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On the complexity of strong and weak total domination in graphs

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Abstract

A set D of vertices in a graph $G = (V, E)$ is a total dominating set if every vertex of G is adjacent to some vertex in D. A total dominating set D of G is said to be weak if every vertex $v \in V - D$ is adjacent to a vertex $u \in D$ such that $d_G(v) \geq d_G(u)$. The weak total domination number $\gamma_{wt}(G)$ of G is the minimum cardinality of a weak total dominating set of G. A total dominating set D of G is said to be strong if every vertex $v \in V - D$ is adjacent to a vertex $u \in D$ such that $d_G(u) \geq d_G(v)$. The strong total domination number $\gamma_{st}(G)$ of G is the minimum cardinality of a strong dominating set of G. We show that the decision problems for these variants are NP-complete, even when restricted to bipartite graphs and chordal graphs. We also show that the decision problem for the strong total domination is NP-complete, even when restricted to split graphs.

1 Introduction

For undefined terminology, the reader is referred to [7]. We consider finite, undirected, simple graphs. Let G be a graph, with vertex set V and edge set E. The *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. For a subset $S \subseteq V$, the *open neighborhood* of S is $N(S) = \cup_{v \in S} N(v)$ and the *closed neighborhood* is $N[S] = N(S) \cup S$. If v is a vertex of V, then the *degree* of v, denoted by $d_G(v)$, is the cardinality of its open neighborhood. A *clique* in G is a subset S of vertices such that its induced subgraph is complete, and an *independent set* in G is a subset of vertices such that its induced subgraph has no edge. A *chordal graph* is a graph that does not contain an induced cycle of length greater than three. A *split graph* is a graph whose vertex set is the disjoint union of a clique and an independent set. It is well-known that a split graph is a chordal graph.

A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ has a neighbor in S and is a *total dominating set* if every vertex in V has a neighbor in S. The *domination number* $\gamma(G)$ (respectively, *total domination number* $\gamma_t(G)$) is the minimum cardinality of a dominating set (respectively, total dominating set) of G. Total domination was introduced by Cockayne, Dawes and Hedetniemi [2]. Sampathkumar and Pushpa Latha [11] have introduced the concept of weak and strong domination in graphs. A subset $D \subseteq V$ is a *weak dominating set* if every vertex $v \in V - S$ is adjacent to a vertex $u \in D$, where $d_G(v) \geq d_G(u)$. The subset D is a *strong dominating set* if every vertex $v \in V - S$ is adjacent to a vertex $u \in D$, where $d_G(u) \geq d_G(v)$. The *weak* (*strong*, respectively) *domination number* is the minimum cardinality of a weak dominating set (a strong dominating set, respectively) of G. Strong and weak domination have been studied in [3, 5, 6, 9, 10]. For more details on domination in graphs and its variations, see the two books [7, 8].

Chellali and Jafari Rad [1] introduced the concept of weak total domination in graphs, and proposed the study of strong total domination. A total dominating set D of G is said to be *weak* if every vertex $v \in V - D$ is adjacent to a vertex $u \in D$ such that $d_G(v) \geq d_G(u)$. The *weak total domination number* of G, denoted by $\gamma_{wt}(G)$, is the minimum cardinality of a weak total dominating set of G. Similarly, a total dominating set D of G is said to be *strong* if every vertex $v \in V - D$ is adjacent to a vertex $u \in D$ such that $d_G(v) \leq d_G(u)$. The *strong total domination number* of G, denoted by $\gamma_{st}(G)$, is the minimum cardinality of a strong total dominating set of G. It is proved in [1] that the decision problem for the weak total domination in NP-complete for general graphs.

Theorem 1 ([1]) *The weak total dominating set is NP-complete for general graphs.*

In this paper we prove that the decision problem for both weak total domination and strong total domination are NP-complete, even when restricted to bipartite graphs, and chordal graphs. We also show that the decision problem for strong total domination is NP-complete, even when restricted to split graphs. In each variant we will state the corresponding decision problem in the standard form [4] and indicate the polynomial-time reduction used to prove that it is NP-complete.

