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Dominance in a Cayley digraph and in its reverse

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Abstract

Let D be a digraph. Its reverse digraph, D^{-1}, is obtained by reversing all arcs of D. We show that the domination numbers of D and D^{-1} can be different if D is a Cayley digraph. The smallest groups admitting Cayley digraphs with this property are the alternating group A_4 and the dihedral group D_6, both on 12 elements. Then, for each $n \geq 6$ we find a Cayley digraph D on the dihedral group D_n such that the domination numbers of D and D^{-1} are different, though D has an efficient dominating set. Analogous results are also obtained for the total domination number.

1 Introduction

Let D be a digraph. The vertex and arc sets of D are denoted by $V(D)$ and $E(D)$, respectively. If there exists a positive integer d such that there are exactly d arcs starting at every vertex and exactly d arcs terminating at every vertex then D is a *regular digraph* of degree d. A digraph which is obtained by reversing all arcs of D is called the *reverse digraph* (or *converse digraph*) of D and is denoted by D^{-1}.

Let $v \in V(D)$. The open and closed neighbourhoods of v in D are denoted by $N_D(v)$ and $N_D[v]$, respectively. That is, $N_D(v) = \{u;\, vu \in E(D)\}$ and $N_D[v] = N_D(v) \cup \{v\}$. For $S \subseteq V(D)$, we set $N_D(S) = \cup_{v\in S} N_D(v)$ and $N_D[S] = \cup_{v\in S} N_D[v]$. Then S is a *dominating set* (*total dominating set*) if $N_D[S] = V(D)$ ($N_D(S) = V(D)$). The smallest size of a dominating set (total dominating set) is the *domination* *number* $\gamma(D)$ (*total domination number* $\gamma_t(D)$) of D. Let S be a dominating set (total dominating set) in D. Then S is an *efficient dominating set* (*efficient total dominating set*) if for every $u, v \in S$, $u \neq v$, we have $N_D[u] \cap N_D[v] = \emptyset$ ($N_D(u) \cap N_D(v) = \emptyset$).

Domination is an intensively studied area in graph theory. Problems of resource allocations and scheduling in networks are frequently formulated as domination problems of underlying (di)graphs; for terminology and survey of results see [5]. Compared with graphs, there is a smaller number of results for domination in digraphs. The domination number in digraphs was introduced in [2] and a survey on domination in digraphs is given in [3].

Let G be a group and let $X \subseteq G$ such that the identity element is not in X. The *Cayley digraph* $\mathrm{Cay}(G, X)$ has vertex set G and there is an arc from v to u in $\mathrm{Cay}(G, X)$ if and only if $va = u$ for some $a \in X$. Observe that $\mathrm{Cay}(G, X)$ is a regular digraph of degree $|X|$. Furthermore, $\mathrm{Cay}(G, X)$ is vertex-transitive, which means that for every pair of its vertices v and u there is an automorphism g of $\mathrm{Cay}(G, X)$ such that $g(v) = u$. Observe that the reverse digraph to $\mathrm{Cay}(G, X)$ is simply $\mathrm{Cay}(G, X^{-1})$.

In [7, 4] it is shown that for every $d \geq 2$ ($d \geq 3$) there is a d-regular digraph D such that the domination numbers (total domination numbers) of D and D^{-1} are different. Can these numbers differ even if D is a Cayley digraph? In [6, text below Theorem 8] the authors state that this is not the case but their conclusion is implied by a wrong assumption that $\mathrm{Cay}(G, X)$ and $\mathrm{Cay}(G, X^{-1})$ are isomorphic digraphs. This wrong assumption was probably caused by the fact that the groups used in [6] are abelian, and in such a case $\mathrm{Cay}(G, X)$ and $\mathrm{Cay}(G, X^{-1})$ are isomorphic. However, even for metacyclic groups G (at least for some of them) we can find $X \subseteq G$ such that $\mathrm{Cay}(G, X)$ and $\mathrm{Cay}(G, X^{-1})$ are not isomorphic digraphs, see [1]. Recall that a group is metacyclic if it is a semidirect product of cyclic groups.

