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Enumeration of tilings of a hexagon with a maximal staircase and a unit triangle removed

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Abstract

In 1988, Proctor presented a result on plane partitions which implied a formula for the number of lozenge tilings of a hexagon with side-lengths a, b, c, a, b, c after removing a "maximal staircase." More recently, Ciucu proved a weighted version of Proctor's result. Here we present weighted and unweighted formulas for a similar region which has an additional unit triangle removed. We use Kuo's graphical condensation method to prove the results. By applying the factorization theorem of Ciucu, we obtain a formula for the number of lozenge tilings of a hexagon with three holes on consecutive edges.

1 Introduction

A *region* in the triangular lattice is any finite union of unit triangles and a *lozenge* is any union of two unit triangles which share an edge. A *lozenge tiling* of a region R is any covering of all unit triangles in R by non-overlapping lozenges. It is clear that a region must have the same number of upward-pointing unit triangles as downward-pointing ones to have any tilings at all, since a lozenge contains one unit triangle of each type. We say that such a region is *balanced*. We can assign to any lozenge that could be used in a tiling a weight, w, which is a positive real number. An *unweighted* region has all weights equal to 1.

The weight of a lozenge tiling of R is the product of all the weights of the lozenges used in the tiling. We denote by $M(R)$ the *tiling generating function* of the region R, which is the sum of the weights of all tilings of R. For an unweighted region, the tiling generating function simply gives the number of tilings of the region.

MacMahon's work in [10] proved that for a hexagonal region with side-lengths a, b, c, a, b, c (in cyclic order), the number of lozenge tilings is given by the formula

$$\frac{H(a)H(b)H(c)H(a+b+c)}{H(a+b)H(a+c)H(b+c)}, \tag{1.1}$$

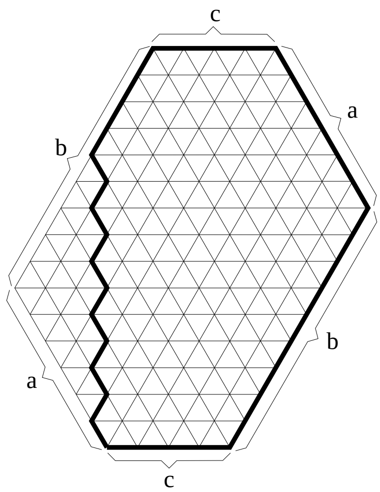

Figure 1.1: The region $P_{a,b,c}$ (inside the bold contour) for $a = 6, b = 9,$ and $c = 4$.

where we define the hyperfactorials $H(n)$ for positive integers n by

$$H(n) := 0!\,1!\ldots(n-1)!\,.$$

The simplicity of (1.1) has inspired many to look for generalizations or similar results. A result of Proctor on plane partitions implies a formula for the number of lozenge tilings of a hexagon with a "maximal staircase" removed, denoted $P_{a,b,c}$ (see Figure 1.1).

Theorem 1.1 (Proctor [11]). *For any non-negative integers $a, b,$ and c with $a \leq b$, we have*

$$M(P_{a,b,c}) = \prod_{i=1}^{a}\left[\prod_{j=1}^{b-a+1}\frac{c+i+j-1}{i+j-1}\prod_{j=b-a+2}^{b-a+i}\frac{2c+i+j-1}{i+j-1}\right],$$

where empty products are taken to be 1. Further, $M(P_{b+1,b,c}) = M(P_{b,b,c})$.

The following result of Ciucu provides a formula for the tiling generating function of the same region, but with each of the vertical lozenges on the west side given weight $\frac{1}{2}$. We denote this region by $P'_{a,b,c}$. In Figure 1.2 (and throughout this paper), lozenges with ovals have weight $\frac{1}{2}$ while those without are unweighted.

Theorem 1.2 (Ciucu [2]). *For any non-negative integers $a, b,$ and c with $a \leq b$ we have*

$$M(P'_{a,b,c}) = \frac{M(P_{a,b,c})}{2^a}\cdot\prod_{i=1}^{a}\frac{2c+b-a+i}{c+b-a+i}.$$

As in Theorem 1.1, $M(P'_{b+1,b,c}) = M(P'_{b,b,c})$.

