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Beans functions of graphs with small beans

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Dedicated to the memory of Dan Archdeacon

Abstract

The beans function $B_G(x)$ of a connected graph G is defined as the maximum number of points on G such that any pair of points have distance at least $x > 0$. We shall exhibit lower and upper bounds of $B_G(x)$ for $x < 1$ and a class of graphs which have the same number of edges and whose beans functions distinguish them. Also, we give some non-isomorphic graphs which have the same beans function.

1 Introduction

Imagine that there are two glasses on a table. Fill each of the glasses with beans and count the number of beans used to fill it. If the numbers of beans are different, then we can conclude that these glasses have different shapes. If we used the same number of beans for both, then we try to fill them again with beans of another size. As the beans get smaller, then the number of beans will tend to be proportional to the volume. Conversely, if a glass has a very narrow neck, then large beans could lead to a significant amount of vacant space, and the number of beans can, to a degree, characterise different glass shapes. This situation leads us to consider a function counting the number of beans of given size in a glass. Such a function might distinguish the shapes of glasses. In this paper, we shall consider a similar notion for graphs, called "the beans function".

Let G be a connected graph without loops and multiple edges. We regard each edge as a line segment having a unit length and focus on not only the vertices but also any point lying along such a line segment. We denote the set of points on G by $X(G)$. This allows us to define the distance $d(p, q)$ between two points p and $q \in X(G)$ to be the length of a shortest curve joining them along G. Define $B_G(x)$ to be the maximum number of points placed on G so that any pair of points have distance at least x, where x ranges over the set of positive real numbers $\boldsymbol{R}^+$.

Then $B_G(x)$ can be regarded as a function $B_G : \boldsymbol{R}^+ \to \boldsymbol{N}$. We call this the *beans function* of G. We often use $U_\varepsilon(p)$ to present the ε-neighborhood of a point p, that is, $U_\varepsilon(p) = \{q \in X(G) : d(p,q) < \varepsilon\}$. This works as a "bean" of radius ε without its skin.

When x is a natural number, $B_G(x)$ is closely related to the combinatorial property of a connected graph G. For example, we have:

$$B_G(1) = \max\{\,|V(G)|, |E(G)|\,\}.$$

Just place points at all vertices or at the midpoints of all edges. If G is not a tree, then the latter case attains the maximum. Also, $B_G(2)$ is related to the maximum size of independent sets and of matchings. It is clear that $B_G(x) = 1$ if x exceeds the diameter of G plus 1.

In this paper, we shall focus on small beans, whose diameter is less than 1. One might suspect that $B_G(1/n) = n|E(G)|$ in most cases and wonder whether $B_G(x)$ depends only on $|E(G)|$. We shall establish sharp lower and upper bounds for $B_G(x)$ to give an affirmative answer to the former in Section 2 and show examples to deny the latter in Section 3. Also, we shall show examples for non-isomorphic graphs which have the same beans function in Section 4.

Our terminology for graph theory is quite standard and can be found in [2].

2 Lower and upper bounds

First, we shall give general lower and upper bound for the beans function $B_G(x)$ of a connected graph G with $0 < x < 1$. For the sake of convenience, we call a set of points on G an *x-set* if any pair of points in S have distance at least x.

A set of edges in a graph G is called a *matching* if any two distinct edges in it have no common endpoints. The *matching number* of G is defined as the number of edges contained in a matching of the maximum size and is denoted by $\mu(G)$ in this paper.

Theorem 1. *Let G be a connected graph and let $x \le 1$ be a positive real number. Put $\varepsilon = 1 - \lfloor 1/x \rfloor x$.*

(i) *If $\varepsilon \ge x/2$, then:*

$$\lfloor 1/x \rfloor \cdot |E(G)| + \mu(G) \;\le\; B_G(x) \;\le\; \lfloor 1/x \rfloor \cdot |E(G)| + |V(G)| - 1.$$

(ii) *If $\varepsilon < x/2$, then:*

$$\lfloor 1/x \rfloor \cdot |E(G)| \;\le\; B_G(x) \;\le\; \lfloor 1/x \rfloor \cdot |E(G)| + \mu(G).$$

If G is a tree, then this lower bound can be replaced with $\lfloor 1/x \rfloor \cdot |E(G)| + 1$.