In this paper we show that $\gamma(\mathrm{Cay}(G, X))$ and $\gamma(\mathrm{Cay}(G, X^{-1}))$ can be different numbers. The smallest groups G admitting $X \subseteq G$ such that $\gamma(\mathrm{Cay}(G, X)) \neq \gamma(\mathrm{Cay}(G, X^{-1}))$ are the alternating group A_4 and the dihedral group D_6, both on 12 elements. Then we show that for every $n \geq 6$ there exists $X_n \subseteq \mathrm{D}_n$ such that $\gamma(\mathrm{Cay}(\mathrm{D}_n, X_n)) \neq \gamma(\mathrm{Cay}(\mathrm{D}_n, X_n^{-1}))$. In this case $|X_n| = n - 1$ and $\mathrm{Cay}(\mathrm{D}_n, X_n)$ has an efficient dominating set. For the total domination number we present analogous results.

As regards further research, it seems that if G is a sufficiently large nonabelian group, then there are $X, Y \subseteq G$ such that $\gamma(\mathrm{Cay}(G, X)) \neq \gamma(\mathrm{Cay}(G, X^{-1}))$ and $\gamma_t(\mathrm{Cay}(G, Y)) \neq \gamma_t(\mathrm{Cay}(G, Y^{-1}))$. However, as this may be a hard problem, we pose the following ones:

Problem 1.1 *Let G be a metacyclic group, $G = \mathbb{Z}_a \rtimes \mathbb{Z}_b$. Is there $X \subseteq G \setminus \{(0, 0)\}$, with $|X| = a - 1$, $\gamma(\mathrm{Cay}(G, X)) = b$ and $\gamma(\mathrm{Cay}(G, X^{-1})) > b$?*

Analogously for the total domination number:

Problem 1.2 *Let G be a metacyclic group, $G = \mathbb{Z}_a \rtimes \mathbb{Z}_b$. Is there $Y \subseteq G \setminus \{(0,0)\}$, with $|Y| = a$, $\gamma(\mathrm{Cay}(G,Y)) = b$ and $\gamma(\mathrm{Cay}(G,Y^{-1})) > b$?*

Of course, we know that for very small groups the answers for the above problems are negative. But if, for fixed b, the value of a is sufficiently large, are the answers to the above problems positive?

The next problem to consider is whether there are digraphs D whose symmetry is higher than that of Cayley digraphs, yet which nevertheless satisfy $\gamma(D) \neq \gamma(D^{-1})$ (or $\gamma_t(D) \neq \gamma_t(D^{-1})$). Here, one can start with searching through the database of small 2-regular arc-transitive digraphs; see [8].

2 Small digraphs

There are exactly seven non-abelian groups of order at most 12, namely the dihedral groups D_3, D_4, D_5 and D_6, then the quaternion group Q, dicyclic group Dic_3 and the alternating group A_4. Denote by Γ the set of these seven groups. By a computer we checked that, if $G \in \Gamma$, $X \subseteq G$ and $\gamma(\mathrm{Cay}(G,X)) \neq \gamma(\mathrm{Cay}(G,X^{-1}))$, then either $G = \mathrm{D}_6$ or $G = \mathrm{A}_4$. If $G = \mathrm{D}_6$ then $|X| = 5$ and if $G = \mathrm{A}_4$ then either $|X| = 3$ or $|X| = 5$. In all these cases, one of $\mathrm{Cay}(G,X)$ and $\mathrm{Cay}(G,X^{-1})$ has an efficient dominating set while the other digraph does not have such a set. Similarly, if $G \in \Gamma$, $Y \subseteq G$ and $\gamma_t(\mathrm{Cay}(G,Y)) \neq \gamma_t(\mathrm{Cay}(G,Y^{-1}))$, then either $G = \mathrm{D}_6$ or $G = \mathrm{A}_4$. If $G = \mathrm{D}_6$ then $|Y| = 6$ and if $G = \mathrm{A}_4$ then either $|Y| = 4$ or $|Y| = 6$. In all these cases, one of $\mathrm{Cay}(G,Y)$ and $\mathrm{Cay}(G,Y^{-1})$ has an efficient total dominating set while the other digraph does not have such a set.

