AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **67(2)** (2017), Pages 304–326

Cyclic m-cycle systems of complete graphs minus a 1-factor

Heather Jordon

Department of Mathematics and Computer Science
Albion College
Albion, MI 49224
U.S.A.
hjordon@albion.edu

Joy Morris

Department of Mathematics and Computer Science
University of Lethbridge
Lethbridge, AB
Canada T1K 3M4
joy@cs.uleth.ca

In honour of Dan Archdeacon.

Abstract

In this paper, we provide necessary and sufficient conditions for the existence of a cyclic m-cycle system of $K_n - I$ when m and n are even and $m \mid n$.

1 Introduction

Throughout this paper, K_n will denote the complete graph on n vertices, $K_n - I$ will denote the complete graph on n vertices with a 1-factor I removed (a 1-*factor* is a 1-regular spanning subgraph), and C_m will denote the m-cycle $(v_1, v_2, \ldots, v_m)$. An *m-cycle system* of a graph G is a set $\mathcal{C}$ of m-cycles in G whose edges partition the edge set of G. An m-cycle system is called *hamiltonian* if $m = |V(G)|$.

Several obvious necessary conditions for an m-cycle system $\mathcal{C}$ of a graph G to exist are immediate: $m \leq |V(G)|$, the degrees of the vertices of G must be even, and m must divide the number of edges in G. A survey on cycle systems is given in [4] and necessary and sufficient conditions for the existence of an m-cycle system of K_n and $K_n - I$ were given in [1, 16] where it was shown that an m-cycle system of K_n or $K_n - I$ exists if and only if $n \geq m$, every vertex of K_n or $K_n - I$ has even degree, and m divides the number of edges in K_n or $K_n - I$, respectively.

Throughout this paper, ρ will denote the permutation $(0\,1\,\ldots\,n-1)$, so $\langle\rho\rangle = \mathbb{Z}_n$. An m-cycle system $\mathcal{C}$ of a graph G with vertex set $V(G) = \mathbb{Z}_n$ is *cyclic* if, for every m-cycle $C = (v_1, v_2, \ldots, v_m)$ in $\mathcal{C}$, the m-cycle $\rho(C) = (\rho(v_1), \rho(v_2), \ldots, \rho(v_m))$ is also in $\mathcal{C}$. A cyclic n-cycle system $\mathcal{C}$ of a graph G with vertex set $\mathbb{Z}_n$ is called a *cyclic hamiltonian cycle system.* Finding necessary and sufficient conditions for cyclic m-cycle systems of K_n is an interesting problem and has attracted much attention (see, for example, [2, 3, 6, 7, 10, 11, 13, 15]). The obvious necessary conditions for a cyclic m-cycle system of K_n are the same as for an m-cycle system of K_n; that is, $n \geq m \geq 3$, n is odd (so that the degree of every vertex is even), and m must divide the number of edges in K_n. However, these conditions are no longer necessarily sufficient. For example, it is not difficult to see that there is no cyclic decomposition of K_{15} into 15-cycles. Also, if p is an odd prime and $\alpha \geq 2$, then K_{p^α} cannot be decomposed cyclically into p^α-cycles [7].

The existence question for cyclic m-cycle systems of K_n has been completely settled in a few small cases, namely $m = 3$ [14], 5 and 7 [15]. For even m and $n \equiv 1 \pmod{2m}$, cyclic m-cycle systems of K_n are constructed for $m \equiv 0 \pmod 4$ in [13] and for $m \equiv 2 \pmod 4$ in [15]. Both of these cases are handled simultaneously in [10]. For odd m and $n \equiv 1 \pmod{2m}$, cyclic m-cycle systems of K_n are found using different methods in [2, 6, 11]. In [3], as a consequence of a more general result, cyclic m-cycle systems of K_n for all positive integers m and $n \equiv 1 \pmod{2m}$ with $n \geq m \geq 3$ are given using similar methods. In [7], it is shown that a cyclic hamiltonian cycle system of K_n exists if and only if $n \neq 15$ and $n \notin \{p^\alpha \mid p \text{ is an odd prime and } \alpha \geq 2\}$. Thus, as a consequence of a result in [6], cyclic m-cycle systems of K_{2mk+m} exist for all $m \neq 15$ and $m \notin \{p^\alpha \mid p \text{ is an odd prime and } \alpha \geq 2\}$. In [17], the last remaining cases for cyclic m-cycle systems of K_{2mk+m} are settled, i.e., it is shown that, for $k \geq 1$, cyclic m-cycle systems of K_{2km+m} exist if $m = 15$ or $m \in \{p^\alpha \mid p \text{ is an odd prime and } \alpha \geq 2\}$. In [19], necessary and sufficient conditions for the existence of cyclic $2q$-cycle and m-cycle systems of the complete graph are given when q is an odd prime power and $3 \leq m \leq 32$. In [5], cycle systems with a sharply vertex-transitive automorphism group that is not necessarily cyclic are investigated. As a result, it is shown in [5] that no cyclic m-cycle system of K_n exists if $m < n < 2m$ with n odd and $\gcd(m, n)$ a prime power. In [18], it is shown that if m is even and $n > 2m$, then there exists a cyclic m-cycle system of K_n if and only if the obvious necessary conditions that n is odd and that $n(n-1) \equiv 0 \pmod{2m}$ hold.

These questions can be extended to the case when n is even by considering the graph $K_n - I$. In [3], it is shown that for all integers $m \geq 3$ and $k \geq 1$, there exists a cyclic m-cycle system of $K_{2mk+2} - I$ if and only if $mk \equiv 0, 3 \pmod 4$. In [12], it is shown that for an even integer $n \geq 4$, there exists a cyclic hamiltonian cycle system of $K_n - I$ if and only if $n \equiv 2, 4 \pmod 8$ and $n \neq 2p^\alpha$ where p is an odd prime and $\alpha \geq 1$. In [8], it was shown that in every cyclic cycle decomposition of $K_{2n} - I$, the number of cycle orbits of odd length must have the same parity as $n(n-1)/2$. As a consequence of this result, in [8], it is shown that a cyclic m-cycle system of $K_{2n} - I$ can not exist if $n \equiv 2, 3 \pmod 4$ and $m \not\equiv 0 \pmod 4$ or $n \equiv 0, 1 \pmod 4$ and m does not divide $n(n-1)$. In this paper we are interested in cyclic m-cycle systems of $K_n - I$ when m and n are even and $m \mid n$. The main result of this paper is the following.

Theorem 1.1 *For an even integer m and integer t, there exists a cyclic m-cycle system of $K_{mt} - I$ if and only if*

(1) $t \equiv 0, 2 \pmod 4$ *when* $m \equiv 0 \pmod 8$,

(2) $t \equiv 0, 1 \pmod 4$ *when* $m \equiv 2 \pmod 8$ *with* $t > 1$ *if* $m = 2p^\alpha$ *for some prime* p *and integer* $\alpha \geq 1$,

(3) $t \geq 1$ *when* $m \equiv 4 \pmod 8$*, and*

(4) $t \equiv 0, 3 \pmod 4$ *when* $m \equiv 6 \pmod 8$.

Our methods involve circulant graphs and difference constructions. In Section 2, we give some basic definitions and lemmas while the proof of Theorem 1.1 is given in Sections 3, 4 and 5. In Section 3, we handle the case when $m \equiv 0 \pmod 8$ and show that there is a cyclic m-cycle system of $K_{mt} - I$ if and only if $t \geq 2$ is even. In Section 4, we handle the case when $m \equiv 4 \pmod 8$ and show that there is a cyclic m-cycle system of $K_{mt} - I$ if and only if $t \geq 1$. In Section 5, we handle the case when $m \equiv 2 \pmod 4$. When $m \equiv 2 \pmod 8$, we show that there is a cyclic m-cycle system of $K_{mt} - I$ if and only if $t \equiv 0, 1 \pmod 4$. When $m \equiv 6 \pmod 8$, we show that there is a cyclic m-cycle system of $K_{mt} - I$ if and only if $t \equiv 0, 3 \pmod 4$. Our main theorem then follows.

2 Preliminaries

The notation $[1, n]$ denotes the set $\{1, 2, \ldots, n\}$. The proof of Theorem 1.1 uses circulant graphs, which we now define. For $x \not\equiv 0 \pmod n$, the *modulo n length* of an integer x, denoted $|x|_n$, is defined to be the smallest positive integer y such that $x \equiv y \pmod n$ or $x \equiv -y \pmod n$. Note that for any integer $x \not\equiv 0 \pmod n$, it follows that $|x|_n \in [1, \lfloor \frac{n}{2} \rfloor]$. If L is a set of modulo n lengths, we define the *circulant graph* $\langle L \rangle_n$ to be the graph with vertex set $\mathbb{Z}_n$ and edge set $\{\{i, j\} \mid |i - j|_n \in L\}$. Notice that in order for a graph G to admit a cyclic m-cycle decomposition, G must be a circulant graph, so circulant graphs provide a natural setting in which to construct cyclic m-cycle decompositions.

The graph K_n is a circulant graph, since $K_n = \langle \{1, 2, \ldots, \lfloor n/2 \rfloor\} \rangle_n$. For n even, $K_n - I$ is also a circulant graph, since $K_n - I = \langle \{1, 2, \ldots, (n-2)/2\} \rangle_n$ (so the edges of the 1-factor I are of the form $\{i, i + n/2\}$ for $i = 0, 1, \ldots, (n-2)/2$).

Let H be a subgraph of a circulant graph $\langle L \rangle_n$. The notation $\ell(H)$ will denote the set of modulo n edge lengths belonging to H, that is,

$$\ell(H) = \{\ell \in L \mid \{g, g + \ell\} \in E(H) \text{ for some } g \in \mathbb{Z}_n\}.$$

Many properties of $\ell(H)$ are independent of the choice of L; in particular, the next lemma in this section does not depend on the choice of L.

Let C be an m-cycle in circulant graph $\langle L\rangle_n$ and recall that the permutation $\rho = (0\,1\,\ldots\,n{-}1)$, which generates $\mathbb{Z}_n$, has the property that $\rho(C) \in \mathcal{C}$ whenever $C \in \mathcal{C}$. We can therefore consider the action of $\mathbb{Z}_n$ as a permutation group acting on the elements of $\mathcal{C}$. Viewing matters this way, the *length of the orbit of* C (under the action of $\mathbb{Z}_n$) can be defined as the least positive integer k such that $\rho^k(C) = C$. Observe that such a k exists since ρ has finite order; furthermore, the well-known orbit-stabilizer theorem (see, for example [9, Theorem 1.4A(iii)]) tells us that k divides n. Thus, if G is a graph with a cyclic m-cycle system $\mathcal{C}$ with $C \in \mathcal{C}$ in an orbit of length k, then it must be that k divides $n = |V(G)|$ and that $\rho(C), \rho^2(C), \ldots, \rho^{k-1}(C)$ are distinct m-cycles in $\mathcal{C}$.