Here we shall present an easy example to show the sharpness of the bounds given in Theorem 1. This will help the reader to undestand the proof below. Let C_k be a cycle of length $k \geq 3$. It is clear that $B_{C_k}(x) = \lfloor k/x \rfloor$ for all $x < 1$. Put $1 = n \cdot x + \varepsilon$ with $n \in \boldsymbol{N}$ and $0 \leq \varepsilon < x$. Then we have:

$$B_{C_k}(x) = \lfloor k/x \rfloor = \left\lfloor \frac{(n \cdot x + \varepsilon)k}{x} \right\rfloor = n \cdot k + \left\lfloor \frac{\varepsilon \cdot k}{x} \right\rfloor = \lfloor 1/x \rfloor \cdot |E(C_k)| + \left\lfloor \frac{\varepsilon \cdot k}{x} \right\rfloor.$$

This implies that $B_{C_k}(x) = \lfloor 1/x \rfloor \cdot |E(C_k)|$ if $\varepsilon = 0$ and that $B_{C_k}(x) = \lfloor 1/x \rfloor \cdot |E(C_k)| + \lfloor k/2 \rfloor$ if $\varepsilon = x/2$. Since $\mu(C_k) = \lfloor k/2 \rfloor$, we can say that C_k attains the lower bounds in (i) and (ii) in Theorem 1. On the other hand, if $\varepsilon = (k-1)/k \cdot x > x/2$, then $B_{C_k}(x) = \lfloor 1/x \rfloor \cdot |E(C_k)| + k - 1$. If ε is smaller than but very close to $x/2$ and if k is odd, then $B_{C_k}(x) = \lfloor 1/x \rfloor \cdot |E(C_k)| + \lfloor k/2 \rfloor$. These coincide with the two upper bounds in the theorem.

Proof of Theorem 1. Let S be an x-set of points of the maximum size for $x \leq 1$. That is, we have $|S| = B_G(x)$. Choose one of the points in S closest to each vertex $v \in V(G)$ and denote it by $\bar{v}$. If $v \in S$, then we set $\bar{v} = v$. We may assume that $\bar{u} \neq \bar{v}$ for distinct vertices u and v. Note that $\bar{v}$ exists at v or on an edge incident to v since $x \leq 1$; otherwise, $S \cup \{v\}$ would be a larger x-set. Put $d_v = d(\bar{v}, v)$. If $d_v > x/2$, then we can make another x-set from S by replacing $\bar{v}$ with the vertex v' lying on the segment between v and $\bar{v}$ with $d(v, v') = x/2$, since $d(p, v) \geq d(\bar{v}, v) > x/2$ for all points p in S around v. Thus, we may assume that $0 \leq d_v \leq x/2$ for each vertex v.

Let $e = uv$ be an edge of G having two endpoints u and v, and let S_e be the set of points in S lying along e. It may happen that $\bar{u}$ or $\bar{v}$ lies on e. If $\bar{u} = u$, then we choose only one of the edges incident to u, say e, and consider that $\bar{u}$ belongs to S_e and to no others. Under this setting, we have $S = \bigcup_{e \in E(G)} S_e$ and $S_e \cap S_{e'} = \emptyset$ for any two distinct edges e and e'. There are the following three cases for an edge $e = uv$.

Case 1: Suppose that S_e contains neither $\bar{u}$ nor $\bar{v}$. Then $e - U_{x-d_u}(u) \cup U_{x-d_v}(v)$ contains all points in S_e. If $2x - d_u - d_v \leq 1$, then the number of such points should be:

$$|S_e| = \left\lfloor \frac{1 - 2x + d_u + d_v}{x} \right\rfloor + 1 = \left\lfloor \frac{1 + d_u + d_v}{x} \right\rfloor - 1.$$

If $2x - d_u - d_v > 1$, then $e - U_{x-d_u}(u) \cup U_{x-d_v}(x) = \emptyset$ and hence $|S_e| = 0$. Since $x \leq 1 + d_u + d_v < 2x$, we have $\lfloor (1 + d_u + d_v)/x \rfloor = 1$. This means that the above formula for $|S_e|$ works in this case, too.

Case 2: Suppose that S_e contains only one of $\bar{u}$ and $\bar{v}$, say $\bar{u}$. Then $e - U_{d_u}(u) \cup U_{x-d_v}(v)$ contains all points in S_e; in particular, if $\bar{u} = u$, then $U_{d_u}(u) = U_0(u) = \emptyset$. If $d_u + x - d_v \leq 1$, then the number of such points should be:

$$|S_e| = \left\lfloor \frac{1 - d_u - x + d_v}{x} \right\rfloor + 1 = \left\lfloor \frac{1 - d_u + d_v}{x} \right\rfloor.$$

If $d_u + x - d_v > 1$, then $e - U_{d_u}(u) \cup U_{x-d_v}(v) = \emptyset$ and $|S_e| = 0$. Since $0 \le d_u, d_v \le x/2$, we have $0 < 1 - x/2 \le 1 - d_u + d_v < x$ and hence the above formula for $|S_e|$ gives 0 as we expect in this case.

