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On the automorphism group of Cayley graphs generated by transpositions

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Abstract

The modified bubble-sort graph of dimension n is the Cayley graph of S_n generated by n cyclically adjacent transpositions. In the present paper, it is shown that the automorphism group of the modified bubble sort graph of dimension n is $S_n \times D_{2n}$, for all $n \geq 5$. Thus, a complete structural description of the automorphism group of the modified bubble-sort graph is obtained. A similar direct product decomposition is seen to hold for arbitrary normal Cayley graphs generated by transposition sets.

1 Introduction

Let $X = (V, E)$ be a simple undirected graph. The (full) automorphism group of X, denoted by $\mathrm{Aut}(X)$, is the set of permutations of the vertex set that preserves adjacency, i.e., $\mathrm{Aut}(X) := \{g \in \mathrm{Sym}(V) : E^g = E\}$. Let H be a group with identity element e, and let S be a subset of H. The Cayley graph of H with respect to S, denoted by $\mathrm{Cay}(H, S)$, is the graph with vertex set H and arc set $\{(h, sh) : h \in H, s \in S\}$. When S satisfies the condition $1 \notin S = S^{-1}$, the Cayley graph $\mathrm{Cay}(H, S)$ has no self-loops and can be considered to be undirected.

A Cayley graph $\mathrm{Cay}(H, S)$ is vertex-transitive since the right regular representation $R(H)$ acts as a group of automorphisms of the Cayley graph. The set of automorphisms of H that fixes S setwise is a subgroup of the stabilizer $\mathrm{Aut}(\mathrm{Cay}(H, S))_e$ (cf. [1], [7]). Let $\mathrm{Aut}(H, S)$ denote the set of automorphisms of the group H that fixes S setwise. A Cayley graph $X := \mathrm{Cay}(H, S)$ is said to be *normal* if $R(H)$ is a normal subgroup of $\mathrm{Aut}(X)$, or equivalently, if $\mathrm{Aut}(X) = R(H) \rtimes \mathrm{Aut}(H, S)$ (cf. [10]).

Let S be a set of transpositions generating the symmetric group S_n. The transposition graph of S, denoted by $T(S)$, is defined to be the graph with vertex set $\{1, \ldots, n\}$, and with two vertices i and j being adjacent in $T(S)$ whenever $(i, j) \in S$.

A set S of transpositions generates S_n if and only if the transposition graph of S is connected.

The bubble-sort graph of dimension n is the Cayley graph of S_n with respect to the generator set $\{(1,2),(2,3),\dots,(n-1,n)\}$. In other words, the bubble-sort graph is the Cayley graph $\mathrm{Cay}(S_n,S)$ corresponding to the case where the transposition graph $T(S)$ is the path graph on n vertices. The reason this Cayley graph is called the bubble-sort graph is that this Cayley graph is closely related to the (inefficient) bubble-sort algorithm for sorting an array. Given a permutation $\pi \in S_n$, expressed as an array $[\pi(1),\pi(2),\dots,\pi(n)]$, the bubble-sort algorithm sorts the array by swapping elements in consecutive positions of the array. Observe that the minimum number of swaps of elements in consecutive positions required to sort a given array π is exactly the distance in the Cayley graph $\mathrm{Cay}(S_n,S)$ between the permutation π and the identity vertex e. The modified bubble-sort graph is obtained by modifying the bubble-sort graph by adding another generator (and hence, by adding extra edges) to the bubble-sort graph, thereby reducing its diameter.

More precisely, when the transposition graph of S is the n-cycle graph, the Cayley graph $\mathrm{Cay}(S_n,S)$ is called the modified bubble-sort graph of dimension n. Thus, the modified bubble-sort graph of dimension n is the Cayley graph of S_n with respect to the set of generators $\{(1,2),(2,3),\dots,(n-1,n),(n,1)\}$. The diameter of the modified bubble-sort graph was investigated in [8]. The modified bubble-sort graph has been investigated for consideration as the topology of interconnection networks (cf. [9]). Many authors have investigated the automorphism group of graphs that arise as the topology of interconnection networks; for example, see [2], [3], [6], [11], [12].

Godsil and Royle [7] showed that if the transposition graph of S is an asymmetric tree, then the automorphism group of the Cayley graph $\mathrm{Cay}(S_n,S)$ is isomorphic to S_n. Feng [4] showed that $\mathrm{Aut}(S_n,S)$ is isomorphic to $\mathrm{Aut}(T(S))$ and that if the transposition graph of S is an arbitrary tree, then the automorphism group of $\mathrm{Cay}(S_n,S)$ is the semidirect product $R(S_n) \rtimes \mathrm{Aut}(S_n,S)$. Ganesan [5] showed that if the girth of the transposition graph of S is at least 5, then the automorphism group of the Cayley graph $\mathrm{Cay}(S_n,S)$ is the semidirect product $R(S_n) \rtimes \mathrm{Aut}(S_n,S)$. The results in the present paper imply that all these automorphism groups in the literature can be factored as a direct product.