The main results of this paper, Theorems 2.1 and 2.2, are similar to (but less general than) work of Ciucu and Krattenthaler in [4].

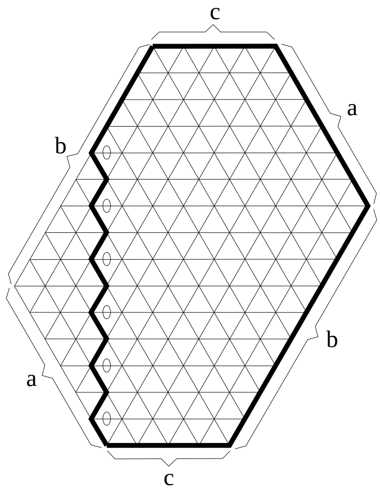

Figure 1.2: The region $P'_{a,b,c}$ has weighted lozenges along its west side.

2 Main Result

Consider a hexagon with side-lengths $a+2, b, c+1, a+1, b+1, c$, with both a maximal staircase and a single upward-pointing unit triangle removed, as in Figure 2.1. The removed unit triangle is the second from the bottom on the northeast side of the original hexagon. We call such a region $S_{a,b,c}$. We denote the corresponding weighted version, with all vertical lozenges on the west side with weight $\frac{1}{2}$, by $S'_{a,b,c}$. We give the formulas for their tiling generating functions below. For ease of notation, define the Pochhammer symbol $(\alpha)_k$ for $k \in \mathbb{Z}$:

$$(\alpha)_k := \begin{cases} \alpha(\alpha+1)\ldots(\alpha+k-1) & \text{if } k > 0, \\ 1 & \text{if } k = 0, \\ 1/(\alpha-1)(\alpha-2)\ldots(\alpha+k) & \text{if } k < 0. \end{cases}$$

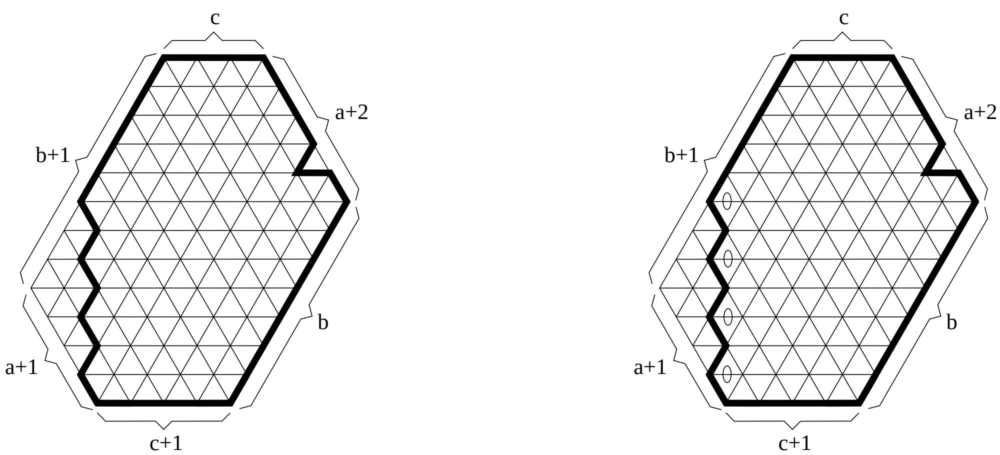

Figure 2.1: The regions $S_{a,b,c}$ and $S'_{a,b,c}$ with $a = 3, b = 7$, and $c = 3$.

Theorem 2.1. *For any non-negative integers a, b, and c with $a \leq b$, we have*

$$M(S_{a,b,c}) = \frac{(c+a+2)_{b-a}(2c+b+3)_{a-1}[(2b-a+2)c+(b+1)(b+2)]}{(a+2)_b} \times \prod_{i=1}^{a} \frac{(c+i)_{b-a+1}(2c+b-a+1+i)_{i-1}}{(i)_{b-a+i}}.$$

If $a = b+1$, $M(S_{b+1,b,c}) = M(S_{b,b,c})$.