The next lemma gives many useful properties of an m-cycle C in a cyclic m-cycle system $\mathcal{C}$ of a graph G with $V(G) = \mathbb{Z}_n$ where C is in an orbit of length k. Many of these properties are also given in [7] in the case that $m = n$. The proofs of the following statements follow directly from the previous definitions and are therefore omitted.

Lemma 2.1 *Let $\mathcal{C}$ be a cyclic m-cycle system of a graph G of order n and let $C \in \mathcal{C}$ be in an orbit of length k. Then*

(1) $|\ell(C)| = mk/n$;

(2) C has n/k edges of length ℓ for each $\ell \in \ell(C)$;

(3) $(n/k) \mid \gcd(m, n)$;

Let $k > 1$ and let $P : v_0 = 0, v_1, \ldots v_{mk/n}$ be a subpath of C of length mk/n. Then

(4) if there exists $\ell \in \ell(C)$ with $k \mid \ell$, then $m = n/\gcd(\ell, n)$,

(5) $v_{mk/n} = kx$ for some integer x with $\gcd(x, n/k) = 1$,

(6) $v_1, v_2, \ldots, v_{mk/n}$ are distinct modulo k,

(7) $\ell(P) = \ell(C)$, and

(8) $P, \rho^k(P), \rho^{2k}(P), \ldots, \rho^{n-k}(P)$ are pairwise edge-disjoint subpaths of C.

Let X be a set of m-cycles in a graph G with vertex set $\mathbb{Z}_n$ such that $\mathcal{C} = \{\rho^i(C) \mid C \in X, i = 0, 1, \ldots, n-1\}$ is an m-cycle system of G. Then X is called a *generating set* for $\mathcal{C}$. Clearly, every cyclic m-cycle system $\mathcal{C}$ of a graph G has a generating set X as we may always let $X = \mathcal{C}$. A generating set X is called a *minimum generating set* if $C \in X$ implies $\rho^i(C) \notin X$ for $1 \le i \le n$ unless $\rho^i(C) = C$.

Let $\mathcal{C}$ be a cyclic m-cycle system of a graph G with $V(G) = \mathbb{Z}_n$. To find a minimum generating set X for $\mathcal{C}$, we start by adding C_1 to X if the length of the orbit of C_1 is maximum among the cycles in $\mathcal{C}$. Next, we add C_2 to X if the length of the orbit of C_2 is maximum among the cycles in $\mathcal{C} \setminus \{\rho^i(C_1) \mid 0 \le i \le n-1\}$. Continuing in this manner, we add C_3 to X if the length of the orbit of C_3 is maximum among the cycles in $\mathcal{C} \setminus \{\rho^i(C_1), \rho^i(C_2) \mid 0 \le i \le n-1\}$. We continue in this manner until $\{\rho^i(C) \mid C \in X, 0 \leq i \leq n-1\} = \mathcal{C}$. Therefore, every cyclic m-cycle system has a minimum starter set. Observe that if X is a minimum generating set for a cyclic m-cycle system $\mathcal{C}$ of the graph $\langle L \rangle_n$, then it must be that the collection of sets $\{\ell(C) \mid C \in X\}$ forms a partition of L.

In this paper, we are interested in the cyclic m-cycle systems of $K_n - I$ where $n = mt$ for some positive integer t. Suppose K_n has a cyclic m-cycle system $\mathcal{C}$ for some $n = mt$. Let X be a minimum generating set for $\mathcal{C}$ and let $C \in X$ be a cycle in an orbit of length k. Then, $\ell(C)$ has $mk/n = k/t$ lengths which implies that $k = \ell t$ for some integer ℓ. Also, since $|\ell(C)| = \ell$, it follows that $\ell \mid m$. The following lemma will be useful in determining the congruence classes of t based on the congruence class of m modulo 8.

Lemma 2.2 *Let m be an even integer and let $K_{mt} - I$ have a cyclic m-cycle system for some positive integer t.*

(1) If $\{1, 2, \ldots, (mt-2)/2\}$ has an odd number of even integers, then t is even.

(2) If $\{1, 2, \ldots, (mt-2)/2\}$ has an odd number of odd integers, then t is odd.

Proof: Let m be even and suppose $K_{mt} - I$ has a cyclic m-cycle system $\mathcal{C}$ for some positive integer t. Let $V(K_{mt}) = \mathbb{Z}_{mt}$, and let X be a minimum generating set for $\mathcal{C}$.

Suppose first that $\{1, 2, \ldots, (mt-2)/2\}$ has an odd number of even integers. Since the set $\{\ell(C) \mid C \in X\}$ is a partition of $\{1, 2, \ldots, (mt-2)/2\}$, there must be an odd number of cycles C in X with $\ell(C)$ containing an odd number of evens. Let $C \in X$ be a cycle in an orbit of length k with an odd number of even edge lengths. Let $|\ell(C)| = \ell$ and note that $k = \ell t$. From Lemma 2.1, we know that the subpath of C starting at vertex 0 of length ℓ ends at vertex jk with $\gcd(j, m/\ell) = 1$.

Suppose first k is odd. Then ℓ and t must both be odd. Thus m/ℓ is even so that jk is odd. Hence, $\ell(C)$ contains an odd number of odd integers and, since $|\ell(C)|$ is odd, an even number of even integers, contradicting the choice of C. Thus, k is even. Since k is even, jk is even. Thus, $\ell(C)$ contains an even number of odd integers. If ℓ is even, then $\ell(C)$ also contains an even number of even integers, contradicting the choice of C. Thus, ℓ is odd. Since k is even and $k = \ell t$, it must be that t is even.

Now suppose $\{1, 2, \ldots, (mt-2)/2\}$ has an odd number of odd integers. Hence there are an odd number of cycles C in X with $\ell(C)$ containing an odd number of odd integers. Again, let $C \in X$ be such a cycle with $|\ell(C)| = \ell$, in an orbit of length $k = \ell t$. Let the subpath of C starting at vertex 0 of length ℓ end at vertex jk with $\gcd(j, m/\ell) = 1$. Now, if k is even, then jk is even so that $\ell(C)$ contains an even number of odd integers, contradicting the choice of C. Thus k is odd. Since $k = \ell t$, we have that t is odd. □

The following corollary is an immediate consequence of Lemma 2.2 and [12].

Corollary 2.3 *For an even integer m and a positive integer t, if there exists a cyclic m-cycle system of $K_{mt} - I$, then*

(1) $t \equiv 0, 2 \pmod 4$ *when* $m \equiv 0 \pmod 8$,

(2) $t \equiv 0, 1 \pmod 4$ *when* $m \equiv 2 \pmod 8$ *with* $t > 1$ *if* $m = 2p^\alpha$ *for some prime* p *and integer* $\alpha \geq 1$,

(3) $t \equiv 0, 3 \pmod 4$ *when* $m \equiv 6 \pmod 8$, *and*

(4) $t \geq 1$ *when* $m \equiv 4 \pmod 8$.

Let $n > 0$ be an integer and suppose there exists an ordered m-tuple $(d_1, d_2, \ldots, d_m)$ satisfying each of the following:

(i) d_i is an integer for $i = 1, 2, \ldots, m$;

(ii) $|d_i| \neq |d_j|$ for $1 \leq i < j \leq m$;

(iii) $d_1 + d_2 + \cdots + d_m \equiv 0 (\bmod n)$; and

(iv) $d_1 + d_2 + \cdots + d_r \not\equiv d_1 + d_2 + \cdots + d_s (\bmod n)$ for $1 \leq r < s \leq m$.

Then an m-cycle C can be constructed from this m-tuple, that is, let $C = (0, d_1, d_1 + d_2, \ldots, d_1 + d_2 + \cdots + d_{m-1})$, and $\{C\}$ is a minimum generating set for a cyclic m-cycle system of $\langle \{d_1, d_2, \ldots, d_m\} \rangle_n$. Thus, in what follows, to find cyclic m-cycle systems of $\langle L \rangle_n$, it suffices to partition L into m-tuples satisfying the above conditions. Hence, an m-tuple satisfying (i)-(iv) above is called a *difference m-tuple* and it *corresponds* to the m-cycle $C = (0, d_1, d_1 + d_2, \ldots, d_1 + d_2 + \cdots d_{m-1})$ in $\langle L \rangle_n$.

3 The Case when $m \equiv 0 \pmod 8$

In this section, we consider the case when $m \equiv 0 \pmod 8$ and show that there exists a cyclic m-cycle system of $K_{mt} - I$ for each even positive integer t. We begin with the case $t = 2$.

Lemma 3.1 *For each positive integer* $m \equiv 0 \pmod 8$, *there exists a cyclic m-cycle system of* $K_{2m} - I$.

Proof: Let m be a positive integer such that $m \equiv 0 \pmod 8$, say $m = 8r$ for some positive integer r. Then $K_{2m} - I = \langle S' \rangle_{2m}$ where $S' = \{1, 2, \ldots, m-1\} = \{1, 2, \ldots, 8r - 1\}$. The proof proceeds as follows. We begin by finding a path P of length $m/2 = 4r$, ending at vertex m, so that $C = P \cup \rho^m(P)$ is an m-cycle. Note that $\langle \{2\} \rangle_{2m}$ consists of two vertex disjoint m-cycles. For the remaining $4r - 2$ edge lengths in $S' \setminus (\ell(P) \cup \{2\})$, we find $2r - 1$ paths P_i of length 2, ending at vertex 4 or -4, so that $C_i = P_i \cup \rho^4(P_i) \cup \rho^8(P_i) \cup \cdots \rho^{2m-4}(P_i)$ is an m-cycle. Then this collection of cycles will give a minimum generating set for a cyclic m-cycle system of $K_{2m} - I$.

Suppose first that r is odd. For $r = 1$, let $P : 0, -3, 3, 7, 8$ and note that the edge lengths of P in the order encountered are $3, 6, 4, 1$. For $r = 3$, let

$$P : 0, -3, 3, -7, 7, -11, 11, 23, 19, 20, -20, -4, 24$$

and note that edge lengths of P in the order encountered are $3, 6, 10, 14, 18, 22, 12, 4, 1, 8, 16, 20$. For $r \geq 5$, let

$$\begin{aligned} P \;:\; & 0, -3, 3, -7, 7, \ldots, -(4r-1), 4r-1, 8r-1, 8r-5, 8r-4, 8r+4, 8r-8, \\ & 8r+8, \ldots, 6r+2, 10r-2, 6r-10, 10r+2, 6r-14, \ldots, 12r-8, 4r-4, 8r \end{aligned}$$

be a path of length $m/2$ whose edge lengths in the order encountered are $3, 6, 10, 14, \ldots, 8r-6, 8r-2, 4r, 4, 1, 8, 12, 16, \ldots, 4r-4, 4r+8, 4r+12, \ldots, 8r-8, 8r-4, 4r+4$.

