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Kernels in Cartesian products of digraphs

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Abstract

A kernel J of a digraph D is an independent set of vertices of D such that for every vertex $w \in V(D) \setminus J$ there exists an arc from w to a vertex in J. In this paper we have obtained results for the existence and non-existence of kernels in Cartesian products of certain families of digraphs, and characterized $T\Box\vec{C}_n$, $T\Box\vec{P}_n$ and $\vec{C}_m\Box\vec{C}_n$ which have kernels, where T is a tournament, and $\vec{P}_n$ and $\vec{C}_n$ are, respectively, the directed path and the directed cycle of order n. Finally, we have introduced and studied kernel-partitionable digraphs.

1 Introduction

For notation and terminology, in general, we follow [1].

Let $D = (V, A)$ be a digraph and let k and ℓ be integers with $k \geq 2$ and $\ell \geq 1$. A set $J \subseteq V$ is a (k, ℓ)-*kernel* of D if

(a) for every ordered pair (x, y) of distinct vertices in J, we have $d_D(x, y) \geq k$,

(b) for each $z \in V \setminus J$, there exists an $x \in J$ such that $d_D(z, x) \leq \ell$.

It follows that every (k, ℓ)-kernel is a $(k, \ell + 1)$-kernel, and for $k \geq 3$, every (k, ℓ)-kernel is a $(k - 1, \ell)$-kernel.

A $(2, 1)$-kernel of D is called a *kernel* of D, or more precisely, a set J of vertices in D is a *kernel* if J is independent (i.e., the subdigraph of D induced by J has no arcs) and the first closed in-neighbourhood of J, namely $N_D^-[J]$, is equal to $V(D)$. A digraph D is *kernel-less* if it has no kernel.

The *Cartesian product* of digraphs D_1 and D_2 is the digraph $D = D_1\Box D_2$ with vertex set $V(D) = \{(u, v) : u \in V(D_1), v \in V(D_2)\}$ and arc set $A(D) = \{((u_1, v_1), (u_2, v_2))$ such that either $u_1 = u_2$ and $(v_1, v_2) \in A(D_2)$ or $v_1 = v_2$ and $(u_1, u_2) \in A(D_1)\}$.

In [2], Kwaśnik proved the following:

If the subset $J_1 \subseteq V(D_1)$ is a (k_1, ℓ_1)-kernel of D_1 and $J_2 \subseteq V(D_2)$ is a (k_2, ℓ_2)-kernel of D_2, for $k_i \geq 2$, $\ell_i \geq 1$, $i \in \{1, 2\}$, then the set $J = J_1 \times J_2$ is a (k, ℓ)-kernel of the digraph $D_1 \Box D_2$, where $k = \min\{k_1, k_2\}$ and $\ell = \ell_1 + \ell_2$.

In [5], Włoch and Włoch generalize the above result for the generalized Cartesian product in which they consider the product for strongly connected digraphs only.

From the above result of Kwaśnik, we have: If $J_1 \subseteq V(D_1)$ and $J_2 \subseteq V(D_2)$ are kernels of the digraphs D_1 and D_2, respectively, then the set $J = J_1 \times J_2$ is a $(2, 2)$-kernel of the digraph $D_1 \Box D_2$. Note that, the set J may not be a kernel of $D_1 \Box D_2$.

In this paper we consider the problem of finding either the existence or the non-existence of kernels in Cartesian products $D_1 \Box D_2$ of certain classes of digraphs D_1 and D_2. Finally, we introduce kernel-partitionable digraphs and provide examples of them.

A digraph D is an *oriented graph* if D contains no directed cycle of length 2.

For any positive integer k, let $V(\vec{P}_k) = V(\vec{C}_k) = \mathbb{Z}_k = \{0, 1, 2, \ldots, k-1\}$, $A(\vec{P}_k) = \{(i, i+1) : i \in \{0, 1, 2, \ldots, k-2\}\}$ and $A(\vec{C}_k) = A(\vec{P}_k) \cup \{(k-1, 0)\}$.

2 Preliminary lemmas

Lemma 2.1 *Let D be a digraph and let $O = \{v \in V(D) : d_D^+(v) = 0\}$. Then any kernel of D, if it exists, contains O.* ■

Lemma 2.2 *If T is a tournament, then there are at most three vertices with out-degree one.*

Proof. By contradiction. Suppose there exist four vertices, say, w, x, y and z of out-degree one.

Case 1. The unique out-neighbour of w is in $V(T) \backslash \{x, y, z\}$.

Then, $\{x, y, z\} \rightarrow w$.