Theorem 2.2. *For any non-negative integers a, b, and c with $a \leq b$, we have*

$$M(S'_{a,b,c}) = \frac{M(S_{a,b,c})}{2^a} \cdot \frac{2c+b+2}{c+b+1} \cdot \prod_{i=1}^{a} \frac{2c+b+1-i}{c+b+1-i}.$$

If $a = b+1$, $M(S'_{b+1,b,c}) = M(S'_{b,b,c})$.

3 Preliminaries

The *dual graph* of a region R is the graph comprising one vertex for each unit triangle in R. Two vertices share an edge in the dual graph if and only if their corresponding unit triangles are edge-adjacent. For regions on the triangular lattice, we have seen that each unit triangle is either pointing upwards or downwards - in particular, there are two types of unit triangles. When creating the dual graph of R, we can recover this information by coloring the vertices corresponding to upward-pointing unit triangles one color, and those corresponding to downward-pointing unit triangles another. The resulting graph is now bipartite, and lozenge tilings of a region R are clearly in one-to-one correspondence with perfect matchings of the bipartite dual graph. If a region has weighted lozenges, these correspond to weighted edges in the dual graph, and the tiling generating functions of the region coincides with the *matching generating function* of the graph. We will also use $M(G)$ to denote the matching generating function of the graph G.

Translating our regions to their dual graphs allows us to make use of the graphical condensation method of Kuo from [8], which provides an effective way to count perfect matchings (or matching generating functions) of bipartite graphs. Many authors have used Kuo's methods in proving results about enumeration of tilings (see, for example, [3, 5, 9]). There are several versions; the one we will use is stated below.

Theorem 3.1 (Kuo Condensation). *Let $G = (V_1, V_2, E)$ be a plane bipartite graph with $|V_1| = |V_2| + 1$, and suppose that vertices t, u, v, and w appear cyclically on a face of G. If $t, u, v \in V_1$ and $w \in V_2$, then*

$$M(G-u)M(G-\{t,v,w\}) = M(G-t)M(G-\{u,v,w\}) + M(G-v)M(G-\{t,u,w\}).$$

In [6], Ciucu and Lai give conditions under which the matching generating function of a bipartite graph is the product of the matching generating function of two induced subgraphs. We will need this result for some special cases in the proofs of Theorems 2.1 and 2.2.

Lemma 3.2 (Graph Splitting Lemma)**.** *Let $G = (V_1, V_2, E)$ be a bipartite graph. Assume H is an induced subgraph of G that satisfies the following condition:*

(i) (Separating condition) There are no edges of G connecting a vertex in $V(H) \cap V_1$ and a vertex in $V(G - H)$.

(ii) (Balancing condition) $|V(H) \cap V_1| = |V(H) \cap V_2|$.

Then

$$M(G) = M(H)M(G - H).$$

4 Proofs of Theorems 2.1 and 2.2

We will prove Theorem 2.1 via induction on a using Theorem 3.1. We will apply Theorem 3.1 to the dual graph of the region $S_{a,b,c}$ without the unit triangle removed from the northeast side, as in Figure 4.1. This region is unbalanced, as required by the theorem, and the locations of the vertices t, u, v, and w are given.

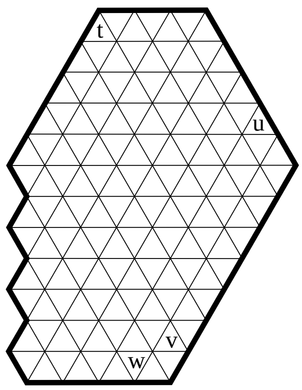

Figure 4.1: The region to which we apply Kuo condensation.

Applying Kuo condensation to such a region gives us a recurrence involving six new regions. They are shown in Figure 4.2. In each subfigure, the triangles corresponding to removed vertices are labelled and any subsequently forced lozenges are shown. As these forced lozenges must be used in any tiling of the given region, the number of tilings of the region is unchanged by their removal.