CASE 3: Suppose that S_e contains both $\bar{u}$ and $\bar{v}$. Then $e - U_{d_u}(u) \cup U_{d_v}(v)$ contains all points in S_e. Since $x \le 1$, we have $1 - d_u - d_v \ge 1 - x \ge 0$. Thus, the number of such points should be:

$$|S_e| = \left\lfloor \frac{1 - d_u - d_v}{x} \right\rfloor + 1.$$

We call an edge "*type i*" if Case i happens for it. Let E_i be the set of edges of type i. Then $E_2 \cup E_3$ covers all vertices of G while any two distinct edges belonging to E_3 have no common endpoints, that is, E_3 forms a matching in G. Then $|S|$ is equal to the summation of these values $|S_e|$ for all edges according to their types. It is clear that $B_G(x)$ is the maximized value of $|S|$ by choosing d_u and $\bar{u}$ suitably for all vertices $u \in V(G)$.

Using $|E_3|$, we can express the others $|E_1|$ and $|E_2|$ as follows:

$$|E_2| = |V(G)| - 2|E_3|;$$
$$|E_1| = |E(G)| - |E_2| - |E_3| = |E(G)| - |V(G)| + |E_3|.$$

First, substitute $d_u = x/2$ for all $u \in V(G)$ although this may not maximize $|S|$. In this case, we have $|S_e| = \lfloor 1/x \rfloor$ for all edges and hence $|S| = \lfloor 1/x \rfloor \cdot |E(G)|$. This gives a tentative lower bound for $B_G(x)$ in both Cases (i) and (ii) since we have never discussed the value of ε yet. Set $d_u = 0$ for all $u \in V(G)$ to consider its possible improvement. Then we have $|S_e| = \lfloor 1/x \rfloor - 1$, $\lfloor 1/x \rfloor$ and $\lfloor 1/x \rfloor + 1$ for edges of types 1, 2 and 3 in order. This implies that:

$$|S| = \lfloor 1/x \rfloor \cdot |E(G)| - |E_1| + |E_3| = \lfloor 1/x \rfloor \cdot |E(G)| - |E(G)| + |V(G)|.$$

If G is not a tree, then the above does not improve the previous lower bound since $|E(G)| \ge |V(G)|$. On the other hand, if G is a tree, then $|E(G)| = |V(G)| - 1$ and we obtain the improved lower bound $\lfloor 1/x \rfloor \cdot |E(G)| + 1$.

Finally, assume that $\varepsilon \ge x/2$ and set $d_u = x/4$ for all $u \in V(G)$. Then $|S_e| = \lfloor 1/x \rfloor$ for edges of types 1 and 2 while $|S_e| = \lfloor 1/x \rfloor + 1$ for edges of type 3. This implies that:

$$|S| = \lfloor 1/x \rfloor \cdot |E(G)| + |E_3|.$$

Take a maximum matching of G as E_3 and place two vertices $\bar{u}$ and $\bar{v}$ on each edge uv in E_3 at distance $x/4$ from its both endpoints. Choose one of edges incident to each vertex v which E_3 does not cover, as an edge in E_2 and put $\bar{v}$ on it at distance $x/4$ from v. Set $E_1 = E(G) - E_2 \cup E_3$. Then $\{\bar{v} : v \in V(G)\}$ extends to S, which attains the maximum of $|S|$ under the assumption here. This gives the lower bound in (i) with $|E_3| = \mu(G)$.

Now evaluate $|S_e|$ to show the upper bounds for $B_G(x)$ in the theorem. In Case 1, we have

$$|S_e| = \left\lfloor \frac{1 + d_u + d_v}{x} \right\rfloor - 1 \le \left\lfloor \frac{1 + x}{x} \right\rfloor - 1 = \left\lfloor \frac{1}{x} \right\rfloor$$

since $d_u \le x/2$ and $d_v \le x/2$. In Case 2, we have:

$$|S_e| = \left\lfloor \frac{1 - d_u + d_v}{x} \right\rfloor \le \left\lfloor \frac{1 + x/2}{x} \right\rfloor.$$

The last value in this inequality is equal to $\lfloor 1/x \rfloor + 1$ if $\varepsilon \ge x/2$, and to $\lfloor 1/x \rfloor$ if $\varepsilon < x/2$. In Case 3, we have simply:

$$|S_e| = \left\lfloor \frac{1 - d_u - d_v}{x} \right\rfloor + 1 \le \left\lfloor \frac{1}{x} \right\rfloor + 1.$$