Now suppose that r is even. For $r = 2$, let $P : 0, -3, 3, -7, 7, -1, -5, -4, 16$ and note that the edge lengths of P in the order encountered are $3, 6, 10, 14, 8, 4, 1, 12$. For $r \geq 4$, let

$$\begin{aligned} P \;:\; & 0, -3, 3, -7, 7, \ldots, -(4r-1), 4r-1, -1, -5, -4, 4, -8, 8, \ldots, -(2r-4), \\ & 2r-4, -2r, 2r+8, -(2r+4), 2r+12, \ldots, -(4r-8), 4r, -(4r-4), 8r \end{aligned}$$

be a path of length $m/2$ whose edge lengths in the order encountered are $3, 6, 10, 14, \ldots, 8r-6, 8r-2, 4r, 4, 1, 8, 12, 16, \ldots, 4r-8, 4r-4, 4r+8, 4r+12, \ldots, 8r-8, 8r-4, 4r+4$.

In each case, let $C = P \cup \rho^m(P)$ and observe that C is an m-cycle C with $\ell(C) = \{1, 3, 4, 6, 8, \ldots, 8r-2\}$. Let $C' = (0, 2, 4, 6, \ldots, 2m-2)$ and note that C' is an m-cycle with $\ell(C') = \{2\}$.

For $0 \leq i \leq r-2$, let $P_i : 0, 9+8i, 4$ be the path of length 2 with edge lengths $9+8i, 5+8i$ and let $P'_i : 0, 11+8i, 4$ be the path of length 2 with edge lengths $11+8i, 7+8i$. Let $C_i = P_i \cup \rho^4(P_i) \cup \rho^8(P_i) \cup \cdots \cup \rho^{2m-4}(P_i)$ and $C'_i = P'_i \cup \rho^4(P'_i) \cup \rho^8(P'_i) \cup \cdots \cup \rho^{2m-4}(P'_i)$ and note that each is an m-cycle with $\ell(C_i) = \{5+8i, 9+8i\}$ and $\ell(C'_i) = \{7+8i, 11+8i\}$.

Finally, let $P'' : 0, 8r-3, -4$ be the path of length 2 with edge lengths $8r-3$ and $8r-1$. Let $C'' = P'' \cup \rho^4(P'') \cup \rho^8(P'') \cup \cdots \cup \rho^{2m-4}(P'')$ and note that C'' is an m-cycle with $\ell(C'') = \{8r-3, 8r-1\}$.

Then $\{C, C', C_0, \ldots, C_{r-2}, C'_0, \ldots, C'_{r-2}, C''\}$ is a minimum generating set for a cyclic m-cycle system of $K_{2m} - I$. □

We now consider the case when t is even and $t > 2$.

Lemma 3.2 *For each positive integer k and each positive integer $m \equiv 0 \pmod 8$, there exists a cyclic m-cycle system of $K_{2mk} - I$.*

Proof: Let m and k be positive integers such that $m \equiv 0 \pmod 8$. Lemma 3.1 handles the case when $k = 1$ and thus we may assume that $k \geq 2$. Then $K_{2km} - I = \langle S' \rangle_{2km}$ where $S' = \{1, 2, \ldots, km-1\}$. Since $K_{2m} - I$ has a cyclic m-cycle system by Lemma 3.1 and $\langle \{k, 2k, \ldots, mk\} \rangle_{2km}$ consists of k vertex-disjoint copies of $K_{2m} - I$, we need only show that $\langle S \rangle_{2km}$ has a cyclic m-cycle system where $S = \{1, 2, \ldots, mk\} \setminus \{k, 2k, \ldots, mk\}$.

Let $A = [a_{i,j}]$ be the $(k-1) \times m$ array

$$\begin{bmatrix} k-1 & 2k-1 & 3k-1 & 4k-1 & & (m-1)k-1 & mk-1 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 2 & k+2 & 2k+2 & 3k+2 & & (m-2)k+2 & (m-1)k+2 \\ 1 & k+1 & 2k+1 & 3k+1 & & (m-2)k+1 & (m-1)k+1 \end{bmatrix}.$$

It is straightforward to verify that A satisfies

$$\sum_{j \equiv 0,1 \pmod 4} a_{i,j} = \sum_{j \equiv 2,3 \pmod 4} a_{i,j},$$

and

$$a_{i,1} < a_{i,2} < \ldots < a_{i,m}$$

for each i with $1 \le i \le k-1$.

For each $i = 1, 2, \ldots, k-1$, the m-tuple

$$(a_{i,1}, -a_{i,3}, a_{i,5}, -a_{i,7}, \ldots, a_{i,m-3}, -a_{i,m-1}, -a_{i,m-2}, a_{i,m-4}, -a_{i,m-6}, \ldots,$$
$$-a_{i,6}, a_{i,4}, -a_{i,2}, a_{i,m})$$

is a difference m-tuple and corresponds to an m-cycle C_i with $\ell(C_i) = \{a_{i,1}, a_{i,2}, \ldots, a_{i,m}\}$. Hence, $X = \{C_1, C_2, \ldots, C_{k-1}\}$ is a minimum generating set for a cyclic m-cycle system of $\langle S \rangle_{2km}$. □

4 The Case when $m \equiv 4 \pmod 8$

In this section, we consider the case when $m \equiv 4 \pmod 8$ and show that there exists a cyclic m-cycle system of $K_{mt} - I$ for each $t \ge 1$. We begin with the case when t is odd, say $t = 2k+1$ for some nonnegative integer k.

Lemma 4.1 *For each nonnegative integer k and each $m \equiv 4 \pmod 8$, there exists a cyclic m-cycle system of $K_{m(2k+1)} - I$.*

Proof: Let m and k be nonnegative integers such that $m \equiv 4 \pmod 8$. Since $K_m - I$ has a cyclic hamiltonian cycle system [12], we may assume that $k \ge 1$. Let $m = 4r$ for some positive integer r. Then $K_{m(2k+1)} - I = \langle S' \rangle_{(2k+1)m}$ where $S' = \{1, 2, \ldots, 4rk + 2r - 1\}$. Again, since $K_m - I$ has a cyclic hamiltonian cycle system [12] and $\langle \{2k+1, 4k+2, \ldots, (2r-1)(2k+1)\} \rangle_{(2k+1)m}$ consists of $2k+1$ vertex-disjoint copies of $K_m - I$, we need only show that $\langle S \rangle_{(2k+1)m}$ has a cyclic m-cycle system where

$$S = \{1, 2, \ldots, 4rk + 2r - 1\} \setminus \{2k+1, 4k+2, \ldots, (2r-1)(2k+1)\}.$$

Let r and k be positive integers. Let $A = [a_{i,j}]$ be the $k \times m$ array

$$\begin{bmatrix} k & 2k & 3k+1 & 4k+1 & 5k+2 & & (4r-2)k+2r-2 & (4r-1)k+2r-1 & 4rk+2r-1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 2 & k+2 & 2k+3 & 3k+3 & 4k+4 & & (4r-3)k+2r & (4r-2)k+2r+1 & (4r-1)k+2r+1 \\ 1 & k+1 & 2k+2 & 3k+2 & 4k+3 & & (4r-3)k+2r-1 & (4r-2)k+2r & (4r-1)k+2r \end{bmatrix}.$$

It is straightforward to verify that A satisfies

$$\sum_{j\equiv 0,1 \pmod 4} a_{i,j} = \sum_{j\equiv 2,3 \pmod 4} a_{i,j},$$

and

$$a_{i,1} < a_{i,2} < \ldots < a_{i,m}$$

for each i with $1 \le i \le k$.

For each $i = 1, 2, \ldots, k$, the m-tuple

$$(a_{i,1}, -a_{i,3}, a_{i,5}, -a_{i,7}, \ldots, a_{i,m-3}, -a_{i,m-1}, -a_{i,m-2}, a_{i,m-4}, -a_{i,m-6}, \ldots,$$
$$-a_{i,6}, a_{i,4}, -a_{i,2}, a_{i,m})$$

is a difference m-tuple and corresponds to an m-cycle C_i with $\ell(C_i) = \{a_{i,1}, a_{i,2}, \ldots, a_{i,m}\}$. Hence, $X = \{C_1, C_2, \ldots, C_k\}$ is a minimum generating set for a cyclic m-cycle system of $K_{m(2k+1)} - I$. □

We now handle the case when t is even, say $t = 2k$ for some positive integer k.

Lemma 4.2 *For each positive integer k and each $m \equiv 4 \pmod 8$, there exists a cyclic m-cycle system of $K_{2mk} - I$.*

Proof: As before, let m and k be positive integers such that $m \equiv 4 \pmod 8$. Thus $m = 4r$ for some positive integer r. Then $K_{2mk} - I = \langle S' \rangle_{2km}$ where $S' = \{1, 2, \ldots, 4rk - 1\}$. Since $K_m - I$ has a cyclic hamiltonian cycle system [12] and $\langle\{2k, 4k, \ldots, (2r-1)(2k)\}\rangle_{2km}$ consists of $2k$ vertex-disjoint copies of $K_m - I$, we need only show that $\langle S \rangle_{2km}$ has a cyclic m-cycle system where

$$S = \{1, 2, \ldots, 4rk - 1\} \setminus \{2k, 4k, \ldots, (2r-1)(2k)\}.$$

Since $|S| = m(k-1) + m/2$, we will start by partitioning a subset $T \subseteq S$ with $|T| = m(k-1)$ into $k-1$ difference m-tuples.

Let $T = \{1, 2, \ldots, 4rk-1\} \setminus \{1, 2k, 4k-1, 4k, 4k+1, 6k, 8k-1, 8k, 8k+1, \ldots, (4r-4)k-1, (4r-4)k, (4r-4)k+1, (4r-2)k, 4rk-1\}$, and observe that $|T| = (k-1)m$. Let $A = [a_{i,j}]$, with entries from the set T, be the $(k-1) \times m$ array

$$\begin{bmatrix} k & 2k-1 & 3k-1 & 4k-2 & 5k & 6k-1 & 7k-1 & 8k-2 & 9k & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 3 & k+2 & 2k+2 & 3k+1 & 4k+3 & 5k+2 & 6k+2 & 7k+1 & 8k+3 & \\ 2 & k+1 & 2k+1 & 3k & 4k+2 & 5k+1 & 6k+1 & 7k & 8k+2 & \end{bmatrix}$$

$$\cdots \begin{array}{llll} (4r-3)k & (4r-2)k-1 & (4r-1)k-1 & 4rk-2 \\ \vdots & \vdots & \vdots & \vdots \\ (4r-4)k+3 & (4r-3)k+2 & (4r-2)k+2 & (4r-1)k+1 \\ (4r-4)k+2 & (4r-3)k+1 & (4r-2)k+1 & (4r-1)k \end{array} \Bigg] .$$

It is straightforward to verify that the array A satisfies

$$\sum_{j \equiv 0,1 \pmod 4} a_{i,j} = \sum_{j \equiv 2,3 \pmod 4} a_{i,j},$$

and

$$a_{i,1} < a_{i,2} < \ldots < a_{i,m}$$

for each i with $1 \leq i \leq k-1$.