Case 2. The unique out-neighbour of w is in $\{x, y, z\}$, say, x, i.e., $w \rightarrow x$.

Then $\{y, z\} \rightarrow w$.

In any case, as $d_T^+(y) = 1 = d_T^+(z)$, we have neither $y \rightarrow z$ nor $z \rightarrow y$, a contradiction. ■

Similarly, we have:

Lemma 2.3 *If T is a tournament, then there are at most three vertices with in-degree one.* ■

Lemma 2.4 *Let D be a digraph with a set X of vertices in D such that $d_D^+(v) = 1$ for every $v \in X$ and $D[X]$, the subdigraph induced by X, is a directed odd cycle. Then D is kernel-less.*

Proof. Assume, by hypothesis, that $D[X] = \vec{C}_{2k+1} := 0 \to 1 \to 2 \to \cdots \to (2k) \to 0$, $k \geq 1$. Suppose J is a kernel of D. Then J contains at most k vertices from X. Consequently, there exist vertices i, $i+1$, where $i \in \{0, 1, 2, \ldots, 2k\}$, such that both i and $i+1$ belong to $V(D) \setminus J$, where $2k+1 = 0$. As $i+1$ is the only vertex dominated by i, no vertex of J is dominated by i, a contradiction. ∎

Lemma 2.5 *Let T be a tournament, D be a digraph, $x \in V(T)$, $y \in V(D)$ and $d_T^+(x) > d_D^+(y)$. Then for any kernel J of $T \Box D$, if it exists, we have $(x, y) \in V(T \Box D) \setminus J$.*

Proof. Let $d_T^+(x) = p$ and $d_D^+(y) = q$. Suppose $(x, y) \in J$. As $d_T^+(x) = p$, there exist vertices $x_1, x_2, \ldots, x_p$ in T such that $x \to x_i$, $i \in \{1, 2, \ldots, p\}$, in T. As T is a tournament, $(x, y) \in J$ implies that, for every $u \in V(T) \setminus \{x\}$, $(u, y) \in V(T \Box D) \setminus J$. In particular, for every $i \in \{1, 2, \ldots, p\}$, $(x_i, y) \in V(T \Box D) \setminus J$. As $d_D^+(y) = q$, there exist vertices $y_1, y_2, \ldots, y_q$ in D such that $y \to y_j$, $j \in \{1, 2, \ldots, q\}$, in D. For each $i \in \{1, 2, \ldots, p\}$, $(x_i, y) \in V(T \Box D) \setminus J$, $(x, y) \to (x_i, y)$, and for every $u \in V(T) \setminus \{x\}$, $(u, y) \notin J$ implies that there exists $j(i) \in \{1, 2, \ldots, q\}$ such that $(x_i, y_{j(i)}) \in J$. Since $q \leq p - 1$, there exist i' and i'' such that $j(i') = j(i'')$, a contradiction, since the adjacent vertices $(x_{i'}, y_{j(i')})$ and $(x_{i''}, y_{j(i'')})$ are in J. ∎

3 Kernel-less Cartesian products

First, we consider $\vec{C}_m \Box \vec{C}_n$.

Lemma 3.1 *Let $m \geq 3$ and $n \geq 3$ be positive integers. If m and n are relatively prime, then $\vec{C}_m \Box \vec{C}_n$ is kernel-less.*

Proof. Suppose $\vec{C}_m \Box \vec{C}_n$ admits a kernel, say, J.
Claim 1. There exists no (i, j) such that

$$\{(i, j-1), (i, j), (i, j+1)\} \subseteq V \setminus J,$$

where $i \in \{0, 1, 2, \ldots, m-1\}$ and $j \in \{0, 1, 2, \ldots, n-1\}$.

Otherwise, there exists a vertex (i, j) such that $\{(i, j-1), (i, j), (i, j+1)\} \subseteq V \setminus J$. Now $(i, j-1) \notin J$ implies that $(i+1, j-1) \in J$, since $(i, j) \notin J$. Also, $(i, j) \notin J$ implies that $(i+1, j) \in J$, since $(i, j+1) \notin J$. Hence the two adjacent vertices $(i+1, j-1)$ and $(i+1, j)$ are in J, a contradiction.

Claim 2. $(i+1, j+1) \in J$ whenever $(i, j) \in J$.

Suppose there exists $(i, j) \in J$ with $(i+1, j+1) \notin J$. As $(i, j) \in J$, both $(i, j+1)$ and $(i+1, j)$ belong to $V \setminus J$. As $(i, j+1) \notin J$, $(i, j+2) \in J$, and hence $(i+1, j+2) \notin J$. Now $\{(i+1, j), (i+1, j+1), (i+1, j+2)\} \subseteq V \setminus J$, a contradiction to Claim 1.