In Zhang and Huang [11], it was shown the automorphism group of the modified bubble-sort graph of dimension n is the group product S_nD_{2n} (groups products are also referred to as Zappa-Szep products). This result was strengthened in Feng [4], where it was proved that the automorphism group of the modified bubble-sort graph of dimension n is the semidirect product $R(S_n) \rtimes D_{2n}$ (cf. [4, p. 72] for an explicit statement of this conclusion).

In the present paper, we obtain a complete structural description of the automorphism group of the modified bubble-sort graph of dimension n (cf. Corollary 2). We shall prove the following more general result:

Theorem 1. *Let S be a set of transpositions generating $S_n (n \geq 3)$ such that the Cayley graph* $\mathrm{Cay}(S_n, S)$ *is normal. Then the automorphism group of the Cayley graph* $\mathrm{Cay}(S_n, S)$ *is the direct product $S_n \times \mathrm{Aut}(T(S))$, where $T(S)$ denotes the transposition graph of S.*

Corollary 2. *The automorphism group of the modified bubble-sort graph of dimension n is $S_n \times D_{2n}$, for all $n \geq 5$.*

In the special case where $T(S)$ is the n-cycle graph, $\mathrm{Aut}(T(S))$ is isomorphic to the dihedral group D_{2n} of order $2n$. Hence, Corollary 2 is a special case of Theorem 1. Also, Ganesan [5] showed that the modified bubble-sort graphs of dimension less than 5 are non-normal; hence, the assumption $n \geq 5$ in Corollary 2 is necessary.

Remark 1. Given a set S of transpositions generating S_n, let $G := \mathrm{Aut}(\mathrm{Cay}(S_n, S))$. In the instances where $G = R(S_n) \rtimes G_e$, the factor $G_e \cong \mathrm{Aut}(T(S))$ is in general not a normal subgroup of G, and so the semidirect product cannot be written immediately as a direct product. For example, for the modified bubble-sort graph of dimension n, $G \cong R(S_n) \rtimes G_e \cong S_n \rtimes D_{2n}$, where G_e is not normal in G. In the present paper, it is shown that $R(S_n)$ has another complement in G which is a normal subgroup of G. In the proof below, we show that the image of $\mathrm{Aut}(T(S))$ under the left regular action of S_n on itself is a normal complement of $R(S_n)$ in G. Thus, the direct factor $\mathrm{Aut}(T(S))$ that arises in $G \cong R(S_n) \times \mathrm{Aut}(T(S))$ is not G_e but is obtained in a different manner.

2 Proof of Theorem 1

Let S be a set of transpositions generating S_n. We first establish that the Cayley graph $\mathrm{Cay}(S_n, S)$ has a particular subgroup of automorphisms. In this section, let L denote the left regular action of S_n on itself, defined by $L : S_n \to \mathrm{Sym}(S_n), a \mapsto L(a)$, where $L(a) : x \mapsto a^{-1}x$. For a subset $K \subseteq S_n$, $L(K)$ denotes the set $\{L(a) : a \in K\}$.

Proposition 3. *Let $T(S)$ denote the transposition graph of S. Then, $\{L(a) : a \in \mathrm{Aut}(T(S))\}$ is a set of automorphisms of the Cayley graph $X := \mathrm{Cay}(S_n, S)$.*

Proof: Let $a \in \mathrm{Aut}(T(S))$. We show that $\{h, g\} \in E(X)$ if and only if $\{h, g\}^{L(a)} \in E(X)$. Suppose $\{h, g\} \in E(X)$. Then $g = sh$ for some transposition $s = (i, j) \in S$. We have $\{h, g\}^{L(a)} = \{h, sh\}^{L(a)} = \{h^{L(a)}, (sh)^{L(a)}\} = \{a^{-1}h, a^{-1}sh\} = \{a^{-1}h, (a^{-1}sa)a^{-1}h\}$. Now $a^{-1}sa = a^{-1}(i, j)a = (i^a, j^a) \in S$ since a is an automorphism of the graph $T(S)$ that has edge set S. Thus, $\{h, sh\}^{L(a)} \in E(X)$. Conversely, suppose $\{h, g\}^{L(a)} \in E(X)$. Then $a^{-1}h = sa^{-1}g$ for some $s \in S$. Hence $h = (asa^{-1})g$. We have $asa^{-1} = a(i, j)a^{-1} = (i, j)^{a^{-1}} \in S$ because a is an automorphism of $T(S)$. Hence, h is adjacent to g. Thus, $L(\mathrm{Aut}(T(S)))$ is a subgroup of $\mathrm{Aut}(X)$. ∎