Therefore, if $\varepsilon < x/2$, then we have:

$$|S| = \sum_{e \in E(G)} |S_e| \ \le\ \lfloor 1/x \rfloor \cdot |E(G)| + |E_3| \ \le\ \lfloor 1/x \rfloor \cdot |E(G)| + \mu(G).$$

This gives the upper bound for $B_G(x)$ in (ii). Also we have the following for (i):

$$|S| = \sum_{e \in E(G)} |S_e| \ \le\ \lfloor 1/x \rfloor \cdot |E(G)| + |E_2 \cup E_3|.$$

Since each edge in $E_2 \cup E_3$ contains $\bar{v}$ for at least one vertex $v \in V(G)$, then $|E_2 \cup E_3| \le |V(G)|$. If $E_3 \ne \emptyset$, then $|E_2 \cup E_3| \le |V(G)| - 1$ since at least one edge contains two $\bar{v}$'s. If there is no cycle consisting of some edges in $E_2 \cup E_3$, then $E_2 \cup E_3$ induces either a tree or a forest and hence $|E_2 \cup E_3| \le |V(G)| - 1$, too.

We may assume that $E_3 = \emptyset$ and that there is a cycle consisting of some edges in E_2. Let $C = u_0 u_1 \cdots u_{k-1}$ be such a cycle of length k with indices taken modulo k. That is, $e_i = u_i u_{i+1}$ is an edge belonging to E_2. We may assume that $\bar{u}_i$ lies on e_i and that $d_i = d(\bar{u}_i, u_i) \le x/2$. Put $\delta = \min\{x/2 - d_i : i = 0, 1, \ldots, k-1\}$ and suppose that d_0 attains this minimum without loss of generality. Thus, $d_0 + \delta = x/2$. Then we have $d_i + \delta \le x/2$ and $x - d_i - \delta \ge x/2$ for $i = 0, 1, \ldots, k-1$.

Move all points in $S \cap C$ by distance δ in the same direction along C so that $d(\bar{u}_0, u_0) = x/2$ afterward. It is clear that S is still an x-set since $d(w, u_i) \ge x - d_i \ge x/2$ for any point w in S on any edge not belonging to C. In particular, we may move the point in S on $u_{k-1}u_0$ nearest u_0 to the point p at distance $x/2$ from u_0 and re-choose p as $\bar{u}_0$ since $d(p, u_0) = d(\bar{u}_0, u_0) = x/2$. Thus, we have $E_3 \ne \emptyset$ after re-choice of $\bar{u}_0$ and hence $|E_2 \cup E_3| \le |V(G)| - 1$ as we have discussed above. ■

To understand the behavior of $B_G(x)$ more clearly, we shall translate the theorem into the following style:

Corollary 2. *Let G be a connected graph and let n be any natural number. If G is a tree, set $\alpha = 1$; otherwise, set $\alpha = 0$.*

(i) *If* $\frac{1}{n+1} < x \leq \frac{2}{2n+1}$, *then* $n|E(G)| + \mu(G) \leq B_G(x) \leq n|E(G)| + |V(G)| - 1$.

(ii) *If* $\frac{2}{2n+1} < x \leq \frac{1}{n}$, *then* $n|E(G)| + \alpha \leq B_G(x) \leq n|E(G)| + \mu(G)$.

(iii) $B_G(1/n) = n|E(G)| + \alpha$.

Proof. Let x be a real number with $\frac{1}{n+1} < x \leq \frac{1}{n}$. Then $n = \lfloor 1/x \rfloor$. Solving $1 - \lfloor 1/x \rfloor x \geq x/2$ for x, we obtain $x \leq \frac{2}{2n+1}$. This implies that Cases (i) and (ii) in the corollary correspond to (i) and (ii) in Theorem 1, respectively.

To evaluate the precise value of $B_G(1/n)$, we follow the notation in the proof of Theorem 1 with $x = 1/n$ and conclude the below:

Case 1:

$$|S_e| = \left\lfloor n + \frac{d_u + d_v}{x} \right\rfloor - 1 = \begin{cases} n & (d_u = d_v = x/2) \\ n-1 & (\text{otherwise}). \end{cases}$$

Case 2:

$$|S_e| = \left\lfloor n - \frac{d_u - d_v}{x} \right\rfloor = \begin{cases} n-1 & (d_u > d_v) \\ n & (\text{otherwise}). \end{cases}$$

Case 3:

$$|S_e| = \left\lfloor n - \frac{d_u + d_v}{x} \right\rfloor + 1 = \begin{cases} n+1 & (d_u = d_v = 0) \\ n & (\text{otherwise}). \end{cases}$$

