AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **67(1)** (2017), Pages 46–64

More classes of super cycle-antimagic graphs

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Abstract

A simple graph $G = (V, E)$ admits an H-covering if every edge in E belongs to a subgraph of G isomorphic to a given graph H. Then the graph G admitting an H-covering is (a, d)-H-antimagic if there exists a bijection $f : V \cup E \to \{1, 2, \dots, |V| + |E|\}$ such that, for all subgraphs

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H' of G isomorphic to H, the H'-weights, $wt_f(H') = \sum_{v\in V(H')} f(v) + \sum_{e\in E(H')} f(e)$, form an arithmetic progression $a, a+d, a+2d, \ldots, a+(t-1)d$ where a is the first term, d is the common difference and t is the number of subgraphs of G isomorphic to H. Such a labeling is called *super* if $f(V) = \{1, 2, \ldots, |V|\}$.

This paper deals with some results on anti-balanced sets and we show the existence of super (a, d)-cycle-antimagic labelings of fans and some square graphs.

1 Introduction

Let $G = (V, E)$ be a finite simple graph. A family of subgraphs $H_1, H_2, \ldots, H_n$ of G is called an *edge-covering* of G if each edge of E belongs to at least one of the subgraphs H_i, $i = 1, 2, \ldots, n$. Then the graph G admitting an H-covering is (a, d)-H-antimagic if there exists a bijection $f : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ such that, for all subgraphs H' of G isomorphic to H, the H'-weights,

$$wt_f(H') = \sum_{v\in V(H')} f(v) + \sum_{e\in E(H')} f(e),$$

form an arithmetic progression $a, a+d, \ldots, a+(t-1)d$ where $a > 0$ is the first term, $d \geq 0$ is the common difference and t is the number of subgraphs of G isomorphic to H. Such a labeling is called *super* if $f(V) = \{1, 2, \ldots, |V|\}$. For $d = 0$ it is called *H-magic* and *H-supermagic*, respectively.

The concept of H-magic graphs was introduced by Gutiérrez and Lladó [7] as an extension of the edge-magic and super edge-magic graphs. They proved that some classes of connected graphs such as the stars $K_{1,n}$ and the complete bipartite graphs $K_{n,m}$ are $K_{1,h}$-supermagic for some h. They also proved that the path P_n and the cycle C_n are P_h-supermagic for some h. Lladó and Moragas [13] proved that wheels, windmills, books and prisms are C_h-magic for some h. Maryati et al. [17] and Salman et al. [20] proved that certain families of trees are path-supermagic. Jeyanthi and Selvagopal [10] proved that one point union of n copies of a 2-connected graph, linear garland of a 2-connected graph are H-supermagic. Interestingly, windmill is a particular case of one point union whereas ladder and triangular ladder are the particular cases of linear garland. Ngurah, Salman and Susilowati [19] proved that chains, wheels, triangles, ladders and grids are cycle-supermagic. Maryati, Salman and Baskoro [16] investigated the G-supermagicness of a disjoint union of c copies of a graph G and showed that the disjoint union of any paths is cP_h-supermagic for some c and h. Muthuraja, Selvagopal and Jeyanthi [18] showed that the square graphs of bistar, path and cycle are cycle-supermagic. They also proved that the middle graph of C_n is also C_3-supermagic. Jeyanthi and Muthuraja [12] proved that the graph $P_{m,n}$ for $m, n \geq 2$ is C_{2m}-supermagic and the splitting graph of C_n is C_4-supermagic for $n \neq 4$.

The (a, d)-H-antimagic labeling was introduced by Inayah, Salman and Simanjuntak [8]. In [9] they investigated the super (a, d)-H-antimagic labelings for some shackles of a connected graph H. In [21] it is proved that wheels W_n, $n \geq 3$, are super (a, d)-C_k-antimagic for every $k = 3, 4, \dots, n-1, n+1$ and $d = 0, 1, 2$.

The (super) (a, d)-H-antimagic labeling is related to a super d-antimagic labeling of type $(1, 1, 0)$ of a plane graph that is the generalization of a face-magic labeling introduced by Lih [14]. Further information on super d-antimagic labelings can be found in [2, 5].

For $H \cong K_2$, (super) (a, d)-H-antimagic labelings are also called (super) (a, d)-edge-antimagic total labelings and have been introduced in [22]. More results on (a, d)-edge-antimagic total labelings, can be found in [4, 15]. The vertex version of these labelings for generalized pyramid graphs is given in [1].

The existence of super (a, d)-H-antimagic labelings for disconnected graphs is studied in [6] where it is proved that if a graph G admits a (super) (a, d)-H-antimagic labeling, where $d = |E(H)| - |V(H)|$, then the disjoint union of m copies of the graph G, denoted by mG, admits a (super) (b, d)-H-antimagic labeling as well. In [3] it is shown that the disjoint union of multiple copies of a (super) $(a, 1)$-tree-antimagic graph is also a (super) $(b, 1)$-tree-antimagic. A natural question is whether the similar result holds also for another differences and another H-antimagic graphs.

A *fan* F_n, $n \geq 2$, is a graph obtained by joining all the vertices of the path P_n on n vertices to a further vertex, called the centre. The vertices on the path we will call the path vertices. The edges adjacent to the central vertex are called the spokes and the remaining edges are called the path edges. The F_n contains $n+1$ vertices and $2n-1$ edges.

For a simple connected graph G, the *square of the graph* G, denoted by G^2, is defined as the graph with the same vertex set as G and two vertices are adjacent in G^2 if they are at a distance 1 or 2 apart in G.

In this paper we investigate the existence of super (a, d)-cycle-antimagic labelings of fans and some square graphs.

2 Known results on (k, δ)-anti-balanced sets

We use the following notation. For two integers a, b, $a < b$, let $[a, b]$ denote the set of all integers from a to b. For any subset $\mathbb{S}$ of the set of integers $\mathbb{Z}$ we write, $\sum \mathbb{S} = \sum_{x \in \mathbb{S}} x$ and for an integer k, let $k + \mathbb{S} = \{k + x : x \in \mathbb{S}\}$. Thus $k + [a, b]$ is the set $\{x \in \mathbb{Z} : k + a \leq x \leq k + b\}$. It can be easily verified that $\sum(k + \mathbb{S}) = k|\mathbb{S}| + \sum \mathbb{S}$.

If $\mathbb{P} = \{X_1, X_2, \dots, X_n\}$ is a partition of a set X of integers with the same cardinality then we say $\mathbb{P}$ is an *n-equipartition of* X. Also we denote the set of subsets sums of the parts of $\mathbb{P}$ by $\sum \mathbb{P} = \{\sum X_1, \sum X_2, \dots, \sum X_n\}$.

A multiset is a generalization of the concept of a set that, unlike a set, allows multiple instances of the multisets elements. If X and Y are two multisets then their union is also a multiset represented by $X \uplus Y$. If an element a appears m times in X and n times in Y, then a appears $m+n$ times in $X \uplus Y$. If X is a multiset of integers and k is an integer then $k \oplus X = \{k + x : x \in X\}$.

Let $k \in \mathbb{N}$ and let X be a multiset containing positive integers. Then X is said to be (k, δ)-*anti-balanced* if there exist k subsets of X, say $X_1, X_2, \ldots, X_k$ such that for every $i \in [1, k]$, $|X_i| = \frac{|X|}{k}$, $\biguplus_{i=1}^{k} X_i = X$ and for $i \in [1, k-1]$, $\sum X_{i+1} - \sum X_i = \delta$ is satisfied.

We use the following results to prove our main results.