As m and n are relatively prime, by Claim 2, $J = V(\vec{C}_m \Box \vec{C}_n)$, a contradiction. ∎

Next, we have an application of Lemma 2.4.

Theorem 3.1 *Let D_1 be a digraph with $\delta^+(D_1) = 0$ and let D_2 be a digraph such that there exists a set Y of vertices in D_2 with $d^+_{D_2}(v) = 1$ for every $v \in Y$ and $D_2[Y]$, the subdigraph induced by Y, is a directed odd cycle. Then $D_1 \Box D_2$ is kernel-less.*

Proof. By hypothesis, $d^+_{D_1}(u) = 0$ for some $u \in V(D_1)$. Apply Lemma 2.4 for $D = D_1 \Box D_2$ and $X = \{u\} \times Y$. ■

Corollary 3.1 *Let D be a digraph with $\delta^+(D) = 0$. If $n \geq 1$ is a positive integer, then $D \Box \vec{C}_{2n+1}$ is kernel-less.* ■

Corollary 3.2 *Let T be a tournament with $\delta^+(T) = 0$. If $n \geq 1$ is a positive integer, then $T \Box \vec{C}_{2n+1}$ is kernel-less.* ■

Since any oriented tree has a vertex of out-degree zero, we have:

Corollary 3.3 *If D is an oriented tree and if $n \geq 1$ is a positive integer, then $D \Box \vec{C}_{2n+1}$ is kernel-less.* ■

Finally, we have some applications of Lemma 2.5.

Corollary 3.4 *Let T be a tournament and D be a digraph. If $\delta^+(T) > \Delta^+(D)$, then $T \Box D$ is kernel-less.* ■

Corollary 3.5 *Let T be a tournament with $\delta^+(T) \geq 2$. If D is a nonempty digraph with $\Delta^+(D) \leq 1$, then $T \Box D$ is kernel-less. In particular, for $n \geq 3$, $T \Box \vec{C}_n$ and for $n \geq 2$, $T \Box \vec{P}_n$ are kernel-less.* ■

Lemma 3.2 *Let T be a tournament with $|\{v \in V(T) : d^+_T(v) \leq 1\}| \leq 1$. If D is a nonempty digraph with $\Delta^+(D) \leq 1$, then $T \Box D$ is kernel-less. In particular, for $n \geq 3$, $T \Box \vec{C}_n$ and for $n \geq 2$, $T \Box \vec{P}_n$ are kernel-less.*

Proof. Let $A = \{v \in V(T) : d^+_T(v) \leq 1\}$. If $A = \emptyset$, then kernel-less follows from Corollary 3.5. Hence, assume that $A = \{v\}$. Then $d^+_T(v) \leq 1$, and for every $u \in V(T) \backslash \{v\}$, $d^+_T(u) \geq 2$. Suppose $T \Box D$ admits a kernel, say, J; then by Lemma 2.5, $J \subseteq \{(v, x) : x \in V(D)\}$. For every $u \in V(T) \setminus \{v\}$ and for every $x \in V(D)$, $(u, x) \notin J$ implies that $(v, x) \in J$ and, in T, we have $u \to v$. But then $d^+_T(v) = 0$ and $J = \{(v, x) : x \in V(D)\}$, a contradiction to J being independent. ■

Lemma 3.3 *Let $n \geq 3$ and T be a tournament with $\delta^+(T) = 1$.*

(1) If T has at most two vertices of out-degree 1, then $T \Box \vec{C}_n$ is kernel-less.

(2) If T has exactly three vertices of out-degree 1 *and if $n \not\equiv 0$* (mod 3), *then $T \Box \vec{C}_n$ is kernel-less.*

Proof. Suppose $T\Box\vec{C}_n$ admits a kernel, say J. As $d^+_{\vec{C}_n}(j) = 1$, $j \in V(\vec{C}_n)$, by Lemma 2.5 we have $(v,j) \in V(T\Box\vec{C}_n) \setminus J$, for every $v \in V(T)$ with $d^+_T(v) \geq 2$. Hence $J \subseteq \{(v,j) : d^+_T(v) = 1,\ 0 \leq j \leq n-1\}$.

Proof of (1). If T has exactly one vertex, say, a of out-degree 1, then $J \subseteq \{(a,j) : 0 \leq j \leq n-1\}$. Let the out-neighbour of a in T be b. As J is nonempty, $(a,j) \in J$ for some j. This implies that $(b,j) \notin J$. As $(a,j) \to (b,j)$, no vertex of J is dominated by (b,j), a contradiction.