First suppose that $V(G) - S \neq \emptyset$. That is, there is a vertex w of G with $\bar{w} \neq w$. Let E_3^0 be the set of edges $e = uv \in E_3$ with $d_u = d_v = 0$. Since G is connected, there is a path joining an edge in E_3^0 and a vertex in $V(G) - S$ if $E_3^0 \neq \emptyset$. Let $P = v_0 v_1 \cdots v_k$ be such a path of the minimum length with $w = v_k \in V(G) - S$ and let $e = uv_0$ be the edge in E_3^0 joined to w by P. Then each edge $v_i v_{i+1}$ does not belong to E_3^0 for $i = 0, 1, \ldots, k-1$ by the minimality of $|P|$.

Since $e \in E_3^0$, its endpoint v_0 belongs to S_e with $\bar{v_0} = v_0$ and not to $S_{v_0 v_1}$, and hence $v_0 v_1$ does not belong to E_3. Remove v_0 from S_e and add v_0 to $S_{v_0 v_1}$. Then e does not belong to E_3^0 afterward. If $\bar{v_1} = v_1$ and if $\bar{v_1} \in S_{v_0 v_1}$, then $v_0 v_1$ belongs to the new E_3^0 and $|E_3^0|$ does not change. Continue this argument, resetting $e = v_0 v_1$ with the shorter path $P = v_1 \cdots v_k$. On the other hand, either if $\bar{v_1} \neq v_1$ or if $\bar{v_1} \notin S_{v_0 v_1}$, then $v_0 v_1$ does not belong to the new E_3^0 and hence $|E_3^0|$ decreased by 1. In particular, the former condition happens when $v_1 = w$. Therefore, we can reduce $|E_3^0|$ by 1 finally, modifying E_3^0 along P.

We can repeat this modification as far as $E_3^0 \neq \emptyset$. If E_3^0 becomes empty finally, then we may assume that $d_v = x/2$ for all vertices $v \in V(G)$ and $|S_e| = n$ for all edges $e \in E(G)$ to maximize $|S|$; this assumption takes "otherwise" in Cases 2 and 3. In this case, we have $|S| = n|E(G)|$.

If $V(G) - S = \emptyset$, then we have $\bar{v} = v$ and $d_v = 0$ for all vertices $v \in V(G)$. In this case, each edge of G should be divided evenly into intervals of length $1/n$ and we have:

$$|S| = (n-1)|E(G)| + |V(G)| = n|E(G)| + |V(G)| - |E(G)|.$$

This is greater than the previous if and only if $|V(G)| - |E(G)| > 0$, which is exactly when G is a tree with $|V(G)| - |E(G)| = 1$. This implies the formula for (iii) in the corollary. ■

3 Distinguishing with beans functions

As Corollary 2 shows, the value of $B_G(1/n)$ depends only on the number of edges of G, but other values may not do. We shall show a class of graphs which have the same number of edges but whose beans functions are all different as functions.

Let $C_m \cdot C_n$ denote the *one-point join* of two cycles C_m and C_n of lengths $m, n \geq 3$. That is, $C_m \cdot C_n$ consists of two cycles which have only one common vertex. The number of edges of $C_m \cdot C_n$ is equal to $m + n$. We can determine its beans function completely by similar arguments in the previous section.

THEOREM 3. *If $3 \leq m \leq n$, then:*

$$B_{C_m \cdot C_n}(x) = \begin{cases} \lfloor \frac{m}{x} \rfloor + \lfloor \frac{n}{x} \rfloor & (0 < x \leq m); \\ \lfloor \frac{m+n}{x} \rfloor & (m < x \leq \frac{m+n}{2}); \\ \;1 & (\frac{m+n}{2} < x). \end{cases}$$

Proof. Let w be the unique vertex of degree 4 in $C_m \cdot C_n$ and let S be an x-set of points on $C_m \cdot C_n$ which attains the value of $B_{C_m \cdot C_n}(x)$ for $x > 0$. If $w \in S$, then we can rotate the points of $S \cap C_m$ slightly along C_m so that S does not contain w and is still an x-set afterward. Thus, we may assume that $w \notin S$. It is clear that $|S \cap C_m| \leq \lfloor \frac{m}{x} \rfloor$ and $|S \cap C_n| \leq \lfloor \frac{n}{x} \rfloor$.

First suppose that $x \leq m$. Then $C_m \cdot C_n - U_{x/2}(w)$ consists of two arcs of length $m - x \geq 0$ and $n - x \geq 0$. We can place $\lfloor \frac{m}{x} \rfloor$ points along the former and $\lfloor \frac{n}{x} \rfloor$ points along the latter to make an x-set. Therefore, we have $|S| = \lfloor \frac{m}{x} \rfloor + \lfloor \frac{n}{x} \rfloor$.