In the rest of this section we consider A_4, the group of even permutations of 4-element set, say $\{1,2,3,4\}$. The group operation is the composition of permutations. Recall that A_4 is one of the two smallest groups admitting a Cayley digraph whose (total) domination number differs from the (total) domination number of its reverse. (The other smallest case, D_6, is considered and generalized in the next section.) For $G = \mathrm{A}_4$ we define $X, Y \subseteq G$ such that $\gamma(\mathrm{Cay}(G,X)) \neq \gamma(\mathrm{Cay}(G,X^{-1}))$ and $\gamma_t(\mathrm{Cay}(G,Y)) \neq \gamma_t(\mathrm{Cay}(G,Y^{-1}))$. Though we did check the above inequalities by a computer, we present rigorous proofs. In fact, we prove that $\gamma(\mathrm{Cay}(G,X)) \neq \gamma(\mathrm{Cay}(G,X^{-1}))$ for a set X of three elements, and then we transform X to Y such that $|Y| = 4$ and $\gamma_t(\mathrm{Cay}(G,Y)) \neq \gamma_t(\mathrm{Cay}(G,Y^{-1}))$.

Theorem 2.1 *Let $X = \{(12)(34), (123), (243)\}$. Then $\gamma(\mathrm{Cay}(\mathrm{A}_4,X)) = 3$ and $\gamma(\mathrm{Cay}(\mathrm{A}_4,X^{-1})) = 4$.*

Proof. We denote $\mathrm{Cay}(\mathrm{A}_4,X)$ and $\mathrm{Cay}(\mathrm{A}_4,X^{-1})$ by D_X and D_X^{-1}, respectively. The digraphs D_X and D_X^{-1} are depicted in Figure 1, where thick edges represent pairs of opposite arcs formed by the involutory generator $(12)(34)$, regular arcs correspond to (123) in D_X and to its reverse (132) in D_X^{-1}, while dashed arcs correspond to (243) in D_X and to its reverse (234) in D_X^{-1}.

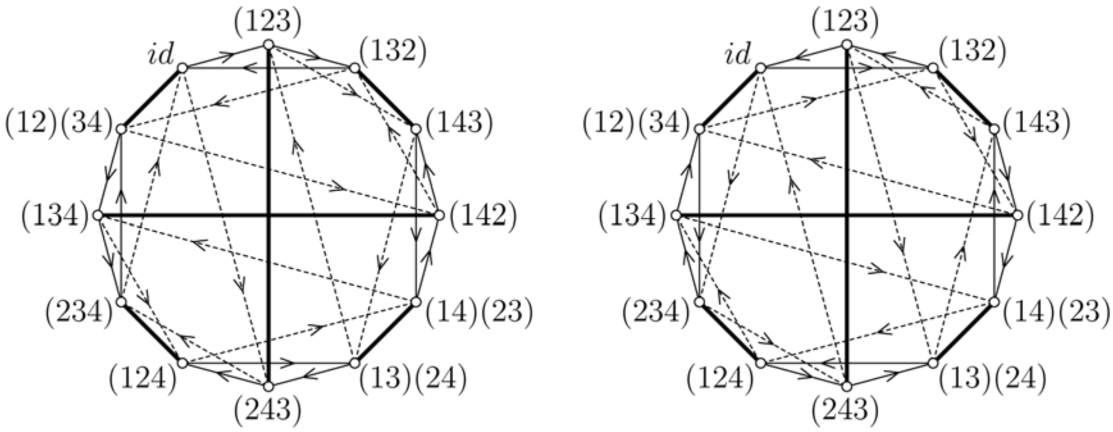

Figure 1: The digraph D_X and its reverse digraph D_X^{-1}.