Lemma 2.1. [7] *Let h and k be two positive integers and let $n = hk$. For each integer $0 \le t \le \left\lfloor \frac{h}{2} \right\rfloor$ there is a k-equipartition $\mathbb{P}$ of $[1, n]$ such that $\sum \mathbb{P}$ is an arithmetic progression of difference $d = h - 2t$.*

Lemma 2.2. [11] *If h is even, then there exists a k-equipartition $\mathbb{P} = \{X_1, X_2, \ldots, X_k\}$ of $X = [1, hk]$ such that $\sum X_r = \frac{h(hk+1)}{2}$ for $1 \le r \le k$. Thus, the subset sums are equal and is equal to $\frac{h(hk+1)}{2}$.*

Lemma 2.3. [11] *Let h and k be two positive integers such that h is even and $k \ge 3$ is odd. Then there exists a k-equipartition $\mathbb{P} = \{X_1, X_2, \ldots, X_k\}$ of $X = [1, hk]$ such that $\sum X_r = \frac{(h-1)(hk+k+1)}{2} + r$ for $1 \le r \le k$. Thus, $\sum \mathbb{P}$ is a set of consecutive integers given by $\sum \mathbb{P} = \frac{(h-1)(hk+k+1)}{2} + [1, k]$.*

The above lemma was proved using permutations on $[1, k]$ in [11]. We can deduce the result for $h = 2$ which is that $\{Y_1, Y_2, \ldots, Y_k\}$ is a k-equipartition of $[1, 2k]$, where

$$Y_i = \begin{cases} \left\{\frac{k-2i+1}{2}, k + 2i\right\} & \text{for } 1 \le r \le \frac{k-1}{2}, \\ \left\{\frac{3k-2i+1}{2}, 2i\right\} & \text{for } \frac{k+1}{2} \le r \le k \end{cases}$$

and $\sum Y_i = \frac{3k+1}{2} + i$ for $1 \le i \le k$.

Lemma 2.4. [11] *Let h and k be two even positive integers and $h \ge 4$. If $X = [1, hk+1] - \{\frac{k}{2} + 1\}$, there exists a k-equipartition $\mathbb{P} = \{X_1, X_2, \ldots, X_k\}$ of X such that $\sum X_r = \frac{h^2k+3h-k-2}{2} + r$ for $1 \le r \le k$. Thus $\sum \mathbb{P}$ is a set of consecutive integers $\frac{h^2k+3h-k-2}{2} + [1, k]$.*

In the proof of this lemma, it is shown that $\{Z_1, Z_2, \ldots, Z_k\}$ is a k-equipartition of $[1, 2k+1] - \{\frac{k}{2} + 1\}$, where

$$Z_i = \begin{cases} \left\{\frac{k}{2} + 1 - i, k + 1 + 2i\right\} & \text{for } 1 \le r \le \frac{k}{2}, \\ \left\{\frac{3k}{2} + 2 - i, 2i\right\} & \text{for } \frac{k}{2} + 1 \le r \le k \end{cases}$$

and $\sum Z_i = \frac{3k}{2} + 2 + i$ for $1 \le i \le k$.

Lemma 2.5. [9] *Let k be an integer such that $k \geq 2$. If*

$$X = \begin{cases} [1, k+1] \uplus [2, k] & \text{for } k \text{ odd,} \\ [1, \frac{k}{2}] \uplus [\frac{k}{2}+2, k+1] \uplus [2, k+1] & \text{for } k \text{ even,} \end{cases}$$

then X is $(k,1)$-anti-balanced.

Lemma 2.6. [9] *Let k be an integer such that $k \geq 2$. If $X = [1, k] \uplus [2, k+1]$ then X is $(k, 2)$-anti-balanced.*

3 New results on (k, δ)-anti-balanced sets

We prove the following lemmas which are useful to prove our main results.

Lemma 3.1. *Let n, r be positive integers, $n \geq 2$, $2 \leq r \leq n-1$ and let $m = \min\{r, n-r+1\}$. If $X = \biguplus_{j=2}^{m} [j, n-j+1] \uplus [1, n]$, then X is $(n-r+1, r)$-anti-balanced.*

Proof. For $1 \leq i \leq n-r+1$, define $X_i = \{i, i+1, \ldots, i+r-1\}$. It can be easily verified that $|X_i| = r$ and $\biguplus_{i=1}^{n-r+1} X_i = X$. Also $\sum X_i = \frac{(r-1)r}{2} + ri$ and hence X is $(n-r+1, r)$-anti-balanced. ∎

Illustration 3.1. For $n = 9$ and $r = 4$, let $X = \biguplus_{j=2}^{4} [j, 9-j+1] \uplus [1, 9] = [2, 8] \uplus [3, 7] \uplus [4, 6] \uplus [1, 9]$. We have $(6, 4)$-anti-balanced subsets $X_1 = \{1, 2, 3, 4\}$, $X_2 = \{2, 3, 4, 5\}$, $X_3 = \{3, 4, 5, 6\}$, $X_4 = \{4, 5, 6, 7\}$, $X_5 = \{5, 6, 7, 8\}$ and $X_6 = \{6, 7, 8, 9\}$. Then the subset sums are 10, 14, 18, 22, 26, 30.

Lemma 3.2. *Let n, r be positive integers, $n \geq 2$, $2 \leq r \leq \lceil \frac{n+1}{2} \rceil$. If $X = [r, n-r+1] \uplus [1, n]$, then X is $(n-r+1, 2)$-anti-balanced, assuming that $[l, k] = \emptyset$ if $l > k$.*

Proof. For $1 \leq i \leq n-r+1$, define $X_i = \{i, i+r-1\}$. It is easy to verify that $|X_i| = 2$ and $\biguplus_{i=1}^{n-r+1} X_i = X$. Also $\sum X_i = r-1+2i$ and hence X is $(n-r+1, 2)$-anti-balanced. ∎

Illustration 3.2. For $n = 10$, $r = 6$ and $X = [6, 5] \uplus [1, 10] = [1, 10]$, we have $(5, 2)$-anti-balanced subsets $X_1 = \{1, 6\}$, $X_2 = \{2, 7\}$, $X_3 = \{3, 8\}$, $X_4 = \{4, 9\}$ and $X_5 = \{5, 10\}$.
For $n = 10$, $r = 5$ and $X = [5, 6] \uplus [1, 10]$, we have $(6, 2)$-anti-balanced subsets $X_1 = \{1, 5\}$, $X_2 = \{2, 6\}$, $X_3 = \{3, 7\}$, $X_4 = \{4, 8\}$, $X_5 = \{5, 9\}$ and $X_6 = \{6, 10\}$.

Lemma 3.3. *Let $n \geq 2$, be a positive integer and let $X = [1, n]$. If r divides n, then X is $\left(\frac{n}{r}, r^2\right)$-anti-balanced.*

Proof. For $1 \leq i \leq \frac{n}{r}$, define $X_i = \{(i-1)r+1, (i-1)r+2, \dots ir\}$. Obviously $|X_i| = r$ and $\bigcup\limits_{i=1}^{\frac{n}{r}} X_i = X$. Also $\sum X_i = \frac{r(r+1)}{2} + (i-1)r^2$ and hence X is $\left(\frac{n}{r}, r^2\right)$-anti-balanced. ∎

Illustration 3.3. Let $n = 14$, $r = 2$ and $X = [1, 14]$. We have $(7, 4)$-anti-balanced subsets $X_1 = \{1, 2\}$, $X_2 = \{3, 4\}$, $X_3 = \{5, 6\}$, $X_4 = \{7, 8\}$, $X_5 = \{9, 10\}$, $X_6 = \{11, 12\}$ and $X_7 = \{13, 14\}$.

Lemma 3.4. *Let n, r, $r < n$, be two relatively prime integers. Then the multiset $X = \biguplus\limits_{1}^{r}[1, n]$ is $(n, 1)$-anti-balanced.*

Proof. Since $\gcd(n, r) = 1$, the linear congruence $rx \equiv 1 \pmod{n}$ has solutions. Let k be the solution such that $k < n$. Actually k is called an inverse of r modulo n and r is called an inverse of k modulo n. Also we have $\gcd(n, k) = 1$.