For each $i = 1, 2, \ldots, k-1$, the m-tuple

$$(a_{i,1}, -a_{i,3}, a_{i,5}, -a_{i,7}, \ldots, a_{i,m-3}, -a_{i,m-1}, -a_{i,m-2}, a_{i,m-4}, -a_{i,m-6}, \ldots,$$
$$-a_{i,6}, a_{i,4}, -a_{i,2}, a_{i,m})$$

is a difference m-tuple and corresponds to an m-cycle C_i with $\ell(C_i) = \{a_{i,1}, a_{i,2}, \ldots, a_{i,m}\}$. Hence, $X = \{C_1, C_2, \ldots, C_{k-1}\}$ is a minimum generating set for a cyclic m-cycle system of $\langle T \rangle_{2km}$.

It now remains to find a minimum generating set for a cyclic m-cycle system of $\langle B \rangle_{2km}$ where $B = \{1, 4k-1, 4k+1, 8k-1, 8k+1, \ldots, (4r-4)k-1, (4r-4)k+1, 4rk-1\}$. For $i = 1, 2, \ldots, r$, define $d_{2i-1} = 4(i-1)k + 1$ and $d_{2i} = 4ik - 1$. Observe that $B = \{d_1, d_2, \ldots, d_{2r}\}$ and $d_{j+2} - d_j = 4k$ for $j = 1, 2, \ldots, 2r-2$. Since $m \equiv 4 \pmod 8$, it follows that r is odd. Let $P_1 : 0, 1, 4k$, and let $P_i : 0, d_{2i+1}, 4k$ if i is even and let $P_i : 0, d_{2i}, 4k$ if i is odd. Let $C'_i = P_i \cup \rho^{4k}(P_i) \cup \rho^{8k}(P_i) \cup \cdots \cup \rho^{(2m-4)k}(P_i)$, and note that C'_i is an m-cycle with $\ell(C'_1) = \{1, 4k-1\}$, $\ell(C'_i) = \{d_{2i-1}, d_{2i+1}\}$ if i is even, and $\ell(C'_i) = \{d_{2i-2}, d_{2i}\}$ if i is odd. Then $\ell(C'_1) \cup \ell(C'_2) \cup \cdots \cup \ell(C'_r) = B$ so that $\{C'_1, C'_2, \ldots, C'_r\}$ is a minimum generating set for $\langle B \rangle_{2km}$. □

5 The Case when $m \equiv 2 \pmod 4$

In this section, we consider the case when $m \equiv 2 \pmod 4$ and prove parts (2) and (4) of Theorem 1.1. We divide this proof into three parts, each dealt with in its own subsection. First we consider the case $t \equiv 0 \pmod 4$. Then we consider the case $m \equiv 2 \pmod 8$ and $t \equiv 1 \pmod 4$. Finally we consider the case $m \equiv 6 \pmod 8$ and $t \equiv 3 \pmod 4$.

5.1 The case when $t \equiv 0 \pmod 4$.

We consider the case $t \equiv 0 \pmod 4$, starting with the special case $t = 4$.

Lemma 5.1 *For each positive integer $m \geq 6$ with $m \equiv 2 \pmod 4$, there exists a cyclic m-cycle system of $K_{4m} - I$.*

Proof: Let $m \geq 6$ be a positive integer with $m \equiv 2 \pmod 4$. Then $K_{4m} - I = \langle S' \rangle_{4m}$ where $S' = \{1, 2, \ldots, 2m-1\}$. The proof proceeds as follows. We begin by finding one difference m-tuple which corresponds to an m-cycle C with $|\ell(C)| = m$. Note that $\langle \{4\} \rangle_{4m}$ consists of four vertex disjoint m-cycles. For the remaining $m-2$ edge lengths in $S' \setminus (\ell(C) \cup \{4\})$, we find $(m-2)/2$ paths P_i of length 2, ending at vertex 8 or -8, so that $C_i = P_i \cup \rho^8(P_i) \cup \rho^{16}(P_i) \cup \cdots \cup \rho^{4m-8}(P_i)$ is an m-cycle. Then this collection of cycles will give a minimum generating set for a cyclic m-cycle system of $K_{4m} - I$.

Consider the difference m-tuple

$$(1, -2, 6, -10, \ldots, 2m-6, -(2m-2), -3, 8, -12, \ldots, 2m-12, -(2m-8), 2m-4)$$

and the corresponding m-cycle C with $\ell(C) = \{1, 2, 3, 6, 8, \ldots, 2m-2\}$. It is straightforward to verify that the odd vertices visited all lie between $-m+1$ and $m-1$ with no duplication. Similarly, the even vertices visited all lie between $-2m+4$ and -4, and have no duplication.

Let $C' = (0, 4, 8, \ldots, 4m-4)$ and note that C' is an m-cycle with $\ell(C') = \{4\}$.

Let $m = 8k + m'$, so m' is either 2 or 6. If $k = 0$, then $m' = 6$ and let $P : 0, 13, 8$ be the path of length 2 with edge lengths $11, 5$. Then, $C'' = P \cup \rho^8(P) \cup \rho^{16}(P)$ is a 6-cycle with $\ell(C'') = \{11, 5\}$. Then $\{C, C', C''\}$ is a minimum generating set for cyclic 6-cycle system of $K_{24} - I$. Now suppose that $k \geq 1$. For $0 \leq i \leq k-1$, let $P_i : 0, 13 + 16i, 8$ be the path of length 2 with edge lengths $13 + 16i, 5 + 16i$; let $P_i' : 0, 15 + 16i, 8$ be the path of length 2 with edge lengths $15 + 16i, 7 + 16i$; let $P_i'' : 0, 17 + 16i, 8$ be the path of length 2 with edge lengths $17 + 16i, 9 + 16i$; and let $P_i''' : 0, 19 + 16i, 8$ with edge lengths $19 + 16i, 11 + 16i$. Let $C_i = P_i \cup \rho^8(P_i) \cup \rho^{16}(P_i) \cup \cdots \cup \rho^{4m-8}(P_i)$, $C_i' = P_i' \cup \rho^8(P_i') \cup \rho^{16}(P_i') \cup \cdots \cup \rho^{4m-8}(P_i')$, $C_i'' = P_i'' \cup \rho^8(P_i'') \cup \rho^{16}(P_i'') \cup \cdots \cup \rho^{4m-8}(P_i'')$, and $C_i''' = P_i''' \cup \rho^8(P_i''') \cup \rho^{16}(P_i''') \cup \cdots \cup \rho^{4m-8}(P_i''')$ and note that each is an m-cycle with $\ell(C_i) = \{5+16i, 13+16i\}$, $\ell(C_i') = \{7+16i, 15+16i\}$, $\ell(C_i'') = \{9+16i, 17+16i\}$, and $\ell(C_i''') = \{11 + 16i, 19 + 16i\}$.

If $m' = 2$, then $\{C, C', C_0, C_0', C_0'', C_0''', \ldots, C_{k-1}, C_{k-1}', C_{k-1}'', C_{k-1}'''\}$ is a minimum generating set for a cyclic m-cycle system of $K_{4m} - I$. If $m' = 6$, then let $P_k : 0, 2m-1, -8$ and $P_k' : 0, 2m-3, -8$ be paths of length 2 with $\ell(P_k) = \{2m-1, 2m-7\}$ and $\ell(P_k') = \{2m-3, 2m-5\}$. Let $C_k = P_k \cup \rho^8(P_k) \cup \rho^{16}(P_k) \cup \cdots \cup \rho^{4m-8}(P_k)$ and $C_k' = P_k' \cup \rho^8(P_k') \cup \rho^{16}(P_k') \cup \cdots \cup \rho^{4m-8}(P_k')$ and observe that each is an m-cycle with $\ell(C_k) = \{2m-1, 2m-7\}$ and $\ell(C_k') = \{2m-3, 2m-5\}$. Thus, $\{C, C', C_0, C_0', C_0'', C_0''', \ldots, C_{k-1}, C_{k-1}', C_{k-1}'', C_{k-1}''', C_k, C_k'\}$ is a minimum generating set for a cyclic m-cycle system of $K_{4m} - I$. □

We now consider the case when $t \equiv 0 \pmod 4$ with $t > 4$.

Lemma 5.2 *For each positive integer k and each positive integer $m \equiv 2 \pmod 4$ with $m \geq 6$, there exists a cyclic m-cycle system of $K_{4mk} - I$.*

Proof: Let $m \geq 6$ and k be positive integers such that $m \equiv 2 \pmod 4$. Lemma 5.1 handles the case when $k = 1$ and thus we may assume that $k \geq 2$. Then

$K_{4km} - I = \langle S' \rangle_{4km}$ where $S' = \{1, 2, \ldots, 2km - 1\}$. Since $K_{4m} - I$ has a cyclic m-cycle system by Lemma 5.1 and $\langle \{k, 2k, \ldots, 2km\} \rangle_{4km}$ consists of k vertex-disjoint copies of $K_{4m} - I$, we need only show that $\langle S \rangle_{2km}$ has a cyclic m-cycle system where $S = \{1, 2, \ldots, 2km\} \setminus \{k, 2k, \ldots, 2km\}$.

Let $A = [a_{i,j}]$ be the $2k \times m$ array

$$\begin{bmatrix} 2k & 4k & 6k & 8k & & (m-1)2k & 2km \\ 2k-1 & 2k+1 & 6k-1 & 8k-1 & & (m-1)2k-1 & 2km-1 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 2 & 4k-2 & 4k+2 & 6k+2 & & (m-2)2k+2 & (m-1)2k+2 \\ 1 & 4k-1 & 4k+1 & 6k+1 & & (m-2)2k+1 & (m-1)2k+1 \end{bmatrix}.$$

(Observe that the second column does not follow the same pattern as the others.)

Let A' be the $(2k-2) \times m$ array obtained from A by deleting rows 1 and $k+1$. Then the entries in A' are precisely the elements of S. Also, it is straightforward to verify that A' satisfies

$$a_{i,j} + a_{i,j+3} = a_{i,j+1} + a_{i,j+2}$$

for each positive integer $j \equiv 3 \pmod 4$ with $j \leq m - 3$,

$$a_{i,1} + a_{i,2} + a_{i,m-3} + a_{i,m-1} = a_{i,m-2} + a_{i,m},$$

and

$$a_{i,1} < a_{i,2} < \ldots < a_{i,m}$$

for each i with $1 \leq i \leq 2k - 2$.

