl in this area. His rotaprint version of lectures [55] is not easily available and thus a less known text.

The concept of a k-closure, and in particular of a 2-closure of a permutation group (G, Ω), was introduced in [55]. In our paper the 2-closure is of a crucial significance. As was demonstrated, some of considered permutation groups naturally appear as full automorphism groups of a suitable color graph with vertex set Ω, provided a special combination of 2-orbits of (G, Ω) is found. Sometimes a detected color graph may be substituted by a unicolor graph. Consideration of the WL-closure in conjunction with the 2-closure turns out to be a very efficient tool.

The concept of a non-Schurian coherent algebra goes back to Schur (1933). Wielandt was considering it for a particular case of Schur rings, that is, coherent algebras W that admit a regular group as a subgroup of $\mathrm{Aut}(W)$. R. Pöschel introduced the term of a non-Schurian Schur ring in 1974. Later on this term was extended to arbitrary coherent algebras, see e.g. [37].

The idea to distinguish between Schurian and non-Schurian objects via the use of local invariants also goes back to Wielandt.

12.4 The Pappus graph

The Pappus graph P is the graph $\mathrm{Inc}(\mathcal{B}_3)$. The name and the concept has a quite long history. In the 19th century it was related to planar configurations, that is, to incidence systems having realization on the Euclidean plane, where blocks are represented by lines. Exactly in this context it was considered by T. Reye (1876) as the incidence graph of one of the three possible symmetric 9_3-configurations, see also a classical book by D. Hilbert and S. Cohn-Vossen (1932).

A literal breakthrough was reached in the seminal paper by Coxeter (1950), where self-dual configurations were considered together with their regular incidence (Levi) graphs. In particular, the Pappus configuration is simultaneously treated (with all details) via its (orthodoxal) planar realization, as well as with the aid of incidence structures. The Hamiltonian diagram of the Pappus graph appears exactly in that paper. Coxeter was considering this configuration again and again in quite diverse contexts. In particular, its full automorphism group was investigated.

The group $\mathrm{Aut}(P)$ is very "nice". The latter property may be expressed in various group-theoretical reformulations. This is why the Pappus graph frequently appears in modern AGT as one of the exceptional examples. To give just one striking case take, for instance, cubic symmetric tricirculants, see [41].

12.5 Graphs, maps and voltage assignments

A regular map is a symmetric tessellation of a closed surface. Typically, it is considered as an embedding of a given graph Γ on a surface. Here "symmetric" means that the full automorphism group of a map acts transitively on its flags. Classification of arc-transitive graphs admitting regular embeddings is one of the central activities in TGT. In fact, the Pappus graph P has a regular embedding on the torus. A diagram, visually giving such an embedding of P, is also presented in Coxeter (1950). It consists of "nice" hexagons in P. The automorphism group of this map is a subgroup of index 2 in $\mathrm{Aut}(P)$.

For graphs with imprimitive automorphism groups the search for possible regular maps is closely related to the consideration of the quotient graphs with respect to a suitable imprimitivity system. Here the techniques of voltage assignments is used. The text by Nedela and Škoviera (1997) is one of the first serious attempts to describe techniques necessary for maps.

12.6 Use of computers

As was mentioned, this project heavily depends on the use of computer tools.

Here we are mainly working with CCs and ASs, as well as with the permutation groups related to them. For this purpose, in 1990–92 a computer package was created in Moscow as a result of the activities of I.A. Faradžev and the second author. This package goes by the name COCO, and was introduced in [19]; see also [20] for deeper consideration of the used methodology and algorithms. COCO is still very helpful for performing initial computational experiments. Nevertheless, nowadays the mainstream of our computer aided activities is based on the use of the free software GAP [23] (Groups, Algorithms and Programming), in particular its share package GRAPE [49] which works in conjunction with nauty [42]. There are a few ongoing activities to transform COCO to a modern package with conditional title COCO II.

12.7 History and style of this paper

A few times in the current text there have been references to its previous version [24], available from the arXiv. Recall that the related research started in the framework of postdoctoral studies of the first author at the Ben-Gurion University of the Negev, beginning in November 2011. Quite soon a successful dialogue between an advisor and a beginning researcher had been transformed into an enjoyable mutual cooperation. The preliminary results of this cooperation were reported at a handful of conferences by both coauthors; see details in [24]. This unusual history of the project determined, in a sense, the style of [24]. We were wishing for the reader to become a witness of all the steps from the first computational experiments up to the final theoretic proofs. Also some of the data not included in this paper are available in [24]. On the other hand, certain presentational ingredients of the initial version are intentionally kept in the current paper. In particular, we mean elements of a heuristical methodology, which were implicitly touched on a few times before. One such topic is the calculation of the spectra of an infinite series of ASs. For metric ASs the methodology for such calculations is presented in [9]. In our case we were forced to combine computer aided activity with further steps on a theoretical level.

12.8 Remaining research challenges

For a long time the second author was acquainted with some ideas developed by Lazebnik, Ustimenko and Woldar; see references in [24], and in particular [57]. Roughly speaking, they were hunting for extremal bipartite graphs with $2p^k$ vertices, p a prime. Our case $k = 2$ is the simplest one in comparison with their efforts. To describe adjacency in a potential candidate graph, a system of equations over the field $\mathbb{Z}_p$, for two vectors from two copies of the vector space $(\mathbb{Z}_p)^k$, is typically used. This leaves a lot of degrees of freedom for future heuristic search.

We hope that the language of basic graphs of detected new ASs provides a new platform for further systematic investigation. The general case $2p^k$ turns out to be much more difficult; for example, even for $p = k = 3$ the use of traditional COCO turns out to be hopeless. Nevertheless, the description of an algebraic group for starting CC, and analysis of suitable algebraic non-Schurian mergings, seems to be an extraordinarily promising tool for a successful hunt for new kinds of clever analogues of the incidence graphs of generalized polygons; cf. again [57].

A reasonably striking goal is to classify all ASs that arise by merging of the master CC $\mathcal{M}$. The arithmetical structure of the number $p - 1$ seems to provide suitable information in attempts to reach such a classification. Another interesting task is to classify maps for basic graphs of detected non-Schurian ASs. Such maps are promising candidates for generalization of the concept of a regular map. Indeed, though the automorphism group of a basic graph Γ does not act transitively on its arcs, nevertheless the arcs of Γ admit combinatorial regularity inside a graph.

We also expect a potential impact on experts in quantum mechanics. Everything around the concept of a non-Schurian AS seems to provide a kind of new promising tool for physicists.

Recently, our attention has been drawn to [29]. Here, starting from an Hadamard matrix H of order n and an AS S of order n, an AS $S(H)$ of order $4n$ is constructed. This allows one to get a lot of schemes $S(H)$, which are isomorphic algebraically, but not combinatorially. There is definite hope that substitution of H by a generalized Hadamard matrix $GH(p,1)$ and of S by the wreath product of two ASs of order p may result in a better understanding of some classes of ASs, which appear as mergings of $\mathcal{M}$. We are glad to share this idea with the reader.

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