Consider the set $A = \{jk \pmod{n}, n : 1 \leq j \leq n-1\}$. Then $A \subseteq [1, n]$. We prove that $A = [1, n]$. If $1 \leq i, j \leq n-1$ then $ik \pmod{n} = jk \pmod{n}$ and thus $n|(i-j)k$. Since $\gcd(n, k) = 1$, we have n divides $i - j$. Now $i - j < n$ implies that $i - j = 0$. Hence all the elements in A are distinct and $|A| = n$ thus we have $A = [1, n]$.

Let $X = \biguplus\limits_{1}^{r}[1, n]$. We define the anti-balanced subsets of X as follows:

$$X_i = \begin{cases} \{jk \pmod{n} : i \leq j \leq i+r-1\} & \text{for } 1 \leq i \leq n-r+1, \\ \{jk \pmod{n} : i \leq j \leq n\} \cup \{jk \pmod{n} : 1 \leq j \leq i+r-n-1\} & \\ & \text{for } n-r+2 \leq i \leq n. \end{cases}$$

It can be easily verified that, $|X_i| = r$, $\biguplus\limits_{1}^{r} X_i = X$ and $|X_i \cap X_{i+1}| = r-1$. Next we prove that $\sum X_{i+1} - \sum X_i = 1$.

Case (i): $1 \leq i \leq n-r$.

$$\sum X_{i+1} - \sum X_i = \big((i+r)k \pmod{n}\big) - \big(ik \pmod{n}\big) = rk \pmod{n} = 1.$$

Case (ii): $i = n-r+1$.

$$\begin{aligned} \sum X_{n-r+2} - \sum X_{n-r+1} &= \big(k \pmod{n}\big) - \big((n-r+1)k \pmod{n}\big) \\ &= (r-n)k \pmod{n} \\ &= rk \pmod{n} = 1. \end{aligned}$$

Case (iii): $n-r+2 \leq i \leq n$.

$$\begin{aligned} \sum X_{i+1} - \sum X_i &= \big((r-n+i)k \pmod{n}\big) - \big(ik \pmod{n}\big) \\ &= (r-n)k \pmod{n} = rk \pmod{n} = 1. \end{aligned}$$

Thus we have $\sum X_{i+1} - \sum X_i = 1$ for $1 \leq i \leq n-1$. Hence X is $(n, 1)$-anti-balanced. ∎

Illustration 3.4. Let $n = 9$, $r = 4$. Then $X = \biguplus_{1}^{4}[1, 9]$. Since $4 \cdot 7 \equiv 1 \pmod 9$, the inverse of 4 modulo 9 is 7 and we have $k = 7$. By Lemma 3.4 we have $A = \{7 \pmod 9,\ 14 \pmod 9,\ 21 \pmod 9,\ 28 \pmod 9,\ 35 \pmod 9,\ 42 \pmod 9,\ 49 \pmod 9,\ 56 \pmod 9,\ 9\}$. That is, $A = \{7, 5, 3, 1, 8, 6, 4, 2, 9\}$. By definition, $X_1 = \{7, 5, 3, 1\}$, $X_2 = \{5, 3, 1, 8\}$, $X_3 = \{3, 1, 8, 6\}$, $X_4 = \{1, 8, 6, 4\}$, $X_5 = \{8, 6, 4, 2\}$, $X_6 = \{6, 4, 2, 9\}$, $X_7 = \{4, 2, 9, 7\}$, $X_8 = \{2, 9, 7, 5\}$ and $X_9 = \{9, 7, 5, 3\}$. Then, the subset sums form an arithmetic sequence 16, 17, 18, 19, 20, 21, 22, 23, 24. Hence X is $(9, 1)$-anti-balanced.

Lemma 3.5. *Let n, r be positive integers, $n \geq 2$, $2 \leq r \leq n - 1$. The elements of $[1, n]$ can be arranged as a sequence $(a_i)_{i=1}^{n}$ such that $\sum X_{i+1} - \sum X_i = 1$ for $1 \leq i \leq n - r + 1$, where $X_i = \{a_i, a_{i+1}, a_{i+2}, \ldots, a_{i+r-1}\}$.*

Proof. Let n be an integer such that $n \geq 2$ and r be an integer such that $2 \leq r \leq n-1$. Let $t = \lceil \frac{n}{r} \rceil$ and $s = n \pmod r$. Then we have $n = (t - 1)r + s$. We rearrange the numbers $[1, n]$ as follows:

$$\left(\begin{array}{llll|llll}
1 & t+1 & \ldots & (s-1)t+1 & st+1 & (s+1)t & \ldots & (r-1)t-r+s+2 \\
2 & t+2 & \ldots & (s-1)t+2 & st+2 & (s+1)t+1 & \ldots & (r-1)t-r+s+3 \\
3 & t+3 & \ldots & (s-1)t+3 & st+3 & (s+1)t+2 & \ldots & (r-1)t-r+s+4 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots \\
t-1 & 2t-1 & \ldots & st-1 & (s+1)t-1 & (s+2)t-2 & \ldots & n \\
t & 2t & \ldots & st & & & &
\end{array}\right)$$

The arrangement of the numbers is given by the function

$$f(i, j) = \begin{cases} (j-1)t + i & \text{for } 1 \leq i \leq t,\ 1 \leq j \leq s, \\ (j-1)(t-1) + s + i & \text{for } 1 \leq i \leq t-1,\ s+1 \leq j \leq r. \end{cases}$$

We arrange the elements of $[1, n]$ as a finite sequence $a_1, a_2, \ldots, a_n$ as follows:

For $1 \leq m \leq n$, let $m = ir + j$, where $0 \leq i \leq t - 1$ and $1 \leq j \leq r$. Define $a_m = f(i + 1, j)$. Note that if $a_m = f(i + 1, j)$ then $a_{m+r} = f(i + 2, j)$ as $m + r = (i + 1)r + j$. Hence, $a_{m+r} - a_m = 1$.

Now, for $1 \leq i \leq n - r + 1$, let $X_i = \{a_i, a_{i+1}, a_{i+2}, \ldots, a_{i+r-1}\}$. Then $\sum X_{i+1} - \sum X_i = a_{i+r} - a_i = 1$.

It can be easily verified that $\sum X_i = \frac{tr(r-1)-(r-s)^2+3r-s-2}{2} + \ddot{u}$ for $1 \leq i \leq n-r+1$. ∎

Illustration 3.5. Let $n = 17$, $r = 5$; $X = \biguplus_{j=1}^{5}[j, 17 - j + 1]$; $t = \lceil \frac{17}{5} \rceil = 4$ and $s = n \pmod r = 2$. Let us arrange the numbers $[1, 17]$ as follows:

$$\begin{bmatrix} 1 & 5 & 9 & 12 & 15 \\ 2 & 6 & 10 & 13 & 16 \\ 3 & 7 & 11 & 14 & 17 \\ 4 & 8 & & & \end{bmatrix}$$

Then by definition, $X_1 = \{1, 5, 9, 12, 15\}$, $X_2 = \{5, 9, 12, 15, 2\}$, $X_3 = \{9, 12, 15, 2, 6\}$, $X_4 = \{12, 15, 2, 6, 10\}$, $X_5 = \{15, 2, 6, 10, 13\}$, $X_6 = \{2, 6, 10, 13, 16\}$, $X_7 = \{6, 10, 13, 16, 3\}$, $X_8 = \{10, 13, 16, 3, 7\}$, $X_9 = \{13, 16, 3, 7, 11\}$, $X_{10} = \{16, 3, 7, 11, 14\}$, $X_{11} = \{3, 7, 11, 14, 17\}$, $X_{12} = \{7, 11, 14, 17, 4\}$ and $X_{13} = \{11, 14, 17, 4, 8\}$. Hence X is $(13, 1)$-anti-balanced as $\sum X_i = \dfrac{tr(r-1)-(r-s)^2+3r-s-2}{2} + i = 41 + i$ for $1 \le i \le 13$.