For each $i = 1, 2, \ldots, 2k - 2$, the m-tuple

$$(a_{i,1}, a_{i,2}, -a_{i,4}, a_{i,6}, -a_{i,8}, a_{i,10}, \ldots, -a_{i,m-2}, -a_{i,m}, a_{i,m-3}, -a_{i,m-5}, a_{i,m-7}, \ldots,$$
$$a_{i,3}, a_{i,m-1})$$

is a difference m-tuple and corresponds to an m-cycle C_i with $\ell(C_i) = \{a_{i,1}, a_{i,2}, \ldots, a_{i,m}\}$. Hence, $X = \{C_1, C_2, \ldots, C_{2k-2}\}$ is a minimum generating set for a cyclic m-cycle system of $\langle S \rangle_{4km}$. □

What remains is to find cyclic m-cycle systems of $K_{mt} - I$ for the appropriate odd values of t, which we do in the following subsections.

5.2 The case when $m \equiv 2 \pmod 8$ and $t \equiv 1 \pmod 4$.

In this subsection, we find a cyclic m-cycle system of $K_{mt} - I$ when $m \equiv 2 \pmod 8$ and $t \equiv 1 \pmod 4$. We begin with two special cases, namely when $m = 10$ or $t = 5$.

Lemma 5.3 *For each positive integer $t \equiv 1 \pmod 4$ with $t > 1$, there exists a cyclic 10-cycle system of $K_{10t} - I$.*

Proof: Let $t \equiv 1 \pmod 4$ with $t > 1$, say $t = 4s+1$ where $s \geq 1$. Then $K_{10t} - I = \langle S' \rangle_{10t}$ where $S' = \{1, 2, \ldots, 20s+4\}$. Consider the paths $P_1 : 0, 5t-1, 2t$ and $P_2 : 0, 5t-2, 2t$. Then, $\ell(P_1) = \{3t-1, 5t-1\}$ and $\ell(P_2) = \{3t-2, 5t-2\}$. For $i \in \{1,2\}$, let $C_i = P_i \cup \rho^{2t}(P_i) \cup \rho^{4t}(P_i) \cup \cdots \cup \rho^{8t}(P_i)$. Then clearly each C_i is an 10-cycle and $X = \{C_1, C_2\}$ is a minimum generating set for $\langle \{3t-2, 3t-1, 5t-2, 5t-1\} \rangle_{10t}$. Since $3t-3 = 12s$ and $5t-2 = 20s+3$, it remains to find a cyclic 10-cycle system of $\langle S \rangle_{10t}$ where $S = \{1, 2, \ldots, 12s, 12s+3, 12s+4, \ldots, 20s+2\}$. Let $A = [a_{i,j}]$ be the $2s \times 10$ array

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 8s+1 & 8s+3 & 12s+3 & 12s+4 & 12s+5 & 12s+6 \\ 5 & 6 & 7 & 8 & 8s+2 & 8s+4 & 12s+7 & 12s+8 & 12s+9 & 12s+10 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\ 8s-3 & 8s-2 & 8s-1 & 8s & 12s-2 & 12s & 20s-1 & 20s & 20s+1 & 20s+2 \end{bmatrix}.$$

Clearly, for each i with $1 \leq i \leq 2s$,

$$a_{i,2} + \sum_{j \equiv 0,1 \pmod 4} a_{i,j} = a_{i,1} + \sum_{j \equiv 2,3 \pmod 4} a_{i,j} \text{ (where } 3 \leq j \leq 10)$$

and

$$a_{i,1} < a_{i,2} < \ldots < a_{i,10}.$$

Thus the 10-tuple

$$(a_{i,1}, -a_{i,2}, a_{i,3}, -a_{i,5}, a_{i,7}, -a_{i,9}, -a_{i,8}, a_{i,6}, -a_{i,4}, a_{i,10})$$

is a difference 10-tuple and corresponds to a 10-cycle C'_i with $\ell(C'_i) = \{a_{i,1}, a_{i,2}, \ldots, a_{i,10}\}$. Hence, $X' = \{C'_1, C'_2, \ldots, C'_{2s}\}$ is a minimum generating set for a cyclic 10-cycle system of $\langle S \rangle_{10t}$. □

We now consider the case when $t = 5$.

Lemma 5.4 *For each positive integer* $m \equiv 2 \pmod 8$*, there exists a cyclic* m*-cycle system of* $K_{5m} - I$*.*

Proof: Let m be a positive integer such that $m \equiv 2 \pmod 8$, say $m = 8r+2$ for some positive integer r. By Lemma 5.3, we may assume $r \geq 2$. Then $K_{5m} - I = \langle S' \rangle_{5m}$ where $S' = \{1, 2, \ldots, 20r+4\}$.

Let $2r = 6q+4+b$ for integers $q \geq 0$ and $b \in \{0, 2, 4\}$. Let a be a positive integer such that $1 + \log_2(q+2) \leq a \leq 1 + \log_2(5q+2)$, and note that a exists since if $q = 0$ then $\log_2(q+2)$ is an integer, while if $q \geq 1$ then $2(q+2) = 2q+4 \leq 4q+2 < 5q+2$. For nonnegative integers i and j, define $d_{i,j} = 10(2r-i)+j$. Consider the path $P_{i,j} : 0, d_{i,j}, 5 \cdot 2^a$ and observe that $\ell(P_{i,j}) = \{10(2r-i)+j, 10(2r-i)+j-5\cdot 2^a\}$. If $0 < j < 10$, then $C_{i,j} = P_{i,j} \cup \rho^{10}(P_{i,j}) \cup \rho^{20}(P_{i,j}) \cup \cdots \cup \rho^{5m-10}(P_{i,j})$ is an m-cycle since $m \equiv 2 \pmod 8$ gives $\gcd(5 \cdot 2^a, 5m) = 10$. Thus, if $0 < j < 10$, $\ell(C_{i,j}) = \{10(2r-i)+j, 10(2r-i)+j-5\cdot 2^a\}$. Let

$$X = \{C_{0,j} \mid 1 \leq j \leq 4\} \cup \{C_{i,j} \mid 1 \leq i \leq q \text{ and } 1 \leq j \leq 6\} \cup \{C_{q+1,j} \mid 6-b+1 \leq j \leq 6\}$$

and let

$$\begin{aligned} B &= \{20r+j, 20r+j-5\cdot 2^a \mid 1 \le j \le 4\} \\ &\cup \{10(2r-i)+j, 10(2r-i)+j-5\cdot 2^a \mid 1 \le i \le q \text{ and } 1 \le j \le 6\} \\ &\cup \{10(2r-q-1)+j, 10(2r-q-1)+j-5\cdot 2^a \mid 6-b+1 \le j \le 6\}, \end{aligned}$$

where if $q = 0$ or $b = 0$, we take the corresponding sets to be empty as necessary. Now B will consist of $4r$ distinct lengths and X will be a minimum generating set for $\langle B\rangle_{5m}$ if $20r+4-5\cdot 2^a \le 10(2r-q-1)+6-b$. Note that $1+\log_2(q+2) \le a \le 1+\log_2(5q+2)$ gives $q+2 \le 2^{a-1} \le 5q+2$. So,

$$20r+4-[10(2r-q-1)+6-b] = 10q+8+b \le 10q+12$$

and

$$(10q+12)/10 < q+2 \le 2^{a-1}.$$

Thus $20r+4-5\cdot 2^a \le 10(2r-q-1)+6-b$ so that B consists of $4r$ distinct lengths, and X is a minimum generating set for $\langle B\rangle_{5m}$.

It remains to find a cyclic m-cycle system of $\langle S' \setminus B\rangle_{5m}$. The smallest length in B is $10(2r-q-1)+6-b+1-5\cdot 2^a$, and we wish to show $10(2r-q-1)+6-b-5\cdot 2^a \ge 12$. So,

$$10(2r-q-1)+6-b-12 = 20r-10q-16-b \ge 20r-10q-20$$

and $(20r-10q-20)/10 \ge 2r-q-2$. Now

$$2r-q-2 = 5q+2+b \ge 5q+2 \ge 2^{a-1}.$$

Hence, $10(2r-q-1)+6-b-5\cdot 2^a \ge 12$. Since $|B| = 4r$, we have $|S' \setminus B| = 20r+4-4r = 2(8r+2)$. Now

$$\begin{aligned} S' \setminus B &= \{1, 2, \dots, 10(2r-q-1)+6-b-5\cdot 2^a\} \\ &\cup \{10(2r-i)-5\cdot 2^a-3, 10(2r-i)-5\cdot 2^a-2, 10(2r-i)-5\cdot 2^a-1, \\ &\qquad 10(2r-i)-5\cdot 2^a \mid 0 \le i \le q\} \\ &\cup \{10(2r)+5-5\cdot 2^a, \dots, 10(2r-q-1)+6-b\} \\ &\cup \{10(2r-i)-3, 10(2r-i)-2, 10(2r-i)-1, 10(2r-i) \mid 0 \le i \le q\}. \end{aligned}$$

Note that each the sets $\{1, 2, \dots, 10(2r-q-1)+6-b-5\cdot 2^a\}$, $\{10(2r-i)-5\cdot 2^a-3, 10(2r-i)-5\cdot 2^a-2, 10(2r-i)-5\cdot 2^a-1, 10(2r-i)-5\cdot 2^a \mid 0 \le i \le q\}$, $\{10(2r)+5-5\cdot 2^a, \dots, 10(2r-q-1)+6-b\}$, and $\{10(2r-i)-3, 10(2r-i)-2, 10(2r-i)-1, 10(2r-i) \mid 0 \le i \le q\}$ has even cardinality and consists of consecutive integers. Therefore, we may partition $S' \setminus B$ into sets $T, S_1, S_2, \dots, S_{8r-4}$ where $T = \{1, 2, \dots, 12\}$ and for $i = 1, 2, \dots, 8r-4$, let $S_i = \{b_i, b_i+1\}$ with $b_1 < b_2 < \cdots < b_{8r-4}$.