First we show $\gamma(D_X) = 3$. Let $S = \{id, (143), (134)\}$. Then

$$\begin{aligned} N_{D_X}[id] &= \{id, (12)(34), (123), (243)\}, \\ N_{D_X}[(143)] &= \{(143), (132), (14)(23), (13)(24)\}, \\ N_{D_X}[(134)] &= \{(134), (142), (234), (124)\}. \end{aligned}$$

So $N_{D_X}[S] = \mathrm{A}_4 = V(D_X)$, and hence S is a dominating set. Since D_X has 12 vertices and every vertex of D_X is a starting vertex of exactly three arcs, the set S is a minimum dominating set. Hence $\gamma(D_X) = 3$. Since $N_{D_X}[id]$, $N_{D_X}[(143)]$ and $N_{D_X}[(134)]$ are disjoint sets, S is an efficient dominating set.

Now consider D_X^{-1} and suppose that $\gamma(D_X^{-1}) = \gamma(D_X)$. Then $\gamma(D_X^{-1}) = 3$ and D_X^{-1} has an efficient dominating set, say T. Since D_X^{-1} is vertex-transitive, without loss of generality we may assume that $id \in T$. Since T is an efficient dominating set, neighbours of id are not in T, and so $(12)(34), (132), (234) \notin T$. For the same reason, T does not contain a vertex v, $v \neq id$, such that v and id dominate a common vertex. Since $(12)(34)$ is dominated by both (134) and id, we have $(134) \notin T$. Analogously $(142), (143), (124) \notin T$. Finally, T does not contain a vertex which dominates id, and so $(123), (243) \notin T$. We excluded all the vertices of D_X^{-1} except $(13)(24)$ and $(14)(23)$. Thus, $T = \{id, (13)(24), (14)(23)\}$. Since $(13)(24)$ and $(14)(23)$ are connected by an arc in D_X^{-1}, T cannot be an efficient dominating set. Thus, $\gamma(D_X^{-1}) > 3$. On the other hand, since $\{id, (143), (13)(24), (234)\}$ is a dominating set in D_X^{-1} (see Figure 1), we have $\gamma(D_X^{-1}) = 4$. □

For the total domination number we have $\gamma_t(\mathrm{Cay}(\mathrm{A}_4, X)) = \gamma_t(\mathrm{Cay}(\mathrm{A}_4, X^{-1})) = 5$. However, modifying X slightly one can obtain a digraph with the total domination number different from the total domination number of its reverse.

The key ingredient in the following proof is the existence of $g, h \in \mathrm{A}_4$ such that $(X \cup \{id\})g$ does not contain id and $g^{-1}(X^{-1} \cup \{id\}) = (X^{-1} \cup \{id\})h$.

Theorem 2.2 *Let* $Y = \{(14)(23), (142), (134), (13)(24)\}$*. Then* $\gamma_t(\mathrm{Cay}(\mathrm{A}_4, Y)) = 3$ *and* $\gamma_t(\mathrm{Cay}(\mathrm{A}_4, Y^{-1})) = 4$*.*

Proof. Let D_Y denote $\mathrm{Cay}(\mathrm{A}_4, Y)$. Observe that $Y = (X \cup \{id\})g$, where $X = \{(12)(34), (123), (243)\}$ as in Theorem 2.1 and $g = (13)(24)$. For every $a \in \mathrm{A}_4$ we have

$$(N_{D_X}[a])g = a[X \cup \{id\}]g = aY = N_{D_Y}(a),$$

and consequently $(N_{D_X}[S])g = N_{D_Y}(S)$ for every $S \subseteq \mathrm{A}_4$. (We remark that D_X is the digraph defined in the proof of Theorem 2.1.) Since g acts on the elements of A_4 as a permutation, S is a dominating set in D_X if and only if it is a total dominating set in D_Y. Hence, $\gamma_t(D_Y) = 3$ by Theorem 2.1.

Now consider $D_Y^{-1} = \mathrm{Cay}(\mathrm{A}_4, Y^{-1})$. Then

$$Y^{-1} = g^{-1}[X^{-1} \cup \{id\}] = \{(14)(23), (124)(143), (13)(24)\} = (X^{-1} \cup \{id\})h,$$

where $h = (14)(23)$. Thus, for every $a \in \mathrm{A}_4$ we have $(N_{D_X^{-1}}[a])h = N_{D_Y^{-1}}(a)$, and consequently $(N_{D_X^{-1}}[S])h = N_{D_Y^{-1}}(S)$ for every $S \subseteq \mathrm{A}_4$. Since h acts on the elements of A_4 as a permutation, S is a dominating set in D_X^{-1} if and only if it is a total dominating set in D_Y^{-1}. Hence, $\gamma_t(D_Y^{-1}) = 4$ by Theorem 2.1. □