4 Some super (a, d)-cycle-antimagic graphs

Now we prove that fans admit super (a, d)-C_k-antimagic labelings for a wide variety of k and d.

Theorem 4.1. *The fan F_n, $n \ge 3$, admits a super (a, d)-C_k-antimagic labeling for $k = 3, 4, \ldots, \lfloor \frac{n}{2} \rfloor + 2$ and $d \in \{1, 2, k-5, k-4, \ldots, k+2, 2k-5, 2k-1\}$.*

Proof. Let $V(F_n) = \{v, v_1, v_2, \ldots, v_n\}$ be the vertex set and let $E(F_n) = \{vv_i : 1 \le i \le n\} \cup \{v_iv_{i+1} : 1 \le i \le n-1\}$ be the edge set of the fan F_n.

Let k be an integer such that $3 \le k \le \lfloor \frac{n}{2} \rfloor + 2$. For $1 \le i \le n-k+2$, let C_k^i be the k-cycle $vv_iv_{i+1}\ldots v_{i+k-2}$. Let V_i and E_i be respectively the vertex and edge set of C_k^i. Thus $V_i = \{v, v_i, v_{i+1}, \ldots, v_{i+k-2}\}$ and $E_i = E_i' \cup E_i''$ where $E_i' = \{v_{i+j}v_{i+j+1} : 0 \le j \le k-3\}$ and $E_i'' = \{vv_i, vv_{i+k-2}\}$. Evidently $\{C_k^i : 1 \le i \le n-k+2\}$ is a C_k-covering of F_n.

Note that the weight of the cycle C_k^i under a total labeling f is $wt_f(C_k^i) = f(V_i) + f(E_i') + f(E_i'') = \sum_{v \in V_i} f(v) + \sum_{e \in E_i'} f(e) + \sum_{e \in E_i''} f(e)$.

We use the following results.

(R1) By Lemma 3.1, the multiset $X = \biguplus_{j=2}^{m} [j, n-j+1] \uplus [1, n]$ is $(n-k+2, k-1)$-anti-balanced with anti-balanced subsets $X_1, X_2, \ldots, X_{n-k+2}$ such that $\sum X_i = \Delta_1 + (k-1)i$, where $m = \min\{k-1, n-k+2\}$ and $\Delta_1 = \frac{(k-2)(k-1)}{2}$.

(R2) By Lemma 3.1, the multiset $X' = (n+1) \oplus \left(\biguplus_{j=2}^{m} [j, n-j] \uplus [1, n-1] \right)$ is $(n-k+2, k-2)$-anti-balanced with anti-balanced subsets $P_1, P_2, \ldots, P_{n-k+2}$ such that $\sum P_i = \Delta_2 + (k-2)i$, where $\Delta_2 = (k-2)(n+1) + \frac{(k-3)(k-2)}{2}$.

(R3) By Lemma 3.5, $Y = [1, n]$ can be arranged as a sequence $(a_i)_{i=1}^{n}$ such that $\sum R_{i+1} - \sum R_i = 1$ for $1 \le i \le n-k+2$, where $R_i = \{a_i, a_{i+1}, a_{i+2}, \ldots, a_{i+k-2}\}$ such that $\sum R_i = \Delta_3 + i$, where $\Delta_3 = \frac{t(k-1)(k-2)-(k-s-1)^2+3(k-1)-s-2}{2}$ and $t = \lceil \frac{n}{k-1} \rceil$, $s = n \pmod{(k-1)}$.

(R4) By Lemma 3.5, $Y' = (n+1) + [1, n-1]$ can be arranged as a sequence $(b_i)_{i=1}^{n-1}$ such that $\sum S_{i+1} - \sum S_i = 1$ for $1 \le i \le n-k+2$, where $S_i =$

$\{b_i, b_{i+1}, b_{i+2}, \dots, b_{i+k-3}\}$ such that $\sum S_i = \Delta_4 + i$, where $\Delta_4 = (k-2)(n+1) + \frac{t'(k-2)(k-3)-(k-s'-2)^2+3(k-1)-s'-5}{2}$ and $t' = \lceil \frac{n-1}{k-2} \rceil$, $s' = (n-1) \pmod{(k-2)}$.

(R5) By Lemma 3.2, the multiset $Z = 2n \oplus ([k-1, n-k+2] \uplus [1, n])$ is $(n-k+2, 2)$-anti-balanced with anti-balanced subsets $Q_1, Q_2, \dots, Q_{n-k+2}$ with $\sum Q_i = \Delta_5 + 2i$, where $\Delta_5 = 4n + k - 2$.

In order to prove the theorem, we use the results (R1) - (R5).

Case (i): F_n is super $(a, 1)$-C_k-antimagic

Define a total labeling f_1 on F_n as follows: For $1 \le i \le n-k+2$, label the vertices in $V_i - \{v\}$ with $X_{n-k+3-i}$, v with $n+1$, the edges in E'_i with P_i and the edges in E''_i with Q_i.

$$\begin{aligned} wt_{f_1}(C_k^i) &= f_1(V_i) + f_1(E'_i) + f_1(E''_i) \\ &= \sum X_{n-k+3-i} + n + 1 + \sum P_i + \sum Q_i \\ &= \Delta_1 + (k-1)(n-k+3-i) + n + 1 + \Delta_2 + (k-2)i + \Delta_5 + 2i \\ &= \Delta_1 + \Delta_2 + \Delta_5 + (k-1)(n-k+3-i) + n + 1 + (k-2)i + 2i \\ &= \Delta_1 + \Delta_2 + \Delta_5 + (k-1)(n-k+3) + n + 1 + i. \end{aligned}$$

Then f_1 is a super $(a, 1)$-C_k-antimagic labeling, where $a = \Delta_1 + \Delta_2 + \Delta_5 + (k-1)(n-k+3) + n + 2$.

Case (ii): F_n is super $(a, 2)$-C_k-antimagic

Define a total labeling f_2 on F_n as follows: For $1 \le i \le n-k+2$, label the vertices in $V_i - \{v\}$ with R_i, v with $n+1$, the edges in E'_i with $S_{n-k+3-i}$ and the edges in E''_i with Q_i.

$$\begin{aligned} wt_{f_2}(C_k^i) &= f_2(V_i) + f_2(E'_i) + f_2(E''_i) \\ &= \sum R_i + n + 1 + \sum S_{n-k+3-i} + \sum Q_i \\ &= \Delta_3 + i + n + 1 + \Delta_4 + n - k + 3 - i + \Delta_5 + 2i \\ &= \Delta_3 + \Delta_4 + \Delta_5 + 2n - k + 4 + 2i. \end{aligned}$$

Therefore, f_2 is a super $(a, 2)$-C_k-antimagic labeling, where $a = \Delta_3 + \Delta_4 + \Delta_5 + 2n - k + 6$.

Case (iii): F_n is super $(a, k-5)$-C_k-antimagic

Define a total labeling f_3 on F_n as follows: For $1 \le i \le n-k+2$, label the vertices in $V_i - \{v\}$ with $R_{n-k+3-i}$, v with $n+1$, the edges in E'_i with P_i and the edges in E''_i with $Q_{n-k+3-i}$.

$$\begin{aligned} wt_{f_3}(C_k^i) &= f_3(V_i) + f_3(E'_i) + f_3(E''_i) \\ &= \sum R_{n-k+3-i} + n + 1 + \sum P_i + \sum Q_{n-k+3-i} \\ &= \Delta_3 + n - k + 3 - i + n + 1 + \Delta_2 + (k-2)i + \Delta_5 + 2(n-k+3-i) \\ &= \Delta_3 + \Delta_2 + \Delta_5 + n - k + 3 - i + n + 1 + (k-2)i + 2(n-k+3-i) \\ &= \Delta_2 + \Delta_3 + \Delta_5 + 4n - 3k + 10 + (k-5)i. \end{aligned}$$

Then f_3 is a super $(a, k-5)$-C_k-antimagic labeling, where $a = \Delta_2 + \Delta_3 + \Delta_5 + 4n - 2k + 5$.