Now suppose that $m < x$. If $S \cap C_m = \emptyset$, we would have $|S| = \lfloor \frac{n}{x} \rfloor$. However, we can improve this value, placing a point of S on C_m, as follows. Since $m < x$, we cannot put two points of S on C_m. Let p denote the unique point of S placed on C_m. Then we may assume that $d(p, w) = m/2$ and that $S - \{p\}$ is contained in $C_n - U_{x-m/2}(w)$. Therefore,

$$|S| = \left\lfloor \frac{n - (2x - m)}{x} \right\rfloor + 2 = \left\lfloor \frac{m+n}{x} \right\rfloor \geq \left\lfloor \frac{n}{x} \right\rfloor.$$

This argument works only when $n - (2x - m) \geq 0$, that is, when $m < x \leq \frac{m+n}{2}$. Otherwise, we cannot place more than one point on $C_m \cdot C_n$ as points of an x-set, and hence $|S| = 1$ if $\frac{m+n}{2} < x$. ■

The beans functions $B_{C_m \cdot C_n}(x)$ can be expressed by a uniform formula. However, they are all different functions as shown below:

Theorem 4. *Given two distinct pair (m_1, n_1) and (m_2, n_2) with $m_i \leq n_i$, there exists a positive real number $x \leq 1$ such that $B_{C_{m_1} \cdot C_{n_1}}(x) \neq B_{C_{m_2} \cdot C_{n_2}}(x)$.*

Proof. Put $E = m + n$, which is equal to the number of edges on $C_m \cdot C_n$, and let $x < 1$ be a positive real number. Then we have $E = Qx + \varepsilon$ for a natural number $Q = \lfloor E/x \rfloor \in \boldsymbol{N}$ and a non-negative real number $\varepsilon < x$. Similarly, $n = qx + \delta$ for $q = \lfloor n/x \rfloor \in \boldsymbol{N}$ and $\delta < x$. Using these quantities, we can evaluate $B_{C_m \cdot C_n}(x)$ as follows:

$$\begin{aligned} B_{C_m \cdot C_n}(x) &= \left\lfloor \frac{E-n}{x} \right\rfloor + \left\lfloor \frac{n}{x} \right\rfloor \\ &= \left\lfloor \frac{(Q-q)x + (\varepsilon - \delta)}{x} \right\rfloor + \left\lfloor \frac{qx + \delta}{x} \right\rfloor \\ &= \left\lfloor \frac{(Q-q)x + (\varepsilon - \delta)}{x} \right\rfloor + q. \end{aligned}$$

Clearly, $|\varepsilon - \delta| < x$, and if $\varepsilon - \delta < 0$ then the first term in the last formula is less that $Q - q$. Therefore:

$$B_{C_m \cdot C_n}(x) = \begin{cases} \lfloor E/x \rfloor & (\varepsilon \geq \delta); \\ \lfloor E/x \rfloor - 1 & (\varepsilon < \delta). \end{cases}$$

Now consider two pair of parameters (m_1, n_1) and (m_2, n_2). Since $B_{C_m \cdot C_n}(1) = m + n = E$, we may assume that $m_1 + n_1 = m_2 + n_2 = E$ and $n_1 > n_2$. Let $x < 1$ be a positive real number and put $n_1 = q_1 x + \delta_1$ and $n_2 = q_2 x + \delta_2$ for natural numbers $q_1, q_2 \in \boldsymbol{N}$ and a non-negative real numbers $\delta_1, \delta_2 < x$. In addition, assume that q_1 is a prime number with $n_1 < q_1$ and $\delta_1 = 0$ after re-choosing x if necessary. Thus, $x = n_1/q_1$ and this is less than 1. If $\delta_2 = 0$, we would conclude that $q_2 = q_1 n_2 / n_1$ and this would not be a natural number since $(q_1, n_1) = 1$ and $n_1 > n_2$. Thus, $\delta_2 > 0$. Furthermore, we can conclude that $\delta_2 \neq \varepsilon$, as follows.

Suppose that $\delta_2 = \varepsilon$. Then we would have:

$$m_2 = E - n_2 = Qx + \varepsilon - (q_2 x + \delta_2) = (Q - q_2)x.$$

Put $q_2' = Q - q_2$. Since $n_1 > n_2 \geq m_2$, q_2' is a natural number less than q_1. Since $(q_1, q_2') = 1$, there exist two integers λ and μ with $\lambda q_1 + \mu q_2' = 1$ and hence $x = \lambda q_1 x + \mu q_2' x = \lambda n_1 + \mu m_2 > 0$ would be an integer, a contradiction; x was a positive real number less than 1.