Let $A = [a_{i,j}]$ be the $2 \times m$ array

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 9 & 11 & b_1 & b_1+1 & b_2 & b_2+1 & \cdots & b_{4r-2} & b_{4r-2}+1 \\ 5 & 6 & 7 & 8 & 10 & 12 & b_{4r-1} & b_{4r-1}+1 & b_{4r} & b_{4r}+1 & \cdots & b_{8r-4} & b_{8r-4}+1 \end{bmatrix}$$

It is straightforward to verify that, for $1 \le i \le 2$,

$$a_{i,2} + \sum_{j \equiv 0,1 \pmod 4} a_{i,j} = a_{i,1} + \sum_{j \equiv 2,3 \pmod 4} a_{i,j} \text{ (where } 3 \le j \le m)$$

and

$$a_{i,1} < a_{i,2} < \ldots < a_{i,m}.$$

Hence, for $1 \le i \le 2$, the m-tuple

$$(a_{i,1}, -a_{i,2}, a_{i,3}, -a_{i,5}, a_{i,7}, \ldots, a_{i,m-3}, -a_{i,m-1}, -a_{i,m-2}, a_{i,m-4}, -a_{i,m-6}, \ldots,$$
$$a_{i,6}, -a_{i,4}, a_{i,m})$$

is a difference m-tuple and corresponds to an m-cycle C_i with $\ell(C_i) = \{a_{i,1}, a_{i,2}, \ldots, a_{i,m}\}$. Hence, $X' = \{C_1, C_2\}$ is a minimum generating set for a cyclic m-cycle system of $\langle S' \setminus B \rangle_{5m}$. □

We are now ready to prove the main result of this subsection, namely, that $K_{mt} - I$ has a cyclic m-cycle system for every $t \equiv 1 \pmod 4$ and $m \equiv 2 \pmod 8$ with $t > 1$ if $m = 2p^\alpha$ for some prime p and integer $\alpha \ge 1$.

Lemma 5.5 *For each positive integer $t \equiv 1 \pmod 4$ and each $m \equiv 2 \pmod 8$ with $t > 1$ if $m = 2p^\alpha$ for some prime p and integer $\alpha \ge 1$, there exists a cyclic m-cycle system of $K_{mt} - I$.*

Proof: Let m and t be positive integers such that $m \equiv 2 \pmod 8$ and $t \equiv 1 \pmod 4$. Thus $m = 8r + 2$ for some positive integer r. Then $K_{mt} - I = \langle S' \rangle_{mt}$ where $S' = \{1, 2, \ldots, (mt - 2)/2\}$. Since $K_m - I$ has a cyclic hamiltonian cycle system [12] if and only if $m \ne 2p^\alpha$ for some prime p and integer $\alpha \ge 1$, we may assume that $t > 1$. Thus, let $t = 4s + 1$ for some positive integer s. By Lemmas 5.3 and 5.4, we may assume that $s \ge 2$ and $r \ge 2$.

The proof proceeds as follows. We begin by finding a set $B \subseteq S'$ such that $|B| = 4r$ and $\langle B \rangle_{mt}$ has a cyclic m-cycle system with a minimum generating set X consisting of cycles each with two distinct lengths and orbit $2t$. We then construct an $(|S' \setminus B|/m) \times m$ array $A = [a_{i,j}]$ with the property that for each i with $1 \le i \le |S' \setminus B|/m$,

$$a_{i,2} + \sum_{j \equiv 0,1 \pmod 4} a_{i,j} = a_{i,1} + \sum_{j \equiv 2,3 \pmod 4} a_{i,j} \text{ (where } 3 \le j \le m)$$

and

$$a_{i,1} < a_{i,2} < \ldots < a_{i,m}.$$

Thus for each $i = 1, 2, \ldots, |S' \setminus B|/m$, the m-tuple

$$(a_{i,1}, -a_{i,2}, a_{i,3}, -a_{i,5}, a_{i,7}, \ldots, a_{i,m-3}, -a_{i,m-1}, -a_{i,m-2}, a_{i,m-4}, -a_{i,m-6}, \ldots,$$
$$a_{i,6}, -a_{i,4}, a_{i,m})$$

is a difference m-tuple and corresponds to an m-cycle C_i with $\ell(C_i) = \{a_{i,1}, a_{i,2}, \dots, a_{i,m}\}$. Hence, $X' = \{C_1, C_2, \dots, C_{|S'\setminus B|/m}\}$ will be a minimum generating set for a cyclic m-cycle system of $\langle S' \setminus B\rangle_{mt}$.

Let $w = \lfloor r/2 \rfloor$, and let $\delta_r = 2(r/2-w)$, so that $\delta_r = 1$ if r is odd and $\delta_r = 0$ if r is even. Write $w = qs+b$ where q and b are non-negative integers with $0 \le b < s$ (note that it may be the case that $q = 0$). For integers i and j, define $d_{i,j} = 4(r-2i)t+j$. Consider the path $P_{i,j} : 0, d_{i,j}, 4t$ and observe that $\ell(P_{i,j}) = \{4(r-2i)t+j, 4(r-2i-1)t+j\}$. If $0 < j < t$, then $C_{i,j} = P_{i,j} \cup \rho^{2t}(P_{i,j}) \cup \rho^{4t}(P_{i,j}) \cup \cdots \cup \rho^{(m-2)t}(P_{i,j})$ is an m-cycle since $m \equiv 2 \pmod 8$ gives $\gcd(4t, mt) = 2t$. Thus, if $0 < j < t$, $\ell(C_{i,j}) = \{4(r-2i)t+j, 4(r-2i-1)t+j\}$. Let

$$X = \{C_{i,j} \mid 0 \le i \le q-1 \text{ and } 1 \le j \le t-1\} \cup \{C_{q,j} \mid t-4b-2\delta_r \le j \le t-1\}$$

and let

$$\begin{aligned} B \;=\; & \{4(r-2i)t+j, 4(r-2i-1)t+j \mid 0 \le i \le q-1 \text{ and } 1 \le j \le t-1\} \\ & \cup \{4(r-2q)t+j, 4(r-2q-1)t+j \mid t-4b-2\delta_r \le j \le t-1\}, \end{aligned}$$

where we take the appropriate sets to be empty if $q = 0$ or $b = 0$. Observe that X is a minimum generating set for $\langle B\rangle_{mt}$, and consider the set $S' \setminus B$. Now $|X| = 4qs+4b$ so that $|B| = 2(4qs+4b) = 4r$. Hence $|S' \setminus B| = (4r+1)t-1-4r = 2s(8r+2)$ and

$$\begin{aligned} S' \setminus B \;=\; & \{1, 2, \dots, 4(r-2q-1)t+t-1-2\delta_r-4b\} \\ & \cup \{4(r-2q-1)t+t, 4(r-2q-1)t+t+1, \dots, \\ & \qquad 4(r-2q)t+t-1-2\delta_r-4b\} \\ & \cup \{4kt+t, 4kt+t+1, \dots, 4(k+1)t \mid r-2q \le k \le r-1\}. \end{aligned}$$

Note that $S' \setminus B$ has been written as the disjoint union of sets, each of which has even cardinality and consists of consecutive integers.

The smallest length in B is $4(r-2q-1)t+t-4b-2\delta_r$, and we wish to show this length is at least $12s+1$. Now $r \ge 2w = 2(qs+b) > 2q+1$ since $s \ge 2$. Next since $0 \le b < s$ and $t = 4s+1$, we have $t-1-4b = 4s-4b \ge 4$. Therefore, $4(r-2q-1)t \ge 4t > 16s$, and thus $4(r-2q-1)t+t-3-4b > 16s+2 > 12s$. Since the smallest length is $S' \setminus B$ is at least $12s+1$ and since $S' \setminus B$ consists of sets of consecutive integers of even cardinality, we may partition $S' \setminus B$ into sets $T, S_1, \dots, S_{8rs-4s}$ where $T = \{1, 2, \dots, 12s\}$, and for $i = 1, 2, \dots, 8rs-4s$, $S_i = \{b_i, b_i+1\}$ with $b_1 < b_2 < \cdots < b_{8rs-4s}$. Let $A = [a_{i,j}]$ be the $2s \times m$ array

$$\left[\begin{array}{cccccccccccccc} 1 & 2 & 3 & 4 & 8s+1 & 8s+3 & b_1 & b_1+1 & & b_2 & b_2+1 & \cdots & b_{4r-2} & b_{4r-2}+1 \\ 5 & 6 & 7 & 8 & 8s+2 & 8s+4 & b_{4r-1} & b_{4r-1}+1 & & b_{4r} & b_{4r}+1 & \cdots & b_{8r-4} & b_{8r-4}+1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 8s-3 & 8s-2 & 8s-1 & 8s & 12s-2 & 12s & b_{8rs-4s-4r+3} & b_{8rs-4s-4r+3}+1 & & b_{8rs-4s-4r+4} & b_{8rs-4s-4r+4}+1 & \cdots & b_{8rs-4s} & b_{8rs-4s}+1 \end{array}\right].$$

Clearly, for each i with $1 \leq i \leq 2s$,

$$a_{i,2} + \sum_{j \equiv 0,1 \pmod 4} a_{i,j} = a_{i,1} + \sum_{j \equiv 2,3 \pmod 4} a_{i,j} \text{ (where } 3 \leq j \leq m)$$

and

$$a_{i,1} < a_{i,2} < \ldots < a_{i,m}.$$

Thus the m-tuple

$$(a_{i,1}, -a_{i,2}, a_{i,3}, -a_{i,5}, a_{i,7}, \ldots, a_{i,m-3}, -a_{i,m-1}, -a_{i,m-2}, a_{i,m-4}, -a_{i,m-6}, \ldots,$$
$$a_{i,6}, -a_{i,4}, a_{i,m})$$

is a difference m-tuple and corresponds to an m-cycle C_i with $\ell(C_i) = \{a_{i,1}, a_{i,2}, \ldots, a_{i,m}\}$. Hence, $X' = \{C_1, C_2, \ldots, C_{2s}\}$ is a minimum generating set for a cyclic m-cycle system of $\langle S' \setminus B \rangle_{mt}$. □

5.3 The Case when $m \equiv 6 \pmod 8$ and $t \equiv 3 \pmod 4$

In this subsection, we find a cyclic m-cycle system of $K_{mt} - I$ when $m \equiv 6 \pmod 8$ and $t \equiv 3 \pmod 4$. We begin with three special cases, namely when $m = 6$, $m = 14$, or $t = 3$. We first consider the case $m = 6$.

Lemma 5.6 *For all positive integers $t \equiv 3 \pmod 4$, there exists a cyclic 6-cycle system of $K_{6t} - I$.*

Proof: Let t be a positive integer such that $t \equiv 3 \pmod 4$, say $t = 4s + 3$ for some non-negative integer s. Then $K_{6t} - I = \langle S' \rangle_{6t}$ where $S' = \{1, 2, \ldots, 12s + 8\}$.