Theorem 4. *Let S be a set of transpositions generating $S_n (n \geq 3)$ such that the Cayley graph* $\mathrm{Cay}(S_n, S)$ *is normal. Then, the automorphism group of the Cayley graph* $\mathrm{Cay}(S_n, S)$ *is $S_n \times \mathrm{Aut}(T(S))$, where $T(S)$ denotes the transposition graph of S.*

Proof: Let X denote the Cayley graph $\mathrm{Cay}(S_n, S)$. Since X is a normal Cayley graph, its automorphism group $\mathrm{Aut}(X)$ is equal to $R(S_n) \rtimes \mathrm{Aut}(S_n, S)$ (cf. [10]). Let $R(a)$ denote the permutation of S_n induced by right multiplication by a, so that $R(S_n) := \{R(a) : a \in S_n\}$ is the right regular representation of S_n. The intersection of the left and right regular representations of a group is the image of the center of the group under either action. The center of S_n is trivial, whence $R(S_n) \cap L(S_n) = 1$. In particular, $L(\mathrm{Aut}(T(S)))$ and $R(S_n)$ have a trivial intersection. By Feng [4], $\mathrm{Aut}(S_n, S) \cong \mathrm{Aut}(T(S))$, and it follows from cardinality arguments that $R(S_n)L(\mathrm{Aut}(T(S)))$ exhausts all the elements of $\mathrm{Aut}(X)$. Thus, $R(S_n)$ and $L(\mathrm{Aut}(T(S)))$ are complements of each other in $\mathrm{Aut}(X)$ and every element in $\mathrm{Aut}(X)$ can be expressed uniquely in the form $R(a)L(b)$ for some $a \in S_n$ and $b \in \mathrm{Aut}(T(S))$. This proves that $\mathrm{Aut}(X) = R(S_n) \rtimes L(\mathrm{Aut}(T(S)))$.

It remains to prove that $L(\mathrm{Aut}(T(S)))$ is a normal subgroup of $\mathrm{Aut}(X)$. Suppose $g \in \mathrm{Aut}(X)$ and $c \in \mathrm{Aut}(T(S))$. We show that $g^{-1}L(c)g \in L(\mathrm{Aut}(T(S)))$. We have $g = R(a)L(b)$ for some $a \in S_n, b \in \mathrm{Aut}(T(S))$. Hence, $g^{-1}L(c)g = (R(a)L(b))^{-1}L(c)(R(a)L(b))$, which maps $x \in S_n$ to $b^{-1}c^{-1}bxa^{-1}a = b^{-1}c^{-1}bx$. Since $b, c \in \mathrm{Aut}(T(S))$, $d^{-1} := b^{-1}c^{-1}b \in \mathrm{Aut}(T(S))$. Thus, $g^{-1}L(c)g = L(d) \in L(\mathrm{Aut}(T(S)))$. Hence, $L(\mathrm{Aut}(T(S)))$ is a normal subgroup of $\mathrm{Aut}(X)$ and $\mathrm{Aut}(X) = R(S_n) \times L(\mathrm{Aut}(T(S)))$. Since $L(\mathrm{Aut}(T(S))) \cong \mathrm{Aut}(T(S))$, the assertion follows. ∎

Remark 2. We recall a particular result from group theory, which can be used to deduce that the semidirect products in the literature can be strengthened to direct products. Let A be a subgroup of a group H and suppose H has a trivial center. Let A act on H by conjugation. Let $L(A)$ denote the image of the left action of A on H. Then the groups $R(H) \rtimes \mathrm{Inn}(A)$ and $R(H) \times L(A)$ are isomorphic, where both groups are internal group products and subgroups of $\mathrm{Sym}(H)$. It follows from this group-theoretic result that the automorphism group of the Cayley graphs mentioned above can be factored as direct products. However, to the best of our knowledge, this group-theoretic result has not been used so far to deduce results in the context of automorphism groups of Cayley graphs generated by transposition sets - the expressions given in the previous literature for the automorphism group of Cayley graphs mentioned above have been only semidirect product factorizations (cf. [4, p.72], [5], [10]). In the present paper, in addition to obtaining a complete structural description of the automorphism group of the modified bubble-sort graph and of a family of normal Cayley graphs, the proof method also includes Proposition 3, which establishes that these graphs possess certain automorphisms.

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