We examine how removal of each of the labelled unit triangles t, u, v, and w affects the region in Figure 4.1. If t is removed, only the north-most row is forced. This causes the length of north side to increase by 1 unit to $c + 1$ while the northeast side decreases by 1 unit. If u is removed, we get a region of S-type because there is a defect of size 1 in the correct position on the northeast side. Removal of v forces

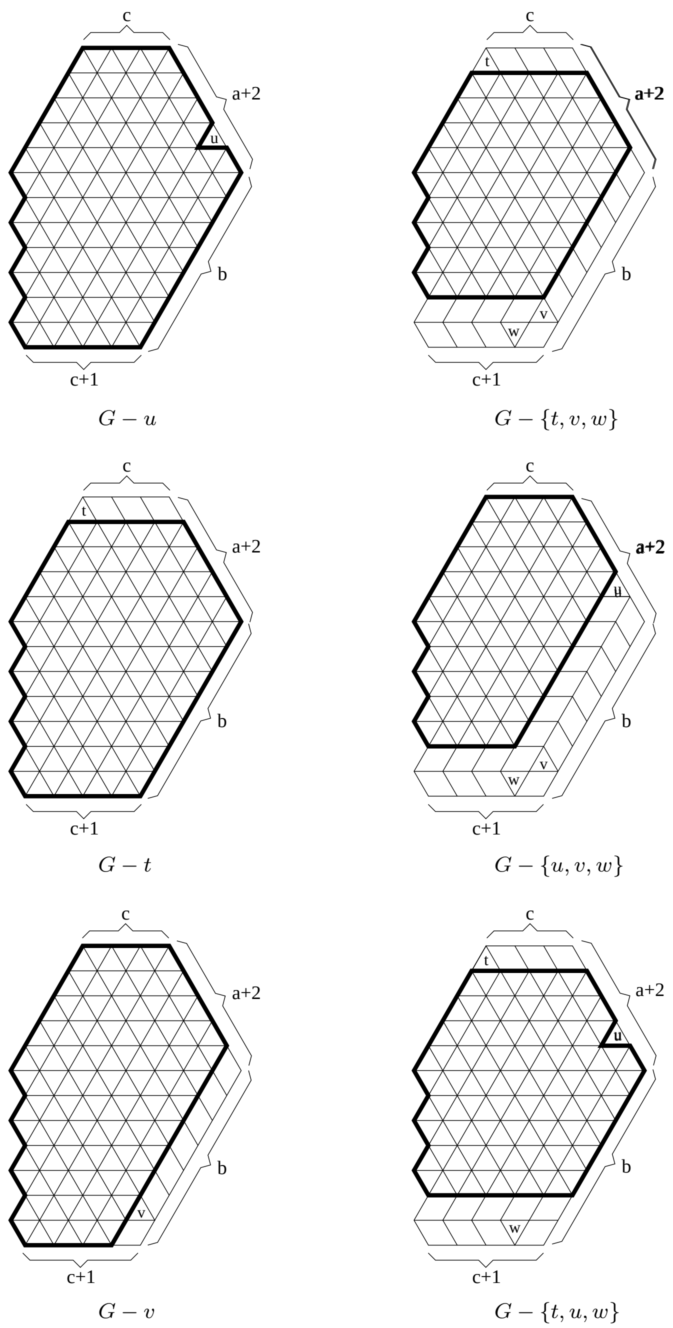

Figure 4.2: The six regions obtained after applying Kuo condensation.

one row of lozenges on the southeast side thereby increasing its length by 1 unit and decreasing the length of the south side by 1 unit. Finally, removal of w forces the south-most two rows, which increases the length of the south side by 1 unit and decreases the length of the southeast side by 2 units. In $G - \{u, v, w\}$, removal of u and v forces two rows on the southeast side so we get a region of P-type. It is now apparent that

$$\begin{aligned} M(G-u) &= M(S_{a,b,c}), \\ M(G-\{t,v,w\}) &= M(P_{a,b-1,c+1}), \\ M(G-t) &= M(P_{a+1,b,c+1}), \\ M(G-\{u,v,w\}) &= M(P_{a,b,c}), \\ M(G-v) &= M(P_{a+1,b+1,c}), \text{ and} \\ M(G-\{t,u,w\}) &= M(S_{a-1,b-2,c+1}). \end{aligned}$$