If T has exactly two vertices, say a' and a'' of out-degree 1, then $J \subseteq \{(a',j), (a'',j) : 0 \leq j \leq n-1\}$. As T is a tournament, we have, in T, either $a' \to a''$ or $a'' \to a'$. Without loss of generality assume that $(a',a'') \in A(T)$. Let the out-neighbour of a'' in T be b. If $(a'',j) \in J$ for some j, then $(a',j),(b,j) \notin J$. Since $(a'',j) \to (b,j) \to (a',j)$ and $(a',j) \notin J$, no vertex of J is dominated by (b,j), a contradiction. Hence $J \subseteq \{(a',j) : 0 \leq j \leq n-1\}$ and so $(a',j) \in J$ for some j; then $(a'',j) \notin J$. Consequently, (a'',j) dominates no vertex of J, a contradiction.

Proof of (2). Let the vertices of out-degree 1 in T be x, y and z. Clearly, they induce a directed cycle of length 3 in T, say, without loss of generality that $x \to y \to z \to x$. By Lemma 2.5, $J \subseteq \{(x,j),(y,j),(z,j) : 0 \leq j \leq n-1\}$. As J is nonempty, by symmetry, assume that $(x,0) \in J$.

Claim. If $(x,j) \in J$, then $(x,j+3) \in J$.

If $(x,j) \in J$, then $\{(y,j),(z,j)\} \subseteq V(T\Box\vec{C}_n) \setminus J$. This shows that $(y,j+1) \in J$, and therefore $\{(x,j+1),(z,j+1)\} \subseteq V(T\Box\vec{C}_n) \setminus J$. Consequently, $(z,j+2) \in J$, and so $\{(x,j+2),(y,j+2)\} \subseteq V(T\Box\vec{C}_n) \setminus J$. Thus $(x,j+3) \in J$.

By the above claim, $\{(x,0),(x,3),(x,6),\dots\} \subseteq J$. This shows that $n \equiv 0 \pmod 3$, a contradiction. ■

Lemma 3.4 *Let $n \geq 2$ and T be a tournament. If T has no pair of vertices u, v with $d^+_T(u) = 0$ and $d^+_T(v) = 1$, then $T\Box\vec{P}_n$ is kernel-less.*

Proof. Suppose $T\Box\vec{P}_n$ admits a kernel, say J. By Lemma 2.5, $J \subseteq \{(w,i) : d^+_T(w) \leq 1$ and $i \in \{0,1,2,\dots,n-1\}\}$.

If $\delta^+(T) \geq 2$, then $J = \emptyset$ and therefore $\delta^+(T) \leq 1$.

By Lemma 2.2, T has at most three vertices with out-degree one.

If T has exactly three vertices, say x, y and z with out-degree one, then without loss of generality assume that $x \to y \to z \to x$. Hence, for every $w \in V(T)\setminus\{x,y,z\}$, $d^+_T(w) \geq 3$. Thus $J \subseteq \{(x,i),(y,i),(z,i) : 0 \leq i \leq n-1\}$. Amongst the three vertices $(x,n-1),(y,n-1),(z,n-1)$, at most one belongs to J and hence at least two must be in $V(T\Box\vec{P}_n) \setminus J$. Assume, by symmetry, that $(x,n-1),(y,n-1) \notin J$. We have a contradiction, since the unique vertex dominated by $(x,n-1)$ is $(y,n-1)$.

If T has exactly two vertices, say x and y with out-degree one, then there exists a vertex z and without loss of generality assume that $x \to y \to z \to x$. Hence, for every $w \in V(T)\setminus\{x,y\}$, $d^+_T(w) \geq 2$. Thus $J \subseteq \{(x,i),(y,i) : 0 \leq i \leq n-1\}$. Amongst the three vertices $(x,n-1),(y,n-1),(z,n-1)$, at most one belongs to J and hence at least two must be in $V(T\Box\vec{P}_n) \setminus J$. If $(x, n-1)$ and $(y, n-1)$ are not in J then we have a contradiction, since the unique vertex dominated by $(x, n-1)$ is $(y, n-1)$. If $(y, n-1)$ and $(z, n-1)$ are not in J, then again we have a contradiction, since the unique vertex dominated by $(y, n-1)$ is $(z, n-1)$. If $(z, n-1)$ and $(x, n-1)$ are not in J, then also we have a contradiction, since the vertices dominated by $(z, n-1)$ are in $V(T\Box\vec{P}_n) \setminus J$.

