AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **64(1)** (2016), Pages 252–276

Deflatability of permutation classes

M. H. Albert M. D. Atkinson*

Department of Computer Science
University of Otago
New Zealand

Cheyne Homberger

Department of Mathematics and Statistics
University of Maryland, Baltimore County
U.S.A.

Jay Pantone†

Department of Mathematics
University of Florida
U.S.A.

Abstract

A permutation class is said to be deflatable if its simple permutations are contained within a proper subclass. Deflatable classes are often easier to describe, analyze and enumerate than their non-deflatable counterparts. This paper presents theorems guaranteeing the non-deflatability of principal classes, constructs an infinite family of deflatable principal classes, and provides examples of each.

1 Introduction

A series of recent enumerative and structural results in the theory of permutation classes make use of a common technique, relying on a sparseness property of the simple permutations in a permutation class. If this property holds, we call the class *deflatable* (formal definitions follow below). Much of the analysis of a deflatable class can be carried out in a substantially smaller, and therefore in principle more easily understood, permutation class. This raises the question, interesting in its own right,

* Corresponding author: mike@cs.otago.ac.nz
† Pantone's research was sponsored by the National Science Foundation under Grant Number DMS-1301692.

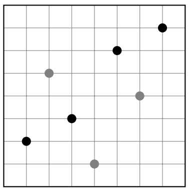


Figure 1: The diagram of the permutation 2531647. The three gray points represent the pattern 312 contained in 2531647.

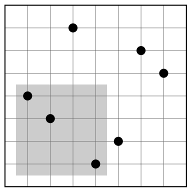


Figure 2: The permutation $4371265 = 2413[21, 1, 12, 21]$. The shaded square represents a box which is cut by two cut points, one by position and one by value.

of characterising deflatable classes. This paper answers that question, at least in part, for principal permutation classes.

The fundamental concept in permutation class theory is the relation of permutation containment. A permutation α is contained as a pattern (or subpermutation) in another permutation β (denoted $\alpha \leqslant \beta$) if, when both are written in one line notation, β has a subsequence whose terms are ordered in the same relative manner as the terms of α. For instance, $312 \leqslant 2531647$ because the entries of the subsequence 514 follow the same relative order as the permutation 312. This relation is more clearly displayed in the diagrams of the permutations where we plot the points $(i, \beta(i))$ of a permutation β in the (x, y) plane. For example, the diagram of 2531647 is shown in Figure 1.

Such diagrams will be used extensively, and we now define some associated terms. A *box* in a permutation diagram is a rectangular region containing a nonempty subset of the points. A *cut point* of a box is a point of the permutation outside the box but whose position is between the leftmost and rightmost points of the box (cut by position), or whose value is between the maximum and minimum points of the box (cut by value). An *interval* of a permutation is a box which is not cut by position nor by value. Equivalently, an interval in a permutation is a set of entries whose indices and values each form a contiguous set, i.e, an interval of the domain and the range. For convenience, we use the convention that the entire permutation is not itself an interval.

Intervals of permutations arise naturally through the process of *inflation*: an inflation of a permutation π is a permutation formed by replacing some of the points of π by other permutations (with appropriate adjustments of values so that the result is a permutation). The permutation resulting from the inflation of a permutation α of length k by permutations $\tau_1, \ldots, \tau_k$ is denoted by $\alpha[\tau_1, \ldots, \tau_k]$. For example, $4371265 = 2413[21, 1, 12, 21]$, as shown in Figure 2.

The pattern containment relation is a partial order on the set of all permutations. It admits eight automorphisms corresponding to the isometries of the square; for example, group-theoretical inversion of permutations corresponds to reflection over the line $y = x$ (see [11] for a more comprehensive discussion). Note that these automorphisms also act on the intervals of a permutation, i.e., the image of any interval of a permutation π is an interval in the image of π. The partial order is studied through its lower ideals, i.e., sets of permutations which are closed downwards under pattern containment. These sets are called *permutation classes* (or sometimes just *classes*). Commonly, a permutation class $\mathcal{C}$ is described by specifying the (unique) set of minimal permutations that do not belong to $\mathcal{C}$. This set B is called the *basis* and we write $\mathcal{C} = \mathrm{Av}(B)$ to signify that $\mathcal{C}$ is the set of permutations that *avoid* (do not contain) any of the permutations of the basis B. Note again that, if a is one of the eight automorphisms mentioned above, and $\mathcal{C}$ is a permutation class then $a(\mathcal{C})$, i.e., the set of images under a of the permutations in $\mathcal{C}$ is also a permutation class, and that if B is the basis of $\mathcal{C}$ then $a(B)$ is the basis of $a(\mathcal{C})$.

The first permutation classes to be studied were the so-called *principal classes* — those whose basis consists of a single permutation. All of our results will be confined to this case. Despite their simple definition, the structure of principal permutation classes is not very well understood. In particular the vast majority of permutation classes which have been enumerated are not principal.