Consider the paths $P_i : 0, 3t - i, 2t$, for $1 \leq i \leq 4$; then $\ell(P_i) = \{3t - i, t - i\}$. Next, let $C_i = P_i \cup \rho^{2t}(P_i) \cup \rho^{4t}(P_i)$. Then each C_i is a 6-cycle and $X = \{C_1, C_2, C_3, C_4\}$ is a minimum generating set for $\langle B \rangle_{6t}$ where $B = \{3t - i, t - i \mid 1 \leq i \leq 4\}$. Now, $t - 5 = 4s - 2$ and thus $S' \setminus B = \{1, 2, \ldots, 4s - 2, 4s + 3, 4s + 4, \ldots, 12s + 4\}$, and so we must find a cyclic 6-cycle system of $\langle S' \setminus B \rangle_{6t}$. Let $A = [a_{i,j}]$ be the $2s \times 6$ array

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 8s+5 & 8s+7 \\ 5 & 6 & 7 & 8 & 8s+6 & 8s+8 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 4s-3 & 4s-2 & 4s+3 & 4s+4 & \alpha & \alpha+2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 8s+1 & 8s+2 & 8s+3 & 8s+4 & 12s+2 & 12s+4 \end{bmatrix}$$

where

$$\alpha = \begin{cases} 10s + 2 & \text{if s is even,} \\ 10s + 3 & \text{if s is odd.} \end{cases}$$

Clearly, for each i with $1 \le i \le 2s$,

$$a_{i,2} + \sum_{j\equiv 0,1 \pmod 4} a_{i,j} = a_{i,1} + \sum_{j \equiv 2,3 \pmod 4} a_{i,j} \quad \text{(where } 3 \le j \le 6\text{)}$$

and

$$a_{i,1} < a_{i,2} < \ldots < a_{i,6}.$$

Thus the 6-tuple

$$(a_{i,1}, -a_{i,2}, a_{i,3}, -a_{i,4}, -a_{i,5}, a_{i,6})$$

is a difference 6-tuple and corresponds to a 6-cycle C_i' with $\ell(C_i') = \{a_{i,1}, a_{i,2}, \ldots, a_{i,6}\}$. Hence, $X' = \{C_1', C_2', \ldots, C_{2s}'\}$ is a minimum generating set for a cyclic 6-cycle system of $\langle S' \setminus B\rangle_{6t}$. □

Next we consider the case when $m = 14$.

Lemma 5.7 *For all positive integers $t \equiv 3 \pmod 4$, there exists a cyclic 14-cycle system of $K_{14t} - I$.*

Proof: Let t be a positive integer such that $t \equiv 3 \pmod 4$, say $t = 4s + 3$ for some non-negative integer s. Then $K_{14t} - I = \langle S'\rangle_{14t}$ where $S' = \{1, 2, \ldots, 28s + 20\}$.

Consider the paths $P_i : 0, 7t - i, 2t$, for $1 \le i \le 10$; then $\ell(P_i) = \{7t - i, 5t - i\}$. Next, let $C_i = P_i \cup \rho^{2t}(P_i) \cup \rho^{4t}(P_i) \cup \cdots \cup \rho^{12t}(P_i)$. Then each C_i is a 14-cycle and $X = \{C_1, C_2, \ldots, C_{10}\}$ is a minimum generating set for $\langle B\rangle_{14t}$ where $B = \{7t-i, 5t-i \mid 1 \le i \le 10\}$. Now, $5t - 10 = 20s + 5$ and thus $S' \setminus B = \{1, 2, \ldots, 20s + 4, 20s + 15, 20s+16, \ldots, 28s+10\}$, and so we must find a cyclic 14-cycle system of $\langle S' \setminus B\rangle_{14t}$. Let $A = [a_{i,j}]$ be the $2s \times 14$ array

$$\left[\begin{array}{cccccccccccccc}
1 & 2 & 3 & 4 & 8s+1 & 8s+3 & 12s+1 & 12s+2 & 12s+3 & 12s+4 & 20s+1 & 20s+2 & 20s+3 & 20s+4 \\
5 & 6 & 7 & 8 & 8s+2 & 8s+4 & 12s+5 & 12s+6 & 12s+7 & 12s+8 & 20s+15 & 20s+16 & 20s+17 & 20s+18 \\
9 & 10 & 11 & 12 & 8s+5 & 8s+7 & 12s+9 & 12s+10 & 12s+11 & 12s+12 & 20s+19 & 20s+20 & 20s+21 & 20s+22 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
8s-3 & 8s-2 & 8s-1 & 8s & 12s-2 & 12s & 20s-3 & 20s-2 & 20s-1 & 20s & 28s+7 & 28s+8 & 28s+9 & 28s+10
\end{array}\right].$$

Clearly, for each i with $1 \le i \le 2s$,

$$a_{i,2} + \sum_{j\equiv 0,1 \pmod 4} a_{i,j} = a_{i,1} + \sum_{j \equiv 2,3 \pmod 4} a_{i,j} \quad \text{(where } 3 \le j \le 14\text{)}$$

and

$$a_{i,1} < a_{i,2} < \ldots < a_{i,14}.$$

Thus the 14-tuple

$$(a_{i,1}, -a_{i,2}, a_{i,3}, -a_{i,5}, a_{i,7}, -a_{i,9}, a_{i,11}, -a_{i,13}, -a_{i,12}, a_{i,10}, -a_{i,8}, a_{i,6}, -a_{i,4}, a_{i,14})$$

is a difference 14-tuple and corresponds to a 14-cycle C'_i with $\ell(C'_i) = \{a_{i,1}, a_{i,2}, \dots, a_{i,14}\}$. Hence, $X' = \{C'_1, C'_2, \dots, C'_{2s}\}$ is a minimum generating set for a cyclic 14-cycle system of $\langle S' \setminus B\rangle_{14t}$. □

We now consider the case when $t = 3$.

Lemma 5.8 *For all positive integers $m \equiv 6 \pmod 8$, there exists a cyclic m-cycle system of $K_{3m} - I$.*

Proof: Let m be a positive integer such that $m \equiv 6 \pmod 8$, say $m = 8r + 6$ for some non-negative integer r. By Lemmas 5.6 and 5.7, we may assume $r \geq 2$. Then $K_{3m} - I = \langle S'\rangle_{mt}$ where $S' = \{1, 2, \dots, 12r+8\}$. Write $2r = 4q+b+2$ for integers $q \geq 0$ and $b \in \{0, 2\}$, and let a be a positive integer such that $1 + \log_2(q + 1) \leq a \leq 1 + \log_2(3q + 4/3 + 5b/6)$. For integers i and j, define $d_{i,j} = 6(2r - i) + j$. Then consider the path $P_{i,j} : 0, d_{i,j}, 3 \cdot 2^a$; so $\ell(P_{i,j}) = \{6(2r - i) + j, 6(2r - i) + j - 3 \cdot 2^a\}$. Now, let $C_{i,j} = P_{i,j} \cup \rho^6(P_{i,j}) \cup \cdots \cup \rho^{3(m-2)}(P_{i,j})$. Then $C_{i,j}$ is an m-cycle since $m \equiv 6 \pmod 8$ implies $\gcd(3 \cdot 2^a, 3m) = 6$. Thus, $\ell(C_{i,j}) = \ell(P_{i,j})$.

Now, let

$$\begin{aligned} X &= \{C_{0,j} \mid j = 7, 8\} \\ &\quad \cup \{C_{i,j} \mid 0 \leq i \leq q - 1 \text{ and } 1 \leq j \leq 4\} \\ &\quad \cup \{C_{q,j} \mid 5 - b \leq j \leq 4\} \end{aligned}$$

and let

$$\begin{aligned} B &= \{12r + 7, 12r + 7 - 3 \cdot 2^a, 12r + 8, 12r + 8 - 3 \cdot 2^a\} \\ &\quad \cup \{6(2r - i) + j, 6(2r - i) - 3 \cdot 2^a + j \mid 0 \leq i \leq q - 1 \text{ and } 1 \leq j \leq 4\} \\ &\quad \cup \{6(2r - q) + j, 6(2r - q) - 3 \cdot 2^a + j \mid 5 - b \leq j \leq 4\} \end{aligned}$$

where, if $q = 0$ or $b = 0$, we take the corresponding sets to be empty as necessary. Now B will consists of $4r$ distinct lengths and X will be a minimum generating set for $\langle B\rangle_{3m}$ if $12r + 8 - 3 \cdot 2^a \leq 6(2r - q) + 5 - b - 1$. Note that $1 + \log_2(q + 1) \leq a$ so that $q + 1 \leq 2^{a-1}$. Next,

$$12r + 8 - [6(2r - q) + 5 - b - 1] = 6q + 4 + b \leq 6q + 6 = 6(q + 1) \leq 6 \cdot 2^{a-1} = 3 \cdot 2^a,$$

and hence $12r + 8 - 3 \cdot 2^a \leq 6(2r - q) + 5 - b - 1$. Thus, B consists of $4r$ distinct lengths, and X is a minimum generating set for $\langle B\rangle_{3m}$. Now, the smallest length in B is $6(2r - q) + 5 - b - 3 \cdot 2^a$ and we want this length to be greater than 8. Recall that $a \leq 1 + \log_2(3q + 3/2 + 5b/6)$ and thus $2^{a-1} \leq 3q + 3/2 + 5b/6$. Hence, $3 \cdot 2^a \leq$

$18q+9+5b = 12r-6q-3-b$ since $2r = 4q+b+2$. Therefore, $6(2r-q)+5-b-3\cdot 2^a \geq 8$. Since $|B| = 4r$, we have $|S' \setminus B| = 8r + 8$. Note that

$$\begin{aligned} S' \setminus B &= \{1, 2, \ldots, 6(2r-q)+5-b-3\cdot 2^a - 1\} \\ &\cup \{6(2r-i) - 3\cdot 2^a + 5, 6(2r-i) - 3\cdot 2^a + 6 \mid 0 \leq i \leq q\} \\ &\cup \{12r - 3\cdot 2^a + 9, \ldots, 6(2r-q)+5-b-1\} \\ &\cup \{6(2r-i)+5, 6(2r-i)+6 \mid 0 \leq i \leq q\}. \end{aligned}$$

Note that $S' \setminus B$ has been written as the disjoint union of sets, each of which has even cardinality and consists of consecutive integers. Therefore, we may partition $S' \setminus B$ into sets $T, S_1, S_2, \ldots, S_{4r}$ where $T = \{1, 2, \ldots, 8\}$ and for $i = 1, 2, \ldots, 4r$, let $S_i = \{b_i, b_i + 1\}$ with $b_1 < b_2 < \cdots < b_{4r}$.

Consider the m-tuple

$$\begin{aligned} (1, -3, 6, -7, b_1, -b_2, b_3, -b_4, \ldots, b_{4r-1}, -b_{4r}, -(b_{4r-1}+1), b_{4r-2}+1, \\ -(b_{4r-3}+1), b_{4r-4}+1, \ldots, b_2+1, -(b_1+1), 8, -5, b_{4r}+1) \end{aligned}$$

which is a difference m-tuple and corresponds to an m-cycle C_1 with

$$\ell(C_1) = \{1, 3, 5, 6, 7, 8, b_1, b_1+1, b_2, b_2+1, \ldots, b_{4r}, b_{4r}+1\}.$$

Then consider the path $P : 0, 2, 6$; so $\ell(P) = \{2, 4\}$. Now, let $C_2 = P \cup \rho^6(P) \cup \cdots \cup \rho^{3(m-2)}(P)$. Then C_2 is an m-cycle since $m \equiv 6 \pmod 8$ implies $\gcd(6, 3m) = 6$. Thus, $\ell(C_2) = \ell(P) = \{2, 4\}$. Hence, $X' = \{C_1, C_2\}$ is a minimum generating set for a cyclic m-cycle system of $\langle S' \setminus B\rangle_{3m}$. □

We now prove the main result of this subsection, namely that $K_{mt} - I$ has a cyclic m-cycle system for every $t \equiv 3 \pmod 4$ and $m \equiv 6 \pmod 8$.