Case (iv): F_n is super $(a, k-4)$-C_k-antimagic

Define a total labeling f_4 on F_n as follows: For $1 \leq i \leq n-k+2$, label the vertices in $V_i - \{v\}$ with X_i, v with $n+1$, the edges in E_i' with $S_{n-k+3-i}$ and the edges in E_i'' with $Q_{n-k+3-i}$.

$$\begin{aligned} wt_{f_4}(C_k^i) &= f_4(V_i) + f_4(E_i') + f_4(E_i'') \\ &= \sum X_i + n + 1 + \sum S_{n-k+3-i} + \sum Q_{n-k+3-i} \\ &= \Delta_1 + (k-1)i + n + 1 + \Delta_4 + n - k + 3 - i + \Delta_5 + 2(n-k+3-i) \\ &= \Delta_1 + \Delta_4 + \Delta_5 + 4n - 3k + 10 + (k-4)i. \end{aligned}$$

Then f_4 is a super $(a, k-4)$-C_k-antimagic labeling, where $a = \Delta_1 + \Delta_4 + \Delta_5 + 4n - 2k + 6$.

Case (v): F_n is super $(a, k-3)$-C_k-antimagic

Define a total labeling f_5 on F_n as follows: For $1 \leq i \leq n-k+2$, label the vertices in $V_i - \{v\}$ with R_i, v with $n+1$, the edges in E_i' with P_i and the edges in E_i'' with $Q_{n-k+3-i}$.

$$\begin{aligned} wt_{f_5}(C_k^i) &= f_5(V_i) + f_5(E_i') + f_5(E_i'') \\ &= \sum R_i + n + 1 + \sum P_i + \sum Q_{n-k+3-i} \\ &= \Delta_3 + i + n + 1 + \Delta_2 + (k-2)i + \Delta_5 + 2(n-k+3-i) \\ &= \Delta_2 + \Delta_3 + \Delta_5 + 3n - 2k + 7 + (k-3)i. \end{aligned}$$

Then f_5 is a super $(a, k-3)$-C_k-antimagic labeling, where $a = \Delta_2 + \Delta_3 + \Delta_5 + 3n - k + 4$.

Case (vi): F_n is super $(a, k-2)$-C_k-antimagic

Define a total labeling f_6 on F_n as follows: For $1 \leq i \leq n-k+2$, label the vertices in $V_i - \{v\}$ with X_i, v with $n+1$, the edges in E_i' with S_i and the edges in E_i'' with $Q_{n-k+3-i}$.

$$\begin{aligned} wt_{f_6}(C_k^i) &= f_6(V_i) + f_6(E_i') + f_6(E_i'') \\ &= \sum X_i + n + 1 + \sum S_i + \sum Q_{n-k+3-i} \\ &= \Delta_1 + (k-1)i + n + 1 + \Delta_4 + i + \Delta_5 + 2(n-k+3-i) \\ &= \Delta_1 + \Delta_4 + \Delta_5 + 3n - 2k + 7 + (k-2)i. \end{aligned}$$

Then f_6 is a super $(a, k-2)$-C_k-antimagic labeling, where $a = \Delta_1 + \Delta_4 + \Delta_5 + 3n - k + 5$.

Case (vii): F_n is super $(a, k-1)$-C_k-antimagic

Define a total labeling f_7 on F_n as follows: For $1 \leq i \leq n-k+2$, label the vertices in $V_i - \{v\}$ with $R_{n-k+3-i}$, v with $n+1$, the edges in E_i' with P_i and the edges in E_i''

with Q_i. Then

$$\begin{aligned} wt_{f_7}(C_k^i) &= f_7(V_i) + f_7(E_i') + f_7(E_i'') \\ &= \sum R_{n-k+3-i} + n + 1 + \sum P_i + \sum Q_i \\ &= \Delta_3 + n - k + 3 - i + n + 1 + \Delta_2 + (k-2)i + \Delta_5 + 2i \\ &= \Delta_3 + \Delta_2 + \Delta_5 + 2n - k + 4 + (k-1)i. \end{aligned}$$

Then f_7 is a super $(a, k-1)$-C_k-antimagic labeling, where $a = \Delta_2 + \Delta_3 + \Delta_5 + 2n + 3$.

Case (viii): F_n is super (a, k)-C_k-antimagic

Define a total labeling f_8 on F_n as follows: For $1 \leq i \leq n-k+2$, label the vertices in $V_i - \{v\}$ with X_i, v with $n+1$, the edges in E_i' with $S_{n-k+3-i}$ and the edges in E_i'' with Q_i.

$$\begin{aligned} wt_{f_8}(C_k^i) &= f_8(V_i) + f_8(E_i') + f_8(E_i'') \\ &= \sum X_i + n + 1 + \sum S_{n-k+3-i} + \sum Q_i \\ &= \Delta_1 + (k-1)i + n + 1 + \Delta_4 + n - k + 3 - i + \Delta_5 + 2i \\ &= \Delta_1 + \Delta_4 + \Delta_5 + 2n - k + 4 + ki. \end{aligned}$$

Then f_8 is a super (a, k)-C_k-antimagic labeling, where $a = \Delta_1 + \Delta_4 + \Delta_5 + 2n + 4$.

Case (ix): F_n is super $(a, k+1)$-C_k-antimagic

Define a total labeling f_9 on F_n as follows: For $1 \leq i \leq n-k+2$, label the vertices in $V_i - \{v\}$ with R_i, v with $n+1$, the edges in E_i' with P_i and the edges in E_i'' with Q_i.

$$\begin{aligned} wt_{f_9}(C_k^i) &= f_9(V_i) + f_9(E_i') + f_9(E_i'') \\ &= \sum R_i + n + 1 + \sum P_i + \sum Q_i \\ &= \Delta_3 + i + n + 1 + \Delta_2 + (k-2)i + \Delta_5 + 2i \\ &= \Delta_3 + \Delta_2 + \Delta_5 + n + 1 + (k+1)i. \end{aligned}$$

Then f_9 is a super $(a, k+1$-C_k-antimagic labeling, where $a = \Delta_2 + \Delta_3 + \Delta_5 + n + k + 2$.

Case (x): F_n is super $(a, k+2)$-C_k-antimagic

Define a total labeling f_{10} on F_n as follows: For $1 \leq i \leq n-k+2$, label the vertices in $V_i - \{v\}$ with X_i, v with $n+1$, the edges in E_i' with S_i and the edges in E_i'' with Q_i.

$$\begin{aligned} wt_{f_{10}}(C_k^i) &= f_{10}(V_i) + f_{10}(E_i') + f_{10}(E_i'') \\ &= \sum X_i + n + 1 + \sum S_i + \sum Q_i \\ &= \Delta_1 + (k-1)i + n + 1 + \Delta_4 + i + \Delta_5 + 2i \\ &= \Delta_1 + \Delta_4 + \Delta_5 + n + 1 + (k+2)i. \end{aligned}$$

Then f_{10} is a super $(a, k+2)$-C_k-antimagic labeling, where $a = \Delta_1 + \Delta_4 + \Delta_5 + n + k + 3$.