Recent successes in permutation class enumeration (for example: [3, 4, 7, 8, 9, 10]) have relied heavily on the notion of *simplicity*. A permutation σ is simple if it has no intervals other than those consisting of single points. Unsurprisingly, given the remarks above, simple permutations are also invariant under the actions of the automorphisms of the containment order. The significance of simple permutations to the theory of permutation classes is due in large part to the following result.

Proposition 1.1 (Albert and Atkinson [2])**.** *Every permutation π is the inflation of a unique simple permutation σ. If $|\sigma| > 2$ then the maximal intervals of π are disjoint and π is obtained from σ by inflating its points with these maximal intervals.*

To see that the requirement $|\sigma| > 2$ in the second part of the proposition is necessary, consider $\pi = 21354$. Its maximal intervals are 213 and 354 and these are not disjoint. However, this can only occur when π is an inflation of either 12 or 21. We consider this special case briefly now. When π is an inflation of 12 we say that π is *sum-decomposable*, and when π is an inflation of 21 we say that π is *skew-decomposable*. We refer to an inflation of 12 as a *sum* and write $12[\alpha, \beta] = \alpha \oplus \beta$, while we refer to an inflation of 21 as a *skew-sum* and write $21[\alpha, \beta] = \alpha \ominus \beta$. These notations extend naturally as follows for any $k \geqslant 2$:

$$\alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_k = 12 \cdots k[\alpha_1, \alpha_2, \ldots, \alpha_k]$$
$$\beta_1 \ominus \beta_2 \ominus \cdots \ominus \beta_k = k\,(k-1) \cdots 1[\beta_1, \beta_2, \ldots, \beta_k]$$

If π is not sum-decomposable then we say that it is *sum-indecomposable* and if it is not skew-decomposable then we say it is *skew-indecomposable*. A permutation that is both sum- and skew- indecomposable will simply be called *indecomposable*. It is to exactly these permutations that the second sentence of the proposition above applies. If a permutation, π is sum-decomposable, then there is a maximum k such that π is an inflation of $12 \cdots k$, say $\pi = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_k$. In this case the sequence α_1, α_2, …, α_k consists of the maximal sum-indecomposable intervals of π (which are disjoint), and these intervals will be called the *sum-components* (or just components) of π. Of course analogous terminology applies when π is skew-decomposable.

Simple permutations are quite common: Albert, Atkinson, and Klazar [6] showed that the number of them of length n is equal to $(n!/e^2)(1+o(1))$, i.e., they have limiting density $1/e^2$. However, it is often the case that within any particular permutation class (apart from the set of all permutations) their density is far lower; no satisfactory explanation of this phenomenon is known. Indeed there is no known permutation class whose simple permutations have positive density within the class as a whole (i.e., a class for which the ratio of the number of simple permutations of length n to the number of all permutations of length n in the class does not go to zero as n goes to infinity). The unexpected low density of simple permutations in a permutation class $\mathcal{C}$ is frequently a consequence of the simple permutations lying in a proper subclass $\mathcal{C}'$ of $\mathcal{C}$. This property may enable the structure of $\mathcal{C}$ to be determined and thereby its enumeration: briefly, $\mathcal{C}'$ is an easier class to work with and the entirety of $\mathcal{C}$ can be recovered by suitably restricted inflations.

Rather than think about the simple permutations of a given class, one can think about classes which contain a given set S of simple permutations. For ease of exposition, we assume that S is closed under taking simple subpermutations, i.e., that if $\sigma \in S$ and if τ is simple with $\tau \leqslant \sigma$ then $\tau \in S$. At one extreme is the downward closure of S, i.e., the set of all subpermutations of elements of S, denoted $\mathrm{Cl}(S)$, which of course is the smallest class containing S. At the other extreme we encounter the notion of the *substitution closure* of a class: the substitution closure of $\mathcal{C}$, denoted $\langle\mathcal{C}\rangle$, is the largest class containing the same simple permutations – it is formed from $\mathcal{C}$ by taking the closure under inflation. With this notation, the largest class containing exactly the simple permutations in S is $\langle\mathrm{Cl}(S)\rangle$.

As mentioned above, it is frequently easier to enumerate and describe a class if its simple permutations are actually contained in a smaller class. This leads us to our key definition. A permutation class $\mathcal{C}$ is said to be *deflatable* if its simple permutations lie in a smaller class, i.e., if $\mathcal{C} \subseteq \langle\mathcal{D}\rangle$ for some proper subclass $\mathcal{D}$ of $\mathcal{C}$. This term is intended to convey that the proper subclass $\mathcal{D}$ can be obtained by, in a sense, reversing the operation of inflating simple permutations. Moreover, it follows that $\mathcal{C}$ is not deflatable if and only if $\mathcal{C}$ is equal to the downward closure of its simple permutations. That is, a class $\mathcal{C}$ is not deflatable if and only if, for every permutation $\pi \in \mathcal{C}$ there is a simple permutation $\sigma \in \mathcal{C}$ with $\pi \leqslant \sigma$. From this it follows that deflatability is also invariant under the automorphisms of the pattern containment relation.