Hence T has at most one vertex of out-degree one. By Lemma 3.2, $|\{v \in V(T) : d_T^+(v) \leq 1\}| \geq 2$. This completes the proof. ■

4 Kernels in Cartesian products

For an integer $n \geq 2$ and a set $S \subseteq \{1, 2, \ldots, n-1\}$, the *circulant digraph* $C_n(S)$ is a digraph with vertex set $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ and arc set $\{(i, (i+j) \pmod n)) : i \in \mathbb{Z}_n$ and $j \in S\}$.

In [3], we have characterized 2-regular circulant digraphs which have kernels: "Let $i + j \neq n$ and let $m = \gcd(i + j, n)$. The oriented graph $C_n(\{i, j\})$ has a kernel if and only if $i \not\equiv 0 \pmod m$, $j \not\equiv 0 \pmod m$ and $m \neq 1$."

First, we consider: $C_m(\{1\})\Box C_n(\{1\})$ and $C_n(S)\Box C_n(S)$ with $|S| \geq 2$.

Lemma 4.1 *Let $m \geq 3$ and $n \geq 3$ be positive integers. If m and n are not relatively prime, then $\vec{C}_m\Box\vec{C}_n$ admits a kernel.*

Proof. Assume without loss of generality that $m \leq n$. For each $\ell \in \{0, 1, 2, \ldots, n-1\}$, define the set P_ℓ inductively as follows: $(0, \ell) \in P_\ell$. If $(a, b) \in P_\ell$, then $((a+1) \pmod m, (b+1) \pmod n) \in P_\ell$. Let $\gcd(m, n) = k$. If k is even, then set $J = P_0 \cup P_2 \cup P_4 \cup \cdots \cup P_{k-2}$, and if k is odd, then set $J = P_0 \cup P_2 \cup P_4 \cup \cdots \cup P_{k-3}$. Note that $P_k = P_0$.

Claim 1. J is independent.

Otherwise, there exists $(x, y) \in J$ such that either $(x+1, y)$ or $(x, y+1)$ belongs to J. Now $(x, y) \in J$ implies that $(x, y) \in P_{2i}$ for some i. But then neither $(x+1, y)$ nor $(x, y+1)$ belongs to J, since $(x+1, y) \in P_{2i-1}$ and $(x, y+1) \in P_{2i+1}$.

Claim 2. J is absorbent.

Suppose $(x, y) \notin J$. Then $(x, y) \in P_1 \cup P_3 \cup P_5 \cup \cdots \cup P_{k-1}$ if k is even, and $(x, y) \in [P_1 \cup P_3 \cup P_5 \cup \cdots \cup P_{k-2}] \cup P_{k-1}$ if k is odd. Except for $(x, y) \in P_{k-2} \cup P_{k-1}$ with k odd, both the out-neighbours $(x+1, y)$ and $(x, y+1)$ of (x, y) are in J. For $(x, y) \in P_{k-2}$ with k odd, the out-neighbour $(x+1, y)$ of (x, y) is in J; for $(x, y) \in P_{k-1}$ with k odd, the out-neighbour $(x, y+1)$ of (x, y) is in J.

By Claims 1 and 2, J is a kernel of $\vec{C}_m\Box\vec{C}_n$. ■

Theorem 4.1 *If $|S| \geq 2$, then $C_n(S)\Box C_n(S)$ admits a kernel.*

Proof. Let $S = \{i_1, i_2, \ldots, i_k\}$, $0 < i_1 < i_2 < \cdots < i_k < n$. For $\ell \in \{0, 1, 2, \ldots, n-1\}$, define the set P_ℓ inductively as follows:

$(0, \ell) \in P_\ell$.

If $(a, b) \in P_\ell$, then $((a+1) \pmod{n}, (b+1) \pmod{n}) \in P_\ell$.

Let A be a maximal independent set of $C_n(S)$. Set $J = \bigcup_{\ell \in A} P_\ell$.

Claim 1. J is independent.

Otherwise, there exists a vertex of J, say, (x, y) such that $\{(x, y+i_1), \ldots, (x, y+i_k), (x+i_1, y), \ldots, (x+i_k, y)\} \cap J \neq \emptyset$. That is, there exists a vertex $y-x$ of A such that $\{y-x+i_1, \ldots, y-x+i_k, y-x-i_1, \ldots, y-x-i_k\} \cap A \neq \emptyset$. In $C_n(S)$, $\{y-x-i_1, \ldots, y-x-i_k\} \to y-x \to \{y-x+i_1, \ldots, y-x+i_k\}$, a contradiction to A being independent.