3 Digraphs on dihedral groups

In this section we show that for every dihedral group D_n, where $n \geq 6$, there are $X_n, Y_n \subseteq \mathrm{D}_n$ such that $\gamma(\mathrm{Cay}(\mathrm{D}_n, X_n)) \neq \gamma(\mathrm{Cay}(\mathrm{D}_n, X_n^{-1}))$ and $\gamma_t(\mathrm{Cay}(\mathrm{D}_n, Y_n)) \neq \gamma_t(\mathrm{Cay}(\mathrm{D}_n, Y_n^{-1}))$. As mentioned above, for $n \leq 5$ such sets of generators do not exist.

The dihedral group D_n is a semidirect product of $\mathbb{Z}_n$ with $\mathbb{Z}_2$, $\mathrm{D}_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$, and so $a \in \mathrm{D}_n$ if and only if $a = (a_1, a_2)$ where $a_1 \in \mathbb{Z}_n$ and $a_2 \in \mathbb{Z}_2$. The multiplication in D_n is given by $(x_1, x_2)(y_1, y_2) = (x_1 + (-1)^{x_2} y_1, x_2 + y_2)$, where the first coordinate is modulo n and the second is modulo 2.

Let $a \in \mathbb{Z}_n$. The set $\{(a, 0), (a, 1)\}$ is called *a pair* of D_n. We use the following simple lemma.

Lemma 3.1 *For arbitrary* $x, y_1, y_2 \in \mathrm{D}_n$*, the set* $x\{y_1, y_2\}$ *is a pair if and only if* $\{y_1, y_2\}$ *is a pair.*

Proof. Let $y_1 = (a_1, b_1)$, $y_2 = (a_2, b_2)$ and let $x = (c, d)$. Then

$$x\{y_1, y_2\} = \{(c + (-1)^d a_1, d + b_1), (c + (-1)^d a_2, d + b_2)\}.$$

Hence, if $d = 0$ then $x\{y_1, y_2\} = \{(c + a_1, d + b_1), (c + a_2, d + b_2)\}$, while if $d = 1$ then $x\{y_1, y_2\} = \{(c - a_1, d + b_1), (c - a_2, d + b_2)\}$. In both cases, $x\{y_1, y_2\}$ is a pair if and only if $a_1 = a_2$ and $b_1 \neq b_2$. That is, $x\{y_1, y_2\}$ is a pair if and only if $\{y_1, y_2\}$ is a pair. □

Now we prove a result for the domination number. If n is even, $n = 2k$, then set

$$X_n = \{(0,1),(1,0),(1,1),(2,0),(2,1),\dots,(k-2,0),(k-2,1),(2k-2,0),(2k-2,1)\}.$$

On the other hand if n is odd, $n = 2k+1$, then set

$$X_n = \{(0,1),(1,0),(1,1),(2,0),\dots,(k-2,1),(k-1,0),(2k-1,0),(2k-1,1)\}.$$

Observe that in both cases, the first $n-3$ elements of X_n are consecutive in lexicographic order, the last two elements of X_n are $(n-2,0)$ and $(n-2,1)$ and $|X_n| = n-1$.

Theorem 3.2 *Let $n \geq 6$. Then $\gamma(\mathrm{Cay}(\mathrm{D}_n, X_n)) \neq \gamma(\mathrm{Cay}(\mathrm{D}_n, X_n^{-1}))$. Particularly, $\mathrm{Cay}(\mathrm{D}_n, X_n)$ has an efficient dominating set of size 2 while $\mathrm{Cay}(\mathrm{D}_n, X_n^{-1})$ does not have such a set.*

Proof. Since $|X_n| = n-1$, $\gamma(\mathrm{Cay}(\mathrm{D}_n, X_n)) \geq 2$. We shall show that the set $\{(0,0),(n-3,1)\}$ is a dominating set, which implies $\gamma(\mathrm{Cay}(\mathrm{D}_n, X_n)) = 2$. Observe that every $x \in \mathrm{D}_n$ dominates exactly the n vertices of $x[\{(0,0)\} \cup X_n]$. We distinguish two cases.