2 Results

We first prove the complexity of strong total domination. Consider the following decision problem.

STRONG TOTAL DOMINATING SET (STDS).
Instance: Graph $G = (V, E)$, positive integer $k \leq |V|$.
Question: Does G have a strong total dominating set of cardinality at most k?

We show that this problem is NP-complete by reducing the well-known NP-complete problem, VERTEX COVER, to strong total dominating set problem.

VERTEX-COVER
Instance: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.
Question: Is there a subset $C \subseteq V$ of size at most k such that for each edge $xy \in E$ either $x \in C$ or $y \in C$?

Theorem 2 *The strong total dominating set problem is NP-complete for bipartite graphs.*

Proof. Clearly, the strong total dominating set problem is in NP, since it is easy to verify that a given set of vertices is a STDS in polynomial time. Now let us show how to transform the vertex cover problem to the strong total dominating set problem. Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = m$. Let H be a graph with $V(H) = V \cup \{y, x_i : 1 \leq i \leq m + n\} \cup \{e_i : e \in E, 1 \leq i \leq n + 1\}$, and $E(H) = \{yx_i : 1 \leq i \leq m + n\} \cup \{yv : v \in V\} \cup \{ve_i : v \in e, e \in E, 1 \leq i \leq n + 1$. Figure 1. shows the graphs $G = P_3$ and H.

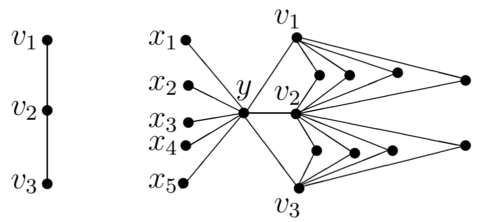

Figure 1. The graphs $G = P_3$ and H.

Clearly H is a bipartite graph. We show that G has a vertex cover of size at most k if and only if H has a strong total dominating set of cardinality at most $k + 1$. If G has a vertex cover S of size at most k then clearly $S \cup \{y\}$ is a strong total dominating set for H. Let D be a strong total dominating set for H of cardinality at most $k + 1$. Clearly $y \in D$. Assume that there is an edge $e = uv \in E$ such that $\{u, v\} \cap D = \emptyset$. Since e_i is dominated by D, for $i = 1, 2, \ldots, n + 1$, we obtain that $e_i \in D$ for $i = 1, 2, \ldots, n + 1$. This implies that $|D| \geq n + 2 > k + 1$, a contradiction. Thus for any $e = uv \in E$, we have $u \in D$ or $v \in D$. Consequently, $D \cap V$ is a vertex cover for G of size at most k, as desired. ■

Theorem 3 *The strong total dominating set problem is NP-complete for split graphs.*

Proof. Let H be the graph in the proof of Theorem 2. We add edges between any pair of vertices of V, and then, if necessary, add pendant edges to y such that in the resulting graph y has maximum degree. Notice that the resulting graph is split, since $V \cup \{y\}$ forms a clique and the remaining vertices form an independent set. Now continuing the proof of Theorem 2 yields the desired result. ■

Since any split graph is chordal, we have the following.

Corollary 4 *The strong total dominating set problem is NP-complete for chordal graphs.*

We next consider the complexity of weak total domination and improve Theorem 1.

WEAK TOTAL DOMINATING SET (WTDS)
Instance: Graph $G = (V, E)$, positive integer $k \leq |V|$.
Question: Does G have a weak total dominating set of cardinality at most k?

We show that this problem is NP-complete by reducing the well-known NP-complete problem, Exact-3-Cover (X3C), to weak total dominating set problem.

EXACT 3-COVER (X3C)
Instance: A finite set X with $|X| = 3q$ and a collection C of 3-element subsets of X.
Question: Is there a subcollection C' of C such that every element of X appears in exactly one element of C'?