Case (xi): F_n is super $(a, 2k-5)$-C_k-antimagic

Define a total labeling f_{11} on F_n as follows: For $1 \leq i \leq n-k+2$, label the vertices in $V_i - \{v\}$ with X_i, v with $n+1$, the edges in E_i' with P_i and the edges in E_i'' with $Q_{n-k+3-i}$.

$$\begin{aligned} wt_{f_{11}}(C_k^i) &= f_{11}(V_i) + f_{11}(E_i') + f_{11}(E_i'') \\ &= \sum X_i + n + 1 + \sum P_i + \sum Q_{n-k+3-i} \\ &= \Delta_1 + (k-1)i + n + 1 + \Delta_2 + (k-2)i + \Delta_5 + 2(n-k+3-i) \\ &= \Delta_1 + \Delta_2 + \Delta_5 + 3n - 2k + 7 + (2k-5)i. \end{aligned}$$

Then f_{11} is a super $(a, 2k-5)$-C_k-antimagic labeling, where $a = \Delta_1+\Delta_2+\Delta_5+3n+2$.

Case (xii): F_n is super $(a, 2k-1)$-C_k-antimagic

Define a total labeling f_{12} on F_n as follows: For $1 \leq i \leq n-k+2$, label the vertices in $V_i - \{v\}$ with X_i, v with $n+1$, the edges in E_i' with P_i and the edges in E_i'' with Q_i.

$$\begin{aligned} wt_{f_{12}}(C_k^i) &= f_{12}(V_i) + f_{12}(E_i') + f_{12}(E_i'') \\ &= \sum X_i + n + 1 + \sum P_i + \sum Q_i \\ &= \Delta_1 + (k-1)i + n + 1 + \Delta_2 + (k-2)i + \Delta_5 + 2i \\ &= \Delta_1 + \Delta_2 + \Delta_5 + n + 1 + (2k-1)i. \end{aligned}$$

Then f_{12} is a super $(a, 2k-1)$-C_k-antimagic labeling, where $a = \Delta_1+\Delta_2+\Delta_5+n+2k$.

Hence the fan F_n, $n \geq 3$ admits a super (a, d)-C_k-antimagic labeling for $k = 3, 4, \ldots, \left\lfloor \frac{n}{2} \right\rfloor + 2$ and $d \in \{1, 2, k-5, k-4, \ldots, k+2, 2k-5, 2k-1\}$. ∎

The bistar graph $B_{n,m}$ is the graph obtained from K_2 by joining n pendent edges to one end and m pendent edges to the other end of K_2. In the next theorem we will prove that the square of bistar graph $B_{n,m}$, denoted by $B^2_{n,m}$ admits cycle antimagic labeling. Note that the graph $B^2_{n,m}$ can be alternatively obtained from the complete bipartite graph $K_{2,r}$, where $r = n+m$, by adding an edge between the vertices of degree r.

Theorem 4.2. *The graph $B^2_{m,n}$ admits a super (a, d)-C_3-antimagic labeling for $m,\ n \geq 1$ and $d \in \{0, 1, 2, 3, 5\}$.*

Proof. We denote the vertices and edges of $B^2_{m,n}$ in the following way $V(B^2_{m,n}) = \{u, v, w_i : 1 \leq i \leq m+n\}$ and $E(B^2_{m,n}) = \{uv, uw_i, vw_i : 1 \leq i \leq m+n\}$. Let C_3^i be the 3-cycle uw_ivu for $1 \leq i \leq m+n$. Then $\{C_3^i : 1 \leq i \leq m+n\}$ is a C_3-covering for $B^2_{m,n}$. The vertex set and edge set of C_3^i are $V_i = V(C_3^i) = \{u, v, w_i\}$ and $E_i = E(C_3^i) = \{uv, uw_i, vw_i\}$, respectively.

Note that the weight of the cycle C_k^i under a total labeling f is $wt_f(C_3^i) = f(V_i)+f(E_i)$, where $f(V_i) = f(u)+f(v)+f(w_i)$ and $f(E_i) = f(uv)+f(uw_i)+f(vw_i)$.

We use the following results.

(R1) By Lemma 2.2, there exists an $(m+n)$-equipartition $\left\{X_1', X_2', \ldots, X_{m+n}'\right\}$ of $[1, 2(m+n)]$ such that $\sum X_i' = 2(m+n)+1$ for $1 \le i \le m+n$. Hence we can find an equipartition $\{X_1, X_2, \ldots, X_{m+n}\}$ of $(m+n+3)+[1, 2(m+n)]$ such that $\sum X_i = \delta_1$ for $1 \le i \le m+n$, where $\delta_1 = 2(m+n+3)+2(m+n)+1 = 4(m+n)+7$.

(R2) If $m+n$ is odd and $h=2$, by Lemma 2.3 there exists an $(m+n)$-equipartition $\{Y_1', Y_2', \ldots, Y_{m+n}'\}$ of $[1, 2(m+n)]$ such that $\sum Y_i' = \frac{3(m+n)+1}{2} + i$ for $1 \le i \le m+n$. Hence we can find an equipartition $\{Y_1, Y_2, \ldots, Y_{m+n}\}$ of $(m+n+3)+[1, 2(m+n)]$ such that $\sum Y_i = \delta_2 + i$ for $1 \le i \le m+n$, where $\delta_2 = 2(m+n+3) + \frac{3(m+n)+1}{2}$.

(R3) If $m+n$ is even and $h=2$, by Lemma 2.4, for $h=2$ there exists an $(m+n)$-equipartition $\{Z_1', Z_2', \ldots, Z_{m+n}'\}$ of $[1, 2(m+n)+1] - \{\frac{m+n}{2}+1\}$ such that $\sum Z_i' = \frac{3(m+n)}{2} + 2 + i$ for $1 \le i \le m+n$. Hence we can find an equipartition $\{Z_1, Z_2, \ldots, Z_{m+n}\}$ of $(m+n+2)+[1, 2(m+n)+1] - \{\frac{m+n}{2}+1\}$ such that $\sum Z_i = \delta_3 + i$ for $1 \le i \le m+n$, where $\delta_3 = 2(m+n+2) + \frac{3(m+n)}{2} + 2 + i$.

(R4) By Lemma 2.1, there exists an $(m+n)$-equipartition $\left\{P_1', P_2', \ldots, P_{m+n}'\right\}$ of $[1, 2(m+n)]$ such that $\sum P_i' = m+n+2i$ for $1 \le i \le m+n$. Hence we can find an equipartition $\{P_1, P_2, \ldots, P_{m+n}\}$ of $(m+n+3)+[1, 2(m+n)]$ such that $\sum P_i = \delta_4 + 2i$ for $1 \le i \le m+n$, where $\delta_4 = 2(m+n+3)+m+n+2i = 3(m+n)+6+2i$.

(R5) By Lemma 3.3, there exists an $(m+n)$-equipartition $\left\{Q_1', Q_2', \ldots, Q_{m+n}'\right\}$ of $[1, 2(m+n)]$ such that $\sum Q_i' = 4i-1$ for $1 \le i \le m+n$. Hence we can find an equipartition $\{Q_1, Q_2, \ldots, Q_{m+n}\}$ of $(m+n+3)+[1, 2(m+n)]$ such that $\sum Q_i = \delta_5 + 4i$ for $1 \le i \le m+n$, where $\delta_5 = 2(m+n+3)-1+4i = 2(m+n)+5$.

In order to prove the theorem, we use the results (R1) - (R5).

Case (i): $B_{m,n}^2$ is super $(a,0)$-C_3-antimagic

We distinguish two subcases.

Subcase (a): $m+n$ is odd.