Claim 2. J is absorbent.

Suppose $(x, y) \notin J$; equivalently, $y-x \notin A$. Claim 2 follows if $\{(x, y+i_1), \ldots, (x, y+i_k), (x+i_1, y), \ldots, (x+i_k, y)\} \cap J \neq \emptyset$. Otherwise, $\{(x, y+i_1), \ldots, (x, y+i_k), (x+i_1, y), \ldots, (x+i_k, y)\} \cap J = \emptyset$. That is, $\{y-x+i_1, \ldots, y-x+i_k, y-x-i_1, \ldots, y-x-i_k\} \cap A = \emptyset$. But then $A \cup \{y-x\}$ is an independent set of $C_n(S)$, a contradiction to the maximality of A.

By Claims 1 and 2, J is a kernel of $C_n(S) \square C_n(S)$. ■

Next, we consider $T \square \vec{C}_n$, where T is a tournament.

Lemma 4.2 *Let T be a tournament with $\delta^+(T) = 1$. If T has exactly three vertices of out-degree 1 and $n \equiv 0 \pmod{3}$, then $T \square \vec{C}_n$ admits a kernel.*

Proof. Let the vertices of out-degree 1 in T be x, y and z. Clearly, they induce a directed cycle of length 3 in T, say, without loss of generality that $x \to y \to z \to x$.

Set $J = \{(x, 3i), (y, 3i+1), (z, 3i+2) : 0 \leq i \leq \frac{n}{3} - 1\}$. Clearly, J is independent.

Claim. J is absorbent.

Let $(r, s) \in V(T \square \vec{C}_n) \setminus J$.

Case 1. $r \in \{x, y, z\}$, say, $r = x$, i.e., $(x, s) \in V(T \square \vec{C}_n) \setminus J$.

Then $s \not\equiv 0 \pmod{3}$. If $s \equiv 1 \pmod{3}$, then $(y, s) \in J$ and $(x, s) \to (y, s)$. If $s \equiv 2 \pmod{3}$, then $(x, s+1) \in J$ and $(x, s) \to (x, s+1)$.

Case 2. $r \notin \{x, y, z\}$.

If $s \equiv 0 \pmod{3}$, then $(x, s) \in J$ and $(r, s) \to (x, s)$. If $s \equiv 1 \pmod{3}$, then $(y, s) \in J$ and $(r, s) \to (y, s)$. If $s \equiv 2 \pmod{3}$, then $(z, s) \in J$ and $(r, s) \to (z, s)$.

Thus J is a kernel of $T \square \vec{C}_n$. ■

Lemma 4.3 *Let T be a tournament with $\delta^+(T) = 0$. If $n \geq 2$ is a positive integer and if T has a vertex of out-degree one, then $T \square \vec{C}_{2n}$ admits a kernel.*

Proof. By hypothesis, there exist vertices x and y in T such that $V(T) \backslash \{x\} \to x$ and $V(T) \backslash \{x, y\} \to y$. Set $J = \{(x, j) : j \in \{0, 2, 4, \ldots, 2n-2\}\} \cup \{(y, j) : j \in \{1, 3, 5, \ldots, 2n-1\}\}$. Clearly, J is independent.

Let $(p,q) \in V(T\Box\vec{C}_{2n})\backslash J$. If $p = x$, then $q \in \{1,3,5,\dots,2n-1\}$ and $(x,q) \to (x,q+1)$, a vertex in J. If $p = y$, then $q \in \{0,2,4,\dots,2n-2\}$ and $(y,q) \to (y,q+1)$, a vertex in J. So assume that $p \notin \{x,y\}$. If $q \in \{0,2,4,\dots,2n-2\}$, then $(p,q) \to (x,q)$, a vertex in J. If $q \in \{1,3,5,\dots,2n-1\}$, then $(p,q) \to (y,q)$, a vertex in J. Here addition is reduced modulo $2n$. Thus J is absorbent. ■

Lemma 4.4 *Let T be a tournament with $\delta^+(T) = 0$. If $n \geq 2$ is a positive integer and if T has a vertex of out-degree one, then $T\Box\vec{P}_n$ admits a kernel.*

Proof. By hypothesis, there exist vertices x and y in T such that $V(T)\backslash\{x\} \to x$ and $V(T)\backslash\{x,y\} \to y$. The set $J = \{(x,j) : j \in \{n-1, n-3, n-5, \dots\}\} \cup \{(y,j) : j \in \{n-2, n-4, n-6, \dots\}\}$ is a kernel of $T\Box\vec{P}_n$. ■