Case 1. If $n = 2k$ then

$$\begin{aligned}(2k-3,1)(\{(0,0)\} \cup X_n) &= \{(2k-3,1),(2k-3,0),(2k-4,1),(2k-4,0),\\ &\quad \dots,(k-1,0),(2k-1,1),(2k-1,0)\}.\end{aligned}$$

Since $(0,0)(\{(0,0)\} \cup X_n) = \{(0,0)\} \cup X_n$ and the union $(\{(0,0)\} \cup X_n) \cup (2k-3,1)(\{(0,0)\} \cup X_n) = \mathrm{D}_n$, the set $\{(0,0),(n-3,1)\}$ is a dominating set in $\mathrm{Cay}(\mathrm{D}_n, X_n)$.

Case 2. If $n = 2k+1$ then

$$\begin{aligned}(2k-2,1)(\{(0,0)\} \cup X_n) &= \{(2k-2,1),(2k-2,0),(2k-3,1),(2k-3,0),\\ &\quad \dots,(k-1,1),(2k,1),(2k,0)\}.\end{aligned}$$

Since $(\{(0,0)\} \cup X_n) \cup (2k-2,1)(\{(0,0)\} \cup X_n) = \mathrm{D}_n$, the set $\{(0,0),(n-3,1)\}$ is a dominating set in $\mathrm{Cay}(\mathrm{D}_n, X_n)$.

Now we show that $\mathrm{Cay}(\mathrm{D}_n, X_n^{-1})$ does not have a dominating set of size 2. First, by an exhaustive computer search we found that $\gamma(\mathrm{Cay}(\mathrm{D}_n, X_n^{-1})) = 3$ if $n \in \{6,7\}$. Hence, assume that $n \geq 8$. Then in both cases, $n = 2k$ and $n = 2k+1$, we have

$$\begin{aligned}\{(0,0)\} \cup X_n^{-1} &= \{(0,0),(0,1),(1,1),(2,0),(2,1),(3,1),\dots,(k-2,1),\\ &\quad (k+2,0),(k+3,0),\dots,(n-2,0),(n-2,1),(n-1,0)\}.\end{aligned}$$

Suppose that $\mathrm{Cay}(\mathrm{D}_n, X_n^{-1})$ has a dominating set S of size 2. Since every Cayley digraph is vertex-transitive, we may assume that $(0,0) \in S$. If we denote by (a_1, a_2) the other element of S, then $(\{(0,0)\} \cup X_n^{-1}) \cup (a_1, a_2)(\{(0,0)\} \cup X_n^{-1}) = \mathrm{D}_n$. Next we consider the pairs in $\{(0,0)\} \cup X_n^{-1}$ and in $(a_1, a_2)(\{(0,0)\} \cup X_n^{-1})$.

If $n \geq 8$, there are exactly three pairs in $\{(0,0)\} \cup X_n^{-1}$, namely $\{(0,0),(0,1)\}$, $\{(2,0),(2,1)\}$ and $\{(n-2,0),(n-2,1)\}$. Denote by X_+^{-1} the set of these three pairs. On the other hand, there are exactly three pairs with empty intersection with $\{(0,0)\} \cup X_n^{-1}$, namely $\{(k-1,0),(k-1,1)\}$, $\{(k,0),(k,1)\}$ and $\{(k+1,0),(k+1,1)\}$. Denote by X_-^{-1} the set of these three pairs. Since $\{(0,0),(a_1,a_2)\}$ is a dominating set in $\mathrm{Cay}(\mathrm{D}_n, X_n^{-1})$, we must have $(a_1,a_2)X_+^{-1} = X_-^{-1}$, by Lemma 3.1. Next we shall show that this cannot be true.