Define a total labeling f_0 on $B_{m,n}^2$ as follows: For $1 \le i \le m+n$, label the vertices $f_0(u) = 1$, $f_0(v)$=2 and $f_0(w_i) = m+n+3-i$ and label the edge uv with $m+n+3$ and the edges in $E_i - \{uv\}$ with Y_i. Then

$$\begin{aligned} wt_{f_0}(C_3^i) &= f_0(V_i) + f_0(E_i) \\ &= 1+2+m+n+3-i+m+n+3+\sum Y_i \\ &= 2(m+n)+9-i+\delta_2+i \\ &= \delta_2 + 2(m+n)+9. \end{aligned}$$

Hence $B_{m,n}^2$ admits a super $(a,0)$-C_3-antimagic labeling with $a = \delta_2 + 2(m+n)+9$.

Subcase (b): $m+n$ is even.

Define a total labeling f_0 on $B^2_{m,n}$ as follows: For $1 \leq i \leq m+n$, label the vertices $f_0(u) = 1$, $f_0(v)$=2 and $f_0(w_i) = m+n+3-i$ and label the edge uv with $m+n+2+\frac{m+n}{2}+1$ and the edges in $E_i - \{uv\}$ with Z_i.Then

$$\begin{aligned} wt_{f_0}(C_3^i) &= f_0(V_i) + f_0(E_i) \\ &= 1+2+m+n+3-i+m+n+2+\tfrac{m+n}{2}+1+\sum Z_i \\ &= 2(m+n)+9-i+\tfrac{m+n}{2}+\delta_3+i \\ &= \delta_3+2(m+n)+\tfrac{m+n}{2}+9. \end{aligned}$$

Hence $B^2_{m,n}$ admits a super $(a,0)$-C_3-antimagic labeling with $a = \delta_3 + 2(m+n) + \frac{m+n}{2} + 9$.

Case (ii): $B^2_{m,n}$ is super $(a,1)$-C_3-antimagic

Define a total labeling f_1 on $B^2_{m,n}$ as follows: For $1 \leq i \leq m+n$, label the vertices $f_1(u) = 1$, $f_1(v)$=2 and $f_1(w_i) = 2+i$ and label the edge uv with $m+n+3$ and the edges in $E_i - \{uv\}$ with X_i.

$$\begin{aligned} wt_{f_1}(C_3^i) &= f_1(V_i) + f_1(E_i) \\ &= 1+2+2+i+m+n+3+\sum X_i \\ &= m+n+8+i+\delta_1 = \delta_1+m+n+8+i. \end{aligned}$$

Hence $B^2_{m,n}$ admits a super $(a,1)$-C_3-antimagic labeling with $a = \delta_1 + m + n + 9$.

Case (iii): $B^2_{m,n}$ is super $(a,2)$-C_3-antimagic

Again we distinguish two subcases according to the parity of $n+m$.

Subcase (a): $m+n$ is odd.

Define a total labeling f_2 on $B^2_{m,n}$ as follows: For $1 \leq i \leq m+n$, label the vertices $f_2(u) = 1$, $f_2(v)$=2 and $f_2(w_i) = 2+i$ and label the edge uv with $m+n+3$ and the edges in $E_i - \{uv\}$ with Y_i.

$$\begin{aligned} wt_{f_2}(C_3^i) &= f_2(V_i) + f_2(E_i) \\ &= 1+2+2+i+m+n+3+\sum Y_i \\ &= m+n+8+i+\delta_2+i \\ &= \delta_2+m+n+8+2i. \end{aligned}$$

Hence $B^2_{m,n}$ admits a super $(a,2)$-C_3-antimagic labeling with $a = \delta_2 + m + n + 10$.

Subcase (b): $m+n$ is even.

Define a total labeling f_2 on $B^2_{m,n}$ as follows: For $1 \leq i \leq m+n$, label the vertices $f_2(u) = 2$, $f_2(v)$=2 and $f_2(w_i) = 2+i$ and label the edge uv with $m+n+2+\frac{m+n}{2}+1$

and the edges in $E_i - \{uv\}$ with Z_i.

$$\begin{aligned} wt_{f_2}(C_3^i) &= f_2(V_i) + f_2(E_i) \\ &= 1 + 2 + 2 + i + m + n + 2 + \frac{m+n}{2} + 1 + \sum Z_i \\ &= m + n + 8 + \frac{m+n}{2} + i + \delta_3 + i \\ &= m + n + \frac{m+n}{2} + 8 + \delta_3 + 2i. \end{aligned}$$

Thus $B^2_{m,n}$ admits a super $(a, 2)$-C_3-antimagic labeling with $a = \delta_3 + m + n + \frac{m+n}{2} + 10$.

Case (iv): $B^2_{m,n}$ is super $(a, 3)$-C_3-antimagic

Define a total labeling f_3 on $B^2_{m,n}$ as follows: For $1 \leq i \leq m + n$, label the vertices $f_3(u) = 1$, $f_3(v)$=2 and $f_3(w_i) = 2 + i$ and label the edge uv with $m + n + 3$ and the edges in $E_i - \{uv\}$ with P_i.

$$\begin{aligned} wt_{f_3}(C_3^i) &= f_3(V_i) + f_3(E_i) \\ &= 1 + 2 + 2 + i + m + n + 3 + \sum P_i \\ &= m + n + 8 + i + \delta_4 + 2i = \delta_4 + m + n + 8 + 3i. \end{aligned}$$

Hence $B^2_{m,n}$ admits a super $(a, 3)$-C_3-antimagic labeling with $a = \delta_4 + m + n + 11$.

Case (v): $B^2_{m,n}$ is super $(a, 5)$-C_3-antimagic

Define a total labeling f_4 on $B^2_{m,n}$ as follows: For $1 \leq i \leq m + n$, label the vertices $f_4(u) = 1$, $f_4(v)$=2 and $f_4(w_i) = 2 + i$ and label the edge uv with $m + n + 3$ and the edges in $E_i - \{uv\}$ with Q_i.

$$\begin{aligned} wt_{f_4}(C_3^i) &= f_4(V_i) + f_4(E_i) \\ &= 1 + 2 + 2 + i + m + n + 3 + \sum Q_i \\ &= m + n + 8 + i + \delta_5 + 4i = \delta_5 + m + n + 8 + 5i. \end{aligned}$$

Hence $B^2_{m,n}$ admits a super $(a, 5)$-C_3-antimagic labeling with $a = \delta_5 + m + n + 13$, which means that the graph $B^2_{m,n}$ admits a super (a, d)-C_3-antimagic labeling for $m,\ n \geq 1$ and $d \in \{0, 1, 2, 3, 5\}$. ∎

In the following theorem we prove that the square of a path is super (a, d)-C_3-antimagic for $1 \leq d \leq 6$.

Theorem 4.3. *The graph P_n^2 admits a super (a, d)-C_3-antimagic labeling for $n \geq 3$ and $d \in \{1, 2, 3, 4, 5, 6\}$.*

Proof. We denote the vertices and edges of P_n^2 such that $V(P_n^2) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n^2) = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_iv_{i+2} : 1 \leq i \leq n-2\}$. Let C_3^i be the 3-cycle $v_iv_{i+1}v_{i+2}$ for $1 \leq i \leq n-2$. Then $\{C_3^i : 1 \leq i \leq n-2\}$ is a C_3-covering for P_n^2. The vertex set and edge set of C_3^i are $V_i = V(C_3^i) = \{v_i, v_{i+1}, v_{i+2}\}$ and $E_i = E(C_3^i) = E'_i \cup \{v_iv_{i+2}\}$, where $E'_i = \{v_iv_{i+1}, v_{i+1}v_{i+2}\}$.

Note that the weight of the cycle C_k^i under a total labeling f is $wt_f(C_3^i) = f(V_i) + f(E_i') + f(v_iv_{i+2})$, where $f(V_i) = f(v_i) + f(v_{i+1}) + f(v_{i+2})$ and $f(E_i') = f(v_iv_{i+1}) + f(v_{i+1}v_{i+2})$.

First we introduce the following rules.