Theorem 5 *The weak total dominating set problem is NP-complete even when restricted to bipartite graphs.*

Proof. Clearly, the weak total dominating set problem is in NP, since it is easy to verify that a given set of vertices is a WTDS in polynomial time. Now let us show how to transform any instance X, C of X3C into an instance G of WTDS so that one of them has a solution if and only if the other has a solution. Let $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, ..., C_m\}$ be an arbitrary instance of X3C. For $i = 1, 2, \ldots, m$, let G'_i be the 6-vertex path $c_i d_i e_i f_i g_i h_i$ (with consecutive vertices $c_i, d_i, e_i, f_i, g_i, h_i$), and let G_i be obtained from G'_i by adding a vertex b_i and joining b_i to d_i, and also adding a 3-vertex path $f'_i g'_i h'_i$ and joining f'_i to f_i. For $i = 1, 2, \ldots, 3q$, let H'_i be the 7-vertex path $w_i z_i y_i x_i y'_i z'_i w'_i$, and let H_i be obtained from H'_i by adding a 3-vertex path $y''_i z''_i w''_i$ and join y''_i to x_i. Figure 2 shows the graphs G_i and H_i.

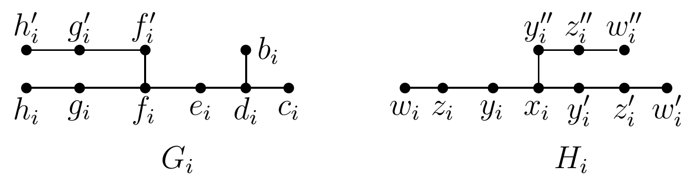

Figure 2. The graphs G_i and H_i.

Corresponding to each variable x_i, we associate the graph H_i, and corresponding to each set C_j we associate the graph G_j. The construction of G is completed by joining x_i and c_j if and only if the variable x_i occurs in the set C_j. Clearly G is bipartite. Set $k = 19q + 7m$.

Suppose that the instance X, C of X3C has a solution C'. It is easily verified that $D = \bigcup_{i=1}^{3q}\{w_i, z_i, w_i', z_i', w_i'', z_i''\} \cup \bigcup_{j=1}^{m}\{h_j, g_j, h_j', g_j', d_j, b_j, e_j\} \cup \bigcup_{C_j \in C'}\{c_j\}$ is a weak total dominating set for G of cardinality k.

Conversely, suppose that G has a weak total dominating set S with $|S| \leq 19q + 7m = k$. An easy observation in [1] assures that S contains every leaf and every support vertex, i.e.,

$$\bigcup_{i=1}^{3q}\{w_i, z_i, w_i', z_i', w_i'', z_i''\} \cup \bigcup_{j=1}^{m}\{h_j, g_j, d_j, b_j, h_j', g_j'\} \subseteq S.$$

Furthermore, $e_j \in S$ for $j = 1, 2, \ldots, m$. To dominate x_i for all $i = 1, 2, \ldots, 3q$, with $|S| \leq 19q + 7m$, it forces that the rest q elements in S are contained in $\{c_j : j = 1, 2, \ldots, m\}$, since at most three x_i's can be dominated by one vertex and each of them must be some of $\{c_j\}$. This implies that $C' = \{C_j : c_j \in S\}$ is an exact cover for C, as desired. ■

Theorem 6 *The weak total dominating set is NP-complete even when restricted to chordal graphs.*

Proof. Let G be the graph in the proof of Theorem 6. We add edges between any pair c_i, c_j for $i, j \in \{1, 2, \ldots, m\}$. Then for each $i = 1, 2, \ldots, 3q$, add $\frac{m(m-1)}{2}$ 3-vertex paths and join x_i to a leaf of each path. Let G^* be the resulting graph. Notice that G^* is chordal. Now continuing the proof of Theorem 6 yields the desired result. ■

We end the paper with a remark on the proof of Theorem 24 of [3]. In the proof of Theorem 24 of [3] the reduction is from X3C with the additional assumption that each variable appears in at least two subsets. However without such additional assumption, for each $i = 1, 2, \ldots, 3q$, one can add a 2-vertex path and join one of its leaves to x_i, set $\ell = 2m + 10q$, and continue the proof.

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