Theorem 4.2 *For any digraph D and for any $n \geq |V(D)|$, $D\Box K_n^*$ admits a kernel, where K_n^* denotes the complete symmetric digraph on n vertices.*

Proof. Let $V(D) = \{v_1, v_2, \dots, v_{|V(D)|}\}$ and $V(K_n^*) = \{1, 2, \dots, n\}$. Then $J = \{(v_i, i) : i \in \{1, 2, \dots, |V(D)|\}\}$ is a kernel of $D\Box K_n^*$. ■

For any digraph D, $\chi(D)$ denotes the chromatic number of the underlying graph of D.

Theorem 4.3 *Let D_1 and D_2 be digraphs. If D_2 contains $\chi(D_1)$ pairwise disjoint kernels, then the Cartesian product $D_1\Box D_2$ contains a kernel.*

Proof. Let $\chi(D_1) = k$. Let $\{U_1, U_2, \dots, U_k\}$ be a chromatic partition of D_1 and, by hypothesis, we have a collection $\{V_1, V_2, \dots, V_k\}$ of pairwise disjoint kernels of D_2. Consider the set $W = (U_1 \times V_1) \cup (U_2 \times V_2) \cup \cdots \cup (U_k \times V_k)$.

As U_i and V_i are, respectively, independent subsets of D_1 and D_2, $U_i \times V_i$ is an independent subset of $D_1\Box D_2$. Suppose W is not independent; then there exist vertices (a,b) and (c,d) such that $(a,b) \in U_i \times V_i$, $(c,d) \in U_j \times V_j$, $i \neq j$ and $(a,b) \to (c,d)$ in $D_1\Box D_2$. But then either $a = c$ and $b \to d$ or $a \to c$ and $b = d$. Consequently $i = j$, a contradiction. Hence W is independent.

If $(x,y) \notin W$, then $x \in U_i$ for some i and so $y \notin V_i$. As V_i is a kernel of D_2, there exists $z \in V_i$ such that $y \to z$ in D_2. Hence $(x,z) \in U_i \times V_i \subseteq W$ and $(x,y) \to (x,z)$ in $D_1\Box D_2$. Thus W is absorbent.

Hence W is a kernel of $D_1\Box D_2$. This completes the proof. ■

5 A few characterizations

Combining Lemmas 3.1 and 4.1, we have:

Theorem 5.1 *Let $m \geq 3$ and $n \geq 3$ be positive integers. Then $\vec{C}_m\Box\vec{C}_n$ admits a kernel if and only if m and n are not relatively prime.* ■

Combining Lemmas 3.3 and 4.2 and Corollary 3.5, we have:

Theorem 5.2 *Let $n \geq 3$ and T be a tournament with $\delta^+(T) \geq 1$. Then $T \Box \vec{C}_n$ admits a kernel if and only if $\delta^+(T) = 1$, T has exactly three vertices with out-degree 1, and $n \equiv 0 \pmod 3$.* ■

Combining Corollary 3.2 and Lemmas 4.3 and 3.2, we have:

Theorem 5.3 *Let $n \geq 3$ and T be a tournament with $\delta^+(T) = 0$. Then $T \Box \vec{C}_n$ admits a kernel if and only if n is even and T has a vertex of out-degree one.* ■

Combining Lemmas 4.4 and 3.4, we have:

Theorem 5.4 *Let $n \geq 2$ and T be a tournament. Then $T \Box \vec{P}_n$ admits a kernel if and only if T has a vertex of out-degree zero and a vertex of out-degree one.* ■

6 Kernel-partitionable digraphs

A digraph D is said to be *kernel-partitionable* if there is a partition $\{J_1, J_2, \ldots, J_\ell\}$ of $V(D)$ such that for each $i \in \{1, 2, \ldots, \ell\}$, J_i is a kernel of the subdigraph induced by $V(D) \setminus (J_1 \cup J_2 \cup \cdots \cup J_{i-1})$.

A digraph for which every induced subdigraph has a kernel is said to be *kernel-perfect*. Clearly, every kernel-perfect digraph is kernel-partitionable but the converse is not true. For example, consider a directed odd cycle and at every vertex of the cycle attach a directed even cycle. This yields a family of digraphs which are kernel-partitionable but not kernel-perfect. Let $\vec{C}_{2n+1} : v_1 \to v_2 \to \cdots \to v_{2n+1} \to v_1$ and let H_i be the directed even cycle attached at v_i. To see that this digraph is kernel-partitionable, set $J_1 =$ [the vertices of the kernel of H_1 containing v_1] $\cup\{\bigcup_{i=2}^{2n+1}$ [the vertices of the kernel of H_i not containing v_i]$\}$.