Under this situation, there are two possibilities: (i) $0 = \delta_1 \leq \varepsilon < \delta_2$, or (ii) $0 = \delta_1 < \delta_2 < \varepsilon$. In Case (i), we have $B_{C_{m_1} \cdot C_{n_1}}(x) = Q$ and $B_{C_{m_2} \cdot C_{n_2}}(x) = Q - 1$ by the previous argument, where $Q = \lfloor E/x \rfloor$. In Case (ii), we need to modify the value of x. Increase x by a sufficiently small value. Then q_1 decreases by 1 and δ_1 becomes very close to x. The other two values δ_2 and ε will change slightly, preserving their order. Thus, we have $\delta_2 < \varepsilon < \delta_1$ afterward and hence $B_{C_{m_1} \cdot C_{n_1}}(x) = Q - 1$ and $B_{C_{m_2} \cdot C_{n_2}}(x) = Q$. In either case, we found a real number x so that $B_{C_{m_1} \cdot C_{n_1}}(x) \neq B_{C_{m_2} \cdot C_{n_2}}(x)$. ■

4 Graphs with the same beans function

Let $T_{m,n}$ be a tree obtained from two adjacent vertices u and v by adding m vertices $u_1, \ldots, u_m$ and n vertices $v_1, \ldots, v_n$ so that each u_i is adjacent to u and each v_j is adjacent to v. Thus, we have $\deg u = m$, $\deg v = n$ and $\deg u_i = \deg v_j = 1$.

Theorem 5. *Let $x \leq 1$ be a positive real number and put $\varepsilon = 1 - \lfloor 1/x \rfloor x$. Then:*

$$B_{T_{m,n}}(x) = \begin{cases} (m+n+1)\lfloor 1/x \rfloor + m + n + 1 & (\frac{2}{3}x \leq \varepsilon); \\ (m+n+1)\lfloor 1/x \rfloor + m + n & (\frac{1}{2}x \leq \varepsilon < \frac{2}{3}x); \\ (m+n+1)\lfloor 1/x \rfloor + 2 & (\frac{1}{3}x \leq \varepsilon < \frac{1}{2}x); \\ (m+n+1)\lfloor 1/x \rfloor + 1 & (0 \leq \varepsilon < \frac{1}{3}x). \end{cases}$$

Proof. Let S be an x-set of the maximum size in $T_{m,n}$; that is, $|S| = B_{T_{m,n}}(x)$. Using the notation introduced in this section, we can arrange the points in S so that all u_i's and all v_j's belong to S and that the points on each edge $u_i u$ (or $v_j v$) are placed at equal intervals from u_i (or v_j). Let S_{uv}, $S_{u_i u}$ and $S_{v_j v}$ be the set of points in S lying along the edges uv, $u_i u$ and $v_j v$, respectively. Let u_i' be the one closest to u in $S_{u_i u}$. and let v_j' be the similar one for v_j. Then we have either (i) $d(u_i', u) = \varepsilon$ or (ii) $d(u_i', u) = x + \varepsilon$ and hence $|S_{u_i u}| = \lfloor 1/x \rfloor + 1$ or $|S_{u_i u}| = \lfloor 1/x \rfloor$ in each case.

First suppose that $\varepsilon \geq x/2$. Assume that Case (i) happens for $u_1 u$. Then the point u' in S_{uv} closest to u is located at distance $x - \varepsilon$ or more from u and clearly Case (i) happens for all $u_i u$'s by the maximality of S. Call this situation "Type A". Thus, if Type A does not happen around u, then Case (ii) happens for all $u_i u$'s. Call it "Type B". In this case, we may assume that $u \in S$, moving u' to u. We can say Types A or B around the vertex v, too. If Type X happens around u and Type Y happens around v, we say that Type XY happens.