Observe that X_-^{-1} contains three consecutive pairs. Since

$$(a_1,0)X_+^{-1} = \{(a_1,0),(a_1,1),(a_1+2,0),(a_1+2,1),(a_1-2,0),(a_1-2,1)\} = (a_1,1)X_+^{-1}$$

for every $a_1 \in \mathbb{Z}_n$ and $a_2 \in \mathbb{Z}_2$ the set $(a_1,a_2)X_+^{-1}$ does not contain three consecutive pairs. Hence $(a_1,a_2)X_+^{-1} \neq X_-^{-1}$, a contradiction. Consequently, $\mathrm{Cay}(\mathrm{D}, X_n^{-1})$ does not have a dominating set of size 2. □

Now we show an analogous result for the total domination number. Denote $Y_n = \mathrm{D}_n \setminus (\{(0,0)\} \cup X_n)$, where X_n is the set defined before Theorem 3.2. Then we have the following result.

Theorem 3.3 *Let $n \geq 6$. Then $\gamma_t(\mathrm{Cay}(\mathrm{D}_n, Y_n)) \neq \gamma_t(\mathrm{Cay}(\mathrm{D}_n, Y_n^{-1}))$. Particularly, $\mathrm{Cay}(\mathrm{D}_n, Y_n)$ has an efficient total dominating set of size 2 while $\mathrm{Cay}(\mathrm{D}_n, Y_n^{-1}))$ does not have such a set.*

PROOF. Since $|Y_n| = n$, we have $\gamma_t(\mathrm{Cay}(\mathrm{D}_n, Y_n)) \geq 2$ and $\gamma_t(\mathrm{Cay}(\mathrm{D}_n, Y_n^{-1})) \geq 2$. By the definition of Y_n, for every $u \in \mathrm{D}_n$ we have $N_{\mathrm{Cay}(\mathrm{D}_n, Y_n)}(u) = \mathrm{D}_n \setminus N_{\mathrm{Cay}(\mathrm{D}_n, X_n)}[u]$. Therefore, $\{a,b\}$ is a total dominating set in $\mathrm{Cay}(\mathrm{D}_n, Y_n)$, with a dominating A and b dominating B, if and only if $\{a,b\}$ is a dominating set in $\mathrm{Cay}(\mathrm{D}_n, X_n)$, with a dominating B and b dominating A. Hence, $\gamma_t(\mathrm{Cay}(\mathrm{D}_n, Y_n)) = \gamma(\mathrm{Cay}(\mathrm{D}_n, X_n)) = 2$, by Theorem 3.2.

Next, $Y_n^{-1} = [\mathrm{D}_n \setminus (\{(0,0)\} \cup X_n)]^{-1} = \mathrm{D}_n \setminus (\{(0,0)\} \cup X_n^{-1})$. So analogously as above, for every $u \in \mathrm{D}_n$ we have $N_{\mathrm{Cay}(\mathrm{D}_n, Y_n^{-1})}(u) = \mathrm{D}_n \setminus N_{\mathrm{Cay}(\mathrm{D}_n, X_n^{-1})}[u]$. Hence, $\{a,b\}$ is a total dominating set in $\mathrm{Cay}(\mathrm{D}_n, Y_n^{-1})$ if and only if $\{a,b\}$ is a dominating set in $\mathrm{Cay}(\mathrm{D}, X_n^{-1})$. Consequently, $\gamma_t(\mathrm{Cay}(\mathrm{D}_n, Y_n^{-1})) > 2$, by Theorem 3.2. □

We remark that if n is odd, then we cannot use a method described in the proof of Theorem 2.2 to find $U_n \subseteq \mathrm{D}_n$ such that $\gamma_t(\mathrm{Cay}(\mathrm{D}_n, U_n)) \neq \gamma_t(\mathrm{Cay}(\mathrm{D}_n, U_n^{-1}))$. The reason is that there are no $g, h \in \mathrm{D}_n$ such that $(\{(0,0)\} \cup X_n)g$ does not contain $(0,0)$ and $g^{-1}(\{(0,0)\} \cup X_n^{-1}) = (\{(0,0)\} \cup X_n^{-1})h$. However, for even n, $n = 2k$, the method of Theorem 2.2 works. It suffices to choose $g = h = (k,0)$ as $(k,0)$ is in the center of D_n.

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