Therefore, we know

$$\begin{aligned} M(S_{a,b,c})M(P_{a,b-1,c+1}) = M(P_{a+1,b,c+1})&M(P_{a,b,c}) \\ &+ M(P_{a+1,b+1,c})M(S_{a-1,b-2,c+1}), \quad (4.1) \end{aligned}$$

as long as $a \geq 1$ and $b \geq 2$. Theorem 1.1 explicitly gives the tiling generating functions for the P-type regions, so Equation (4.1) is merely a recurrence between the tiling generating functions of S-type regions. We will prove Theorem 2.1 by induction on a. Therefore, it suffices to show that the claimed formula in Theorem 2.1 satisfies Equation (4.1) and holds for $a = 0$. For completeness, we will also show that this formula holds when $b = 0$ or $b = 1$.

If $a = 0$, the formula from Theorem 2.1 implies

$$\begin{aligned} M(S_{0,b,c}) = \frac{(c+2)_b(2c+b+3)_{-1}[(2b+2)c+(b+1)(b+2)]}{(2)_b} \\ = \binom{b+c+1}{b}. \end{aligned}$$

On the other hand, if $a = 0$, then the north-most row is forced and the resulting region is a hexagon with side-lengths $c+1, 1, b, c+1, 1, b$ as in Figure 4.3. MacMahon's formula (1.1) verifies that the number of tilings of this hexagon is indeed $\binom{b+c+1}{b}$.

We now need to check that Equation (4.1) holds. First, we rewrite Theorem 1.1 as

$$M(P_{a,b,c}) = \prod_{i=1}^{a} \frac{(c+i)_{b-a+1}(2c+b-a+1+i)_{i-1}}{(i)_{b-a+i}}.$$

Using this formula for the P-type regions and the formula from Theorem 2.1 for the S-type regions, Equation (4.1) becomes

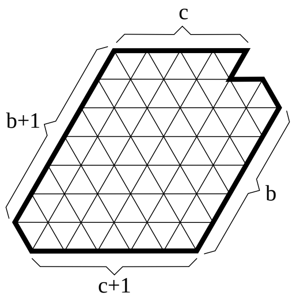

Figure 4.3: When $a = 0$, it is clear that the north-most row is forced.

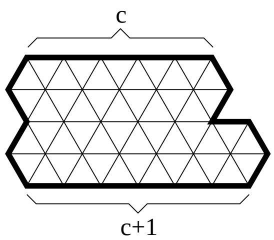

Figure 4.4: The $a = b = 1$ case.

$$\frac{(c+a+2)_{b-a}(2c+b+3)_{a-1}[(2b-a+2)c+(b+1)(b+2)]}{(a+2)_b} \times \prod_{i=1}^{a} \frac{(c+i)_{b-a+1}(2c+b-a+1+i)_{i-1}}{(i)_{b-a+i}} \times \prod_{i=1}^{a} \frac{(c+1+i)_{b-a}(2c+b-a+2+i)_{i-1}}{(i)_{b-a+i-1}}$$
$$= \prod_{i=1}^{a+1} \frac{(c+1+i)_{b-a}(2c+b-a+2+i)_{i-1}}{(i)_{b-a-1+i}} \times \prod_{i=1}^{a} \frac{(c+i)_{b-a+1}(2c+b-a+1+i)_{i-1}}{(i)_{b-a+i}}$$
$$+ \frac{(c+a+2)_{b-a-1}(2c+b+3)_{a-2}[(2b-a-1)(c+1)+b(b-1)]}{(a+1)_{b-2}} \times \prod_{i=1}^{a-1} \frac{(c+1+i)_{b-a}(2c+b-a+2+i)_{i-1}}{(i)_{b-a+i-1}} \times \prod_{i=1}^{a+1} \frac{(c+i)_{b-a+1}(2c+b-a+1+i)_{i-1}}{(i)_{b-a+i}}. \quad (4.2)$$

Through straightforward algebraic manipulation, one can verify that Equation (4.2) is true.