Lemma 5.9 *For all positive integers $t \equiv 3 \pmod 4$ and $m \equiv 6 \pmod 8$, there exists a cyclic m-cycle system of $K_{mt} - I$.*

Proof: Let m and t be positive integers such that $m \equiv 6 \pmod 8$ and $t \equiv 3 \pmod 4$. Then $m = 8r + 6$ and $t = 4s + 3$ for some non-negative integers r and s. Then $K_{mt} - I = \langle S'\rangle_{mt}$ where $S' = \{1, 2, \ldots, (4r+3)t - 1\}$.

By Lemmas 5.6, 5.7, and 5.8, we may assume $s \geq 1$ and $r \geq 2$. First, write $6r+4 = (2t-2)q + (t-1)\ell + b$ for integers q, ℓ and b with $q \geq 0, 0 \leq b < 2t-2$, and $\ell = 0$ if $6r+4 < t-1$, or $\ell = 1$ otherwise. For integers i and j, define $d_{i,j} = 2t(2r-2i-1)+j$. Consider the path $P_{i,j} : 0, d_{i,j}, 2t$ and note that $\ell(P_{i,j}) = \{2t(2r-2i-1)+j, 2t(2r-2i-2)+j\}$. If $0 < j < 2t$, then $C_{i,j} = P_{i,j} \cup \rho^{2t}(P_{i,j}) \cup \rho^{4t}(P_{i,j}) \cup \cdots \cup \rho^{(m-2)t}(P_{i,j})$ is an m-cycle since $m \equiv 6 \pmod 8$ implies $\gcd(2t, mt) = 2t$. Thus, if $0 < j < 2t$, then $\ell(C_{i,j}) = \ell(P_{i,j})$.

Now, let

$$\begin{aligned} X &= \{C_{-1,j} \mid 1 \leq j \leq t-1\} \\ &\cup \{C_{i,j} \mid 0 \leq i \leq q-1 \text{ and } 1 \leq j \leq 2t-2\} \\ &\cup \{C_{q,j} \mid 2t-1-b \leq j \leq 2t-2\} \end{aligned}$$

and let

$$\begin{aligned} B &= \{2t(2r+1)+j, 2t(2r)+j \mid 1 \le j \le t-1\} \\ &\cup \{2t(2r-2i-1)+j, 2t(2r-2i-2)+j \mid 1 \le j \le 2t-2 \text{ and } 0 \le i \le q-1\} \\ &\cup \{2t(2r-(2q+1))+2t-1-b+j, 2t(2r-2q-2)+2t-1-b+j \\ &\qquad \mid 0 \le j \le b-1\}. \end{aligned}$$

where we take the first set to be empty if $\ell = 0$, the second to be empty if $q = 0$, and the third to be empty if $b = 0$. Then X is a minimum generating set for $\langle B \rangle_{mt}$. Now we must find a cyclic m-cycle system of $\langle S' \setminus B \rangle_{mt}$. First, $|B| = 2[(2t-2)q + (t-1)\ell + b] = 12r + 8$ so that $|S' \setminus B| = (4r+3)t - 1 - 12r - 8 = (8r+6)(2s)$. Moreover,

$$\begin{aligned} S' \setminus B &= \{1, 2, \ldots, 2t(2r-2q-1)-b-2\} \\ &\cup \{2t(2r-2q-1)-1, 2t(2r-2q-1), \ldots, 2t(2r-2q)-b-2\} \\ &\cup \{2t(2r-i)-1, 2t(2r-i) \mid 0 \le i \le 2q\} \\ &\cup \{4rt+t, 4rt+t+1, \ldots, 4rt+2t\}. \end{aligned}$$

The smallest length in B is $4t(r-q-1) + (2t-1) - b$, and we must verify that this length is at least $12s+1$. Note that we have $2t-1-b > 1$. Thus, it is sufficient to prove that $4t(r-q-1) \ge 12s$, or $t(r-q-1) \ge 3s$. This inequality follows if $r > q+1$. Clearly, this is true if $q = 0$ since $r \ge 2$, so assume $q \ge 1$. Then $\ell = 1$, and so $6r + 4 = 2q(4s+2) + (4s+2) + b$, or

$$\begin{aligned} 3r + 2 &= q(4s+2) + 2s + 1 + b/2 \\ &= 4qs + 2q + 2s + 1 + b/2 \\ &\ge 6q + 3 \;\; (\text{since } s \ge 1). \end{aligned}$$

So, $r \ge 2q + 1/3 > q + 1$ since $q \ge 1$. Since the smallest length in B is at least $12s+1$ and $S' \setminus B$ consists of sets of consecutive integers of even cardinality, we may partition $S' \setminus B$ into sets $T, S_1, \ldots, S_{8rs}$ where $T = \{1, 2, \ldots, 12s\}$, and for $i = 1, 2, \ldots, 8rs$, $S_i = \{b_i, b_i + 1\}$ with $b_1 \le b_2 \le \cdots \le b_{8rs}$. Let $A = [a_{i,j}]$ be the $2s \times m$ array

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 8s+1 & 8s+3 & b_1 & b_1+1 & \\ 5 & 6 & 7 & 8 & 8s+2 & 8s+4 & b_{4r+1} & b_{4r+1}+1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 8s-3 & 8s-2 & 8s-1 & 8s & 12s-2 & 12s & b_{8rs-4r+1} & b_{8rs-4r+1}+1 & \end{bmatrix}$$

$$\begin{matrix} b_2 & b_2+1 & \cdots & b_{4r} & b_{4r}+1 \\ b_{4r+2} & b_{4r+2}+1 & \cdots & b_{8r} & b_{8r}+1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ b_{8rs-4r+2} & b_{8rs-4r+2}+1 & \cdots & b_{8rs} & b_{8rs}+1 \end{matrix} \Bigg].$$

Clearly, for each i with $1 \le i \le 2s$,

$$a_{i,2} + \sum_{j \equiv 0,1 \pmod 4} a_{i,j} = a_{i,1} + \sum_{j \equiv 2,3 \pmod 4} a_{i,j} \text{ (where } 3 \le j \le m)$$

and

$$a_{i,1} < a_{i,2} < \ldots < a_{i,m}.$$

Thus the m-tuple

$$(a_{i,1}, -a_{i,2}, a_{i,3}, -a_{i,5}, a_{i,7}, \ldots, a_{i,m-3}, -a_{i,m-1}, -a_{i,m-2}, a_{i,m-4}, -a_{i,m-6}, \ldots,$$
$$a_{i,6}, -a_{i,4}, a_{i,m})$$

is a difference m-tuple and corresponds to an m-cycle C_i with $\ell(C_i) = \{a_{i,1}, a_{i,2}, \ldots, a_{i,m}\}$. Hence, $X' = \{C_1, C_2, \ldots, C_{2s}\}$ is a minimum generating set for a cyclic m-cycle system of $\langle S' \setminus B \rangle_{mt}$. □

References

[1] B. Alspach and H. Gavlas, Cycle decompositions of K_n and $K_n - I$, *J. Combin. Theory Ser. B* **81**(2001), 77–99.

[2] A. Blinco, S. El-Zanati and C. Vanden Eynden, On the cyclic decomposition of complete graphs into almost-bipartite graphs, *Discrete Math.* **284** (2004), 71–81.

[3] D. Bryant, H. Gavlas and A. Ling, Skolem-type difference sets for cycles, *Electronic J. Combin.* **10** (2003), #R38.

[4] D. Bryant and C. A. Rodger, Cycle Decompositions, in the *CRC Handbook of Combinatorial Designs, 2nd Ed.*, (eds. C. J. Colbourn and J. H. Dinitz), Chapman & Hall/CRC, Boca Raton FL (2007).

[5] M. Buratti, Cycle decompositions with a sharply vertex transitive automorphism group, *Le Matematiche* **59** (2004), 91–105.

[6] M. Buratti and A. Del Fra, Existence on cyclic k-cycle systems of the complete graph. *Discrete Math.* **261** (2003), 113–125.

[7] M. Buratti and A. Del Fra, Cyclic Hamiltonian cycle systems of the complete graph, *Discrete Math.*, **279** (2004), 107–119.

[8] M. Buratti and G. Rinaldi, A non-existence result on cyclic cycle-decompositions of the cocktail party graph, *Discrete Math.* **309** (2009) 4722–4726.

[9] J. D. Dixon and B. Mortimer, *Permutation Groups*, Springer-Verlag New York, Berlin, Heidelberg, Graduate Texts in Mathematics, **163**, 1996.

[10] S. I. El-Zanati, N. Punnim and C. Vanden Eynden, On the cyclic decomposition of complete graphs into bipartite graphs, *Australas. J. Combin.* **24** (2001), 209–219.

[11] H.-L. Fu and S.-L. Wu, Cyclically decomposing the complete graph into cycles, *Discrete Math.* **282** (2004), 267–273.

[12] H. Jordon and J. Morris, Cyclic hamiltonian cycle systems of the complete graph minus a 1-factor, *Discrete Math.* **308** (2008), 2440–2449.

[13] A. Kotzig, On decompositions of the complete graph into $4k$-gons, *Mat.-Fyz. Cas.* **15** (1965), 227–233.

[14] R. Peltesohn, Eine Losung der beiden Heffterschen Differenzenprobleme, *Compos. Math.* **6** (1938), 251-257.

[15] A. Rosa, On cyclic decompositions of the complete graph into $(4m+2)$-gons, *Mat.-Fyz. Cas.* **16** (1966), 349–352.

[16] M. Šajna, Cycle Decompositions III: Complete graphs and fixed length cycles, *J. Combin. Des.* **10** (2002), 27–78.

[17] A. Veitri, Cyclic k-cycles systems of order $2kn+k$; a solution of the last open cases, *J. Combin. Des.* **12** (2004), 299–310.

[18] S.-L. Wu, Cyclic even cycle systems of the complete graph, *J. Combin. Des.* **20** (2012), 23–29.

[19] S.-L. Wu and H.-L. Fu, Cyclic m-cycle systems with $m \leq 32$ or $m = 2q$ with q a prime power, *J. Combin. Des.* **14** (2006), 66–81.

(Received 13 Nov 2015; revised 19 May 2016)