Theorem 6.1 *If D_1 and D_2 are kernel-partitionable digraphs, then $D_1 \Box D_2$ admits a kernel.*

Proof. Let $\{J_1, J_2, \ldots, J_r\}$ and $\{L_1, L_2, \ldots, L_s\}$ be, respectively, the partitions of $V(D_1)$ and $V(D_2)$ obtained from the definition of kernel-partitionable. Consider the set $J = \bigcup_{i=1}^{min\{r,s\}} (J_i \Box L_i)$.

Claim 1. J is independent.

If two adjacent vertices (a, b) and (c, d) are in J, then either $a = c \in J_i$ for some i and the two adjacent vertices b and d of D_2 are in L_i, or $b = d \in L_i$ for some i and the two adjacent vertices a and c of D_1 are in J_i. In any case, we have a contradiction to the independent property of L_i or J_i.

Claim 2. J is absorbent.

Let $(x_i, y_j) \in V(D_1 \Box D_2) \setminus J$. Then $x_i \in J_\ell$ and $y_j \in L_m$ for some $\ell \neq m$. Clearly, (x_i, y_j) dominates a vertex of $J_p \Box L_p$, where $p = min\{\ell, m\}$. Therefore J is absorbent.

This completes the proof. ■

The converse of Theorem 6.1 is not true. For if m and n are odd integers and if $gcd(m, n) \neq 1$, then $\vec{C}_m \Box \vec{C}_n$ admits a kernel (see Theorem 5.1), but neither $\vec{C}_m$ nor $\vec{C}_n$ is kernel-partitionable.

Corollary 6.1 *If D_1 and D_2 are kernel-perfect digraphs, then $D_1 \Box D_2$ admits a kernel.* ■

Theorem 6.2 *If $|S| = 2$ and if $C_n(S)$ admits a kernel, then $C_n(S)$ is kernel-partitionable.*

Proof. Let $S = \{i, j\}$ and let J be a kernel of $C_n(S)$. Also, let $D = C_n(S) \setminus J$. As J is a kernel of $C_n(\{i, j\})$, for every $x \in V(D)$, $d^+_D(x) \leq 1$.

Suppose there exists a directed k-cycle $\vec{C}_k : u_0 \to u_1 \to u_2 \to \cdots \to u_{k-1} \to u_0$, in D, where $u_k = u_0$ and $u_{-1} = u_{k-1}$.

Claim. If $(u_{\ell-1}+i) \equiv u_\ell \pmod n$, then $(u_\ell+j) \equiv u_{\ell+1} \pmod n$, $\ell \in \{0, 1, 2, \ldots, k-1\}$.

Otherwise $(u_\ell + i) \equiv u_{\ell+1} \pmod n$. From the definition of kernel, the adjacent vertices $(u_{\ell-1} + j) \pmod n$ and $(u_\ell + j) \pmod n$ are in J, a contradiction.

Similarly, if $(u_{\ell-1} + j) \equiv u_\ell \pmod n$, then $(u_\ell + i) \equiv u_{\ell+1} \pmod n$, $\ell \in \{0, 1, 2, \ldots, k-1\}$.

Hence the directed cycle $\vec{C}_k$, in D, is of even length. Consequently, D contains no directed odd cycle. Thus D is kernel-perfect [4] and therefore $C_n(S)$ is kernel-partitionable. ■

We have another family of digraphs which are kernel-partitionable but not kernel-perfect. For, let n be odd and take i, j such that $C_n(\{i, j\})$ admits a kernel. By the above theorem, $C_n(\{i, j\})$ is kernel-partitionable. Restrict i, j such that there exist integers ℓ and m with $(\ell + m) \equiv 1 \pmod 2$ and $(\ell i + mj) \equiv 0 \pmod n$. But then $C_n(\{i, j\})$ is not kernel-perfect, since it contains a directed odd cycle.

7 Conclusion

The main problem is: characterize digraphs D_1 and D_2 such that the Cartesian product $D_1 \Box D_2$ has a kernel.

In this paper, we have solved the above problem for the Cartesian products $T \Box \vec{C}_n$, $T \Box \vec{P}_n$, $\vec{C}_m \Box \vec{C}_n$ and $H_1 \Box H_2$, where H_1 and H_2 are kernel-partitionable digraphs.

We propose the following:

Problem. Characterize kernel-partitionable digraphs.

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