If Type AA happens, then the points in S_{uv} must be contained in an interval of length $1 - 2(x - \varepsilon)$. Thus, we can evaluate $|S|$ as follows:

$$|S| = (m+n)(\lfloor 1/x \rfloor + 1) + \left(\left\lfloor \frac{1 - 2x + 2\varepsilon}{x} \right\rfloor + 1 \right).$$

The inside of the last blackets is equal to $\lfloor 1/x \rfloor + 1$ if $\varepsilon \geq \frac{2}{3}x$, and to $\lfloor 1/x \rfloor$ otherwise. Thus, the above formula for Type AA splits into two:

$$|S| = (m+n)(\lfloor 1/x \rfloor + 1) + \lfloor 1/x \rfloor + 1 \quad (\varepsilon \geq \tfrac{2}{3}x); \tag{AA$_1$}$$

$$|S| = (m+n)(\lfloor 1/x \rfloor + 1) + \lfloor 1/x \rfloor \quad (\varepsilon < \tfrac{2}{3}x). \tag{AA$_2$}$$

If Type AB happens, then we can calculate $|S|$ as follows:

$$|S| = m(\lfloor 1/x \rfloor + 1) + n\lfloor 1/x \rfloor + \left(\left\lfloor \frac{1-x+\varepsilon}{x} \right\rfloor + 1 \right).$$

Since $\varepsilon \geq x/2$, we have $\lfloor (1+\varepsilon)/x \rfloor = \lfloor 1/x \rfloor + 1$ and hence:

$$|S| = m(\lfloor 1/x \rfloor + 1) + n\lfloor 1/x \rfloor + (\lfloor 1/x \rfloor + 1). \tag{AB}$$

If Type BB happens, then both endpoints of uv belong to S and we have:

$$|S| = (m+n)\lfloor 1/x \rfloor + \left(\left\lfloor \frac{1}{x} \right\rfloor + 1 \right). \tag{BB}$$

Comparing these formulas, we find that the formulas for Type AA attain the maximum and hence they give the actual value of $|S|$.

Now suppose that $\varepsilon < x/2$. It is clear that Case (i) happens for at most one of the m edges $u_i u$'s. Call such a case "Type A" in turn. If Type A happens around u, then we may assume that $d(u'_1, u) = \varepsilon$, $d(u'_i, u) = x + \varepsilon$ for $i \neq 1$ and $d(u', u) = x - \varepsilon$. On the other hand, Type B has the same situation as in the previous.

If Type AA happens, then:

$$|S| = (m+n)\lfloor 1/x \rfloor + 2 + \left(\left\lfloor \frac{1-2x+2\varepsilon}{x} \right\rfloor + 1 \right).$$

This splits into two, depending on the value of ε:

$$|S| = (m+n)\lfloor 1/x \rfloor + 2 + \lfloor 1/x \rfloor \quad (\varepsilon \geq \tfrac{1}{3}x); \tag{AA$_1$}$$

$$|S| = (m+n)\lfloor 1/x \rfloor + 2 + \lfloor 1/x \rfloor - 1 \quad (\varepsilon < \tfrac{1}{3}x). \tag{AA$_2$}$$

If Type AB happens, then:

$$|S| = (m+n)\lfloor 1/x \rfloor + 1 + \left(\left\lfloor \frac{1-x+\varepsilon}{x} \right\rfloor + 1 \right).$$

Since $\varepsilon < x/2$, we have $\lfloor (1+\varepsilon)/x \rfloor = \lfloor 1/x \rfloor$ and hence:

$$|S| = (m+n)\lfloor 1/x \rfloor + 1 + \lfloor 1/x \rfloor. \tag{AB}$$

If Type BB happens, then:

$$|S| = (m+n)\lfloor 1/x \rfloor + \lfloor 1/x \rfloor + 1. \tag{BB}$$

Comparing these, we conclude that the formulas for Type AA give $|S|$. ■

By easy arguments, we can determine the values of $B_{T_{m,n}}(x)$ for big beans $x > 1$ and conclude that the whole beans function $B_{T_{m,n}}(x)$ is completely determined by the values of $m + n$:

$$B_{T_{m,n}}(x) = \begin{cases} m+n+1 & (1 < x \leq 1.5); \\ m+n & (1.5 < x \leq 2); \\ 2 & (2 < x \leq 3); \\ 1 & (3 < x). \end{cases}$$

It is clear that $T_{m,n}$ is isomorhpic to $T_{m',n'}$ if and only if $\{m, n\} = \{m', n'\}$ and that they have the same beans function if and only if $m + n = m' + n'$.

5 For further study

We could give lower and upper bounds for the beans function $B_G(x)$ within each interval $(\frac{1}{n+1}, \frac{1}{n}]$ and some examples to show their sharpness. In fact, we have already known that the values of $B_G(x)$ over one interval $(\frac{1}{n+1}, \frac{1}{n}]$ determine its all values for $x \leq 1$ and that the upper bound $n \cdot |E(G)| + |V(G)| - 1$ given in Theorem 1 is attained for all connected graphs; the latter has been proved in [1]. We would like to establish an algorithm to decide the value of $B_G(x)$ for a given $x \leq 1$ in a combinatorial way.

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