To complete the proof of Theorem 2.1, we need to show the result for $b = 0, 1$. As $0 \le a \le b$, the only remaining case is when $a = b = 1$. In this case, Theorem 2.1 says that $M(S_{1,1,c}) = (c + 1)(c + 2)$.

We get a region as in Figure 4.4 when $a = b = 1$. At this point we apply Lemma 3.2, and we take H to be the dual subgraph to the top two rows of $S_{1,1,c}$ (which makes $G - H$ the dual graph to the bottom two rows). It is clear that $M(S_{1,1,c})$ is the product of the tiling generating functions of two hexagons - one with side-lengths $c, 1, 1, c, 1, 1$ and the other $c + 1, 1, 1, c + 1, 1, 1$. Using (1.1), we see that the tiling generating functions are $c + 1$ and $c + 2$ respectively. □

The proof of Theorem 2.2 is similar to that of Theorem 2.1. Theorem 3.1 is applied in exactly the same way, yielding a recurrence identical to that of Equation (4.1) but with S and P replaced by S' and P', respectively. Verifying that the formula in Theorem 2.2 satisfies the new recurrence is done similarly, as are the few special cases.

5 Symmetric triply-dented hexagons

By symmetrizing our region along the "maximal staircase" we obtain a symmetric triply-dented hexagon, $STDH_{a,b,c}$ as in Figure 5.1. Notice that the removed triangle on the north edge is centrally located and that the removed unit triangles on the northwest and northeast edges are at distance one away from the west and east corners of the region, respectively.

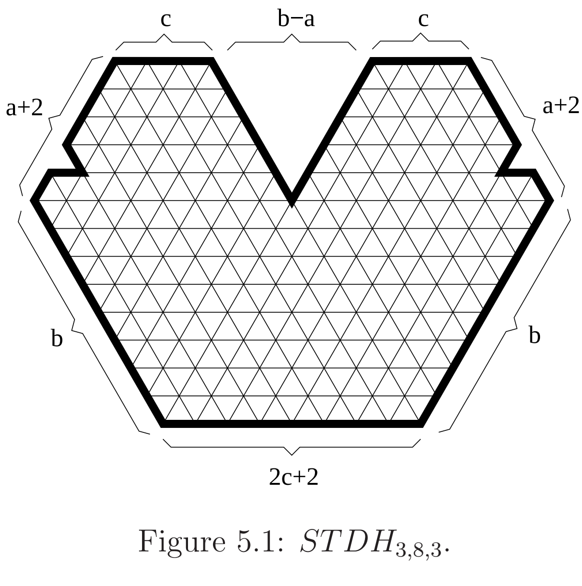

Figure 5.1: $STDH_{3,8,3}$.

We now present Ciucu's factorization theorem from [1] which allows us to easily compute the number of tilings of $STDH_{a,b,c}$ using Theorems 2.1 and 2.2.

A planar graph G is a *weighted, symmetric* graph if their exists an embedding of G in the plane such that its vertex set, edge set, and edge weight function are invariant under reflection across some straight line. We call this line the *axis of symmetry*. Define the *width* of a symmetric graph G, written $w(G)$, to be half the

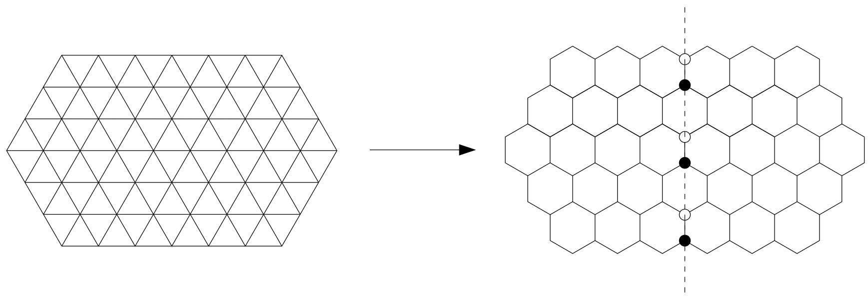

Figure 5.2: We obtain G by taking the dual graph of the region on the left. All the white vertices on ℓ are a_i and all the black vertices are b_i.

number of vertices on its axis of symmetry. The number of vertices on the axis of symmetry must be even otherwise the total number of vertices in G would be odd and G would have no perfect matchings.