(R1) By Lemma 3.5, $[1, n]$ can be arranged as a sequence $(a_i)_{i=1}^n$ such that $\sum X_{i+1} - \sum X_i = 1$ for $1 \le i \le n-2$, where $X_i = \{a_i, a_{i+1}, a_{i+2}\}$ with $\sum X_i = \delta_1 + i$ for $1 \le i \le n-2$ where $\delta_1 = \frac{6t-(3-s)^2-s+7}{2}$, $t = \left\lceil \frac{n}{3} \right\rceil$ and $s = n \pmod 3$.

(R2) By Lemma 3.1, the multiset $X = \biguplus_{j=2}^{m} [j, n-j+1] \uplus [1, n]$ is $(n-2, 3)$-anti-balanced with anti-balanced subsets $Y_1, Y_2, \ldots, Y_{n-2}$ with $\sum Y_i = 3 + 3i$, where $m = \min\{3, n-2\}$.

(R3) Let $X = \begin{cases} [1, n-1] \uplus [2, n-2] & \text{if } n-1 \text{ is even,} \\ [1, \frac{n-2}{2}] \uplus [\frac{n-2}{2} + 2, n-1] \uplus [2, n-1] & \text{if } n-1 \text{ is odd,} \end{cases}$

By Lemma 2.5, the multiset $X = [1, n-1]$ is $(n-2, 1)$-anti-balanced with anti-balanced subsets $Y_1, Y_2, \ldots, Y_{n-2}$ defined by $Y_i = \left\{ \left\lceil \frac{i+1}{2} \right\rceil, \left\lfloor \frac{n-2}{2} \right\rfloor + \left\lceil \frac{i}{2} \right\rceil + 1 \right\}$. Correspondingly, the multiset $n \oplus X$ is also $(n-2, 1)$-anti-balanced with anti-balanced subsets $P_1, P_2, \ldots, P_{n-2}$ defined by $P_i = \left\{ n + \left\lceil \frac{i+1}{2} \right\rceil, n + \left\lfloor \frac{n-2}{2} \right\rfloor + \left\lceil \frac{i}{2} \right\rceil + 1 \right\}$ and $\sum P_i = \delta_2 + i$, where $\delta_2 = 2n + \left\lfloor \frac{n-2}{2} \right\rfloor + 2$.

(R4) By Lemma 2.6, the multiset $Y = [1, n-2] \uplus [2, n-1]$ is $(n-2, 2)$-anti-balanced with anti-balanced subsets $Y_1, Y_2, \ldots, Y_{n-2}$ defined by $Y_i = \{i, i+1\}$. Correspondingly, the multiset $n \oplus Y$ is also $(n-2, 2)$-anti-balanced with anti-balanced subsets $Q_1, Q_2, \ldots, Q_{n-2}$ defined by $Q_i = \{n+i, n+i+1\}$ and $\sum Q_i = 2n + 1 + 2i$.

Now we prove that P_n^2 is super (a, d)-C_3-antimagic for $1 \le d \le 6$.

Case (i): $d = 1$

Define a total labeling f_1 on P_n^2 as follows: For $1 \le i \le n-2$, label the vertices in V_i with X_i, the edges in E_i' with P_{n-1-i} and the edge v_iv_{i+2} with $2n-1+i$. Then

$$
\begin{aligned}
wt_{f_1}(C_3^i) &= f_1(V_i) + f_1(E_i) \\
&= \sum X_i + \sum P_{n-1-i} + 2n - 1 + i \\
&= \delta_1 + i + \delta_2 + n - 1 - i + 2n - 1 + i \\
&= \delta_1 + \delta_2 + 3n - 2 + i.
\end{aligned}
$$

Hence P_n^2 admits a super $(a, 1)$-C_3-antimagic labeling with $a = \delta_1 + \delta_2 + 3n - 1$.

Case (ii): $d = 2$

Define a total labeling f_2 on P_n^2 as follows: For $1 \le i \le n-2$, label the vertices in

V_i with X_i, the edges in E'_i with Q_i and the edge v_iv_{i+2} with $3n-2-i$. Then

$$\begin{aligned} wt_{f_2}(C_3^i) &= f_2(V_i) + f_2(E_i) \\ &= \sum X_i + \sum Q_i + 3n - 2 - i \\ &= \delta_1 + i + 2n + 1 + 2i + 3n - 2 - i \\ &= \delta_1 + 5n - 1 + 2i. \end{aligned}$$

Hence P_n^2 admits a super $(a, 2)$-C_3-antimagic labeling with $a = \delta_1 + 5n + 1$.

Case (iii): $d = 3$

Define a total labeling f_3 on P_n^2 as follows: For $1 \leq i \leq n-2$, label the vertices in V_i with Y_i, the edges in E'_i with P_i and the edge v_iv_{i+2} with $3n-2-i$.

$$\begin{aligned} wt_{f_3}(C_3^i) &= f_3(V_i) + f_3(E_i) \\ &= \sum Y_i + \sum P_i + 3n - 2 - i \\ &= 3 + 3i + \delta_2 + i + 3n - 2 - i \\ &= \delta_2 + 3n + 1 + 3i. \end{aligned}$$

Hence P_n^2 admits a super $(a, 3)$-C_3-antimagic labeling with $a = \delta_2 + 3n + 4$.

Case (iv): $d = 4$

Define a total labeling f_4 on P_n^2 as follows: For $1 \leq i \leq n-2$, label the vertices in V_i with Y_i, the edges in E'_i with Q_i and the edge v_iv_{i+2} with $3n-2-i$.

$$\begin{aligned} wt_{f_4}(C_3^i) &= f_4(V_i) + f_4(E_i) \\ &= \sum Y_i + \sum Q_i + 3n - 2 - i \\ &= 3 + 3i + 2n + 1 + 2i + 3n - 2 - i \\ &= 5n + 2 + 4i. \end{aligned}$$

Hence P_n^2 admits a super $(a, 4)$-C_3-antimagic labeling with $a = 5n + 6$.

Case (v): $d = 5$

Define a total labeling f_5 on P_n^2 as follows: For $1 \leq i \leq n-2$, label the vertices in V_i with Y_i, the edges in E'_i with P_i and the edge v_iv_{i+2} with $2n-1+i$.

$$\begin{aligned} wt_{f_5}(C_3^i) &= f_5(V_i) + f_5(E_i) \\ &= \sum Y_i + \sum P_i + 2n - 1 + i \\ &= 3 + 3i + \delta_2 + i + 2n - 1 + i \\ &= \delta_2 + 2n + 2 + 5i. \end{aligned}$$

Hence P_n^2 admits a super $(a, 5)$-C_3-antimagic labeling with $a = \delta_2 + 2n + 7$.

Case (vi): $d = 6$

Define a total labeling f_6 on P_n^2 as follows: For $1 \leq i \leq n-2$, label the vertices in V_i with Y_i, the edges in E_i' with Q_i and the edge $v_i v_{i+2}$ with $2n-1+i$. Then

$$\begin{aligned} wt_{f_6}(C_3^i) &= f_6(V_i) + f_6(E_i) \\ &= \sum Y_i + \sum Q_i + 2n - 1 + i \\ &= 3 + 3i + 2n + 1 + 2i + 2n - 1 + i \\ &= 4n + 3 + 6i. \end{aligned}$$

Hence P_n^2 admits a super $(a, 6)$-C_3-antimagic labeling with $a = 4n + 9$.

This concludes the proof. ∎

Note that Muthuraja, Selvagopal and Jeyanthi [18] showed that the square graph of a path is cycle-supermagic. Thus combining these results we find that P_n^2 is super (a, d)-C_3-antimagic for $0 \leq d \leq 6$.

Acknowledgements

The research for this article was supported by APVV-15-0116.

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(Received 19 Aug 2016; revised 22 Nov 2016)