Let G be a weighted, symmetric, bipartite graph with axis of symmetry ℓ. Suppose that the set of vertices on ℓ *separates* G – that is, if these vertices (and incident edges) are removed from G, then G would be disconnected. Let us label the vertices on the symmetric axis $a_1, b_1, a_2, b_2, \ldots, a_{w(G)}, b_{w(G)}$ in this order from top to bottom. We color the vertices in the two vertex classes white and black, and without loss of generality, let the topmost vertex on ℓ be white. See Figure 5.2 for an example.

We will define two subgraphs of G, called $G+$ and $G-$ as follows:

1. remove edges adjacent to and to the left side of all white a_i and black b_i,
2. remove edges adjacent to and to the right side of all black a_i and white b_i,
3. multiply the weights of all edges lying on ℓ by $\frac{1}{2}$.

Given an edge e lying on ℓ, the first two steps remove edges adjacent to and on the same side of e. Because the vertices on ℓ separate G, we now have two subgraphs of G (with new weights on any edges along ℓ). We call the one to the right of ℓ, $G-$, and the one to the left, $G+$. We see the results of cutting and weighting the previous graph in Figure 5.3.

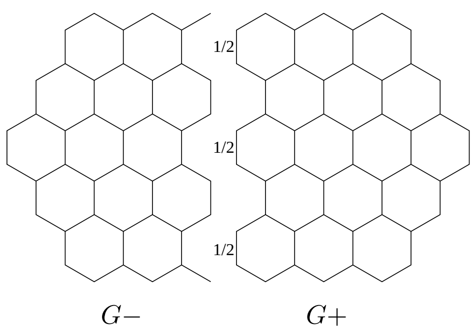

Figure 5.3: The three edges on ℓ have weight $\frac{1}{2}$.

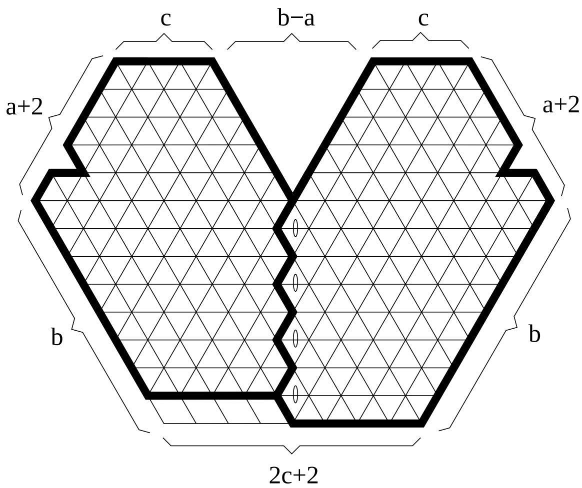

Figure 5.4: $STDH_{3,8,3}$.

Theorem 5.1 (Ciucu's Factorization Theorem). *Let G be a weighted, symmetric, bipartite graph separated by its axis of symmetry. Then*

$$M(G) = 2^{w(G)} M(G+) M(G-).$$

We have the following result in the vein of Eisenkölbl [7] and Lai [9].

Corollary 5.2. *For non-negative integers $a, b,$ and c with $a \leq b$ we have*

$$M(STDH_{a,b,c}) = 2^{a+1} M(S'_{a,b,c}) M(S_{a,b-1,c}).$$

Proof. When applying the factorization theorem we must cut the region into the two subregions shown in Figure 5.4. After forcing, we have one region of type S and another of type S'. The result follows immediately. □

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