Examples of deflatable and non-deflatable classes are readily given. The class $\mathrm{Av}(231)$ is deflatable since 12 and 21 are its only simple permutations. On the other hand $\mathrm{Av}(321)$ is not deflatable, a result which is more or less folkloric, but appears (essentially) in a paper by Albert, Atkinson, Brignall, Ruškuc, Smith, and West [5, Proposition 6].

Now we can pose our central question: *for which permutations* π *is* $\mathrm{Av}(\pi)$ *deflatable*?

We can only provide partial answers to this question. In particular, for indecomposable π we can only show that $\mathrm{Av}(2413)$ and symmetrically (or rather "automorphically" but we will use the more common term) Av(3142) are not deflatable. However, for decomposable π we can say much more. Considering the sum-decomposable case only (which by symmetry is sufficient) we can say that $\mathrm{Av}(\pi)$ is not deflatable unless π has exactly two components, one of which has length 1. Even more holds: even in this case, if $|\pi| \geqslant 4$ unless the non-trivial component contains both an increasing and decreasing consecutive pair of consecutive values (like 34 and 65 in 134652), then $\mathrm{Av}(\pi)$ is still not deflatable. These results are proven in Section 3. In the converse direction, Section 4 presents a test for deflatability and uses this test to provide several examples of deflatable principal permutation classes, including an infinite family. Finally, we summarize some remaining open questions in Section 5.

2 Preliminary Lemmas

In the following sections, we will sometimes restrict our focus to indecomposable permutations. The next lemma shows that this does not lose us much generality.

Lemma 2.1. *Every permutation in* $\mathrm{Av}(\pi)$ *can be embedded into an indecomposable permutation in* $\mathrm{Av}(\pi)$ *unless* $\pi \in \{1, 12, 21, 132, 213, 231, 312\}$.

Proof. Let $\omega \in \mathrm{Av}(\pi)$. We first handle the case where π has a corner point, i.e., π has one of the forms $1 \oplus \tau$, $\tau \oplus 1$, $1 \ominus \tau$, or $\tau \ominus 1$. Assume further that π starts with 1, i.e., the first case applies; the other three cases follow symmetrical arguments.

We first embed ω into the sum-indecomposable permutation $\hat{\omega} = \omega \ominus 1$. By the assumption that π has the form $1 \oplus \tau$, it is clear that $\hat{\omega} \in \mathrm{Av}(\pi)$. Let us be explicit, just once, about this sort of remark. Suppose that $1 \oplus \tau \leqslant \omega \ominus 1$. Then there is a subset of $\omega \ominus 1$ whose elements have pattern $1 \oplus \tau$. In particular, the leftmost element of this subset is its least element. Therefore, the last element of $\omega \ominus 1$ cannot be in the set, and in fact $1 \oplus \tau \leqslant \omega$.

Consider the skew-decomposition $\hat{\omega} = \omega_1 \ominus \cdots \ominus \omega_k$ such that each ω_i is itself skew-indecomposable. Form a new skew-indecomposable permutation $\bar{\omega} = \overline{\omega_1} \ominus \cdots \ominus \overline{\omega_k}$, where $\bar{\omega_i} = 12$ if $\omega_i = 1$, and $\bar{\omega_i} = \omega_i$ otherwise. Lastly, obtain an indecomposable permutation ζ containing ω by taking each pair $(\bar{\omega_i}, \overline{\omega_{i+1}})$ of skew components of $\bar{\omega}$ and linking them together by inserting an entry just before the final point of $\bar{\omega_i}$ and just below the topmost point of $\overline{\omega_{i+1}}$. The only permutations beginning with 1 that can be introduced by this step are 1, 12, and 132. Therefore $\zeta \in \mathrm{Av}(\pi)$. Figure 3 gives an example of performing these steps to $\omega = 564213$. This completes the proof in the case that π has a corner point.

Assume now that π has no corner points. It follows that the outer points (the top- bottom- left- and right-most entries) of π form one of the patterns 2143, 2413, 3142, or 3412. By appealing to a symmetry if necessary, we can assume that these outer points do not form a 2413 pattern. Form an indecomposable permutation ζ by adding outer points to ω to form a 2413, as shown in Figure 4. For convenience, we refer to these points as the 2, the 4, the 1, and the 3. Suppose toward a contradiction that ζ contains an occurrence of π. Since $\omega \in \mathrm{Av}(\pi)$, at least one of the new outer points must be involved in the occurrence of π. We will assume that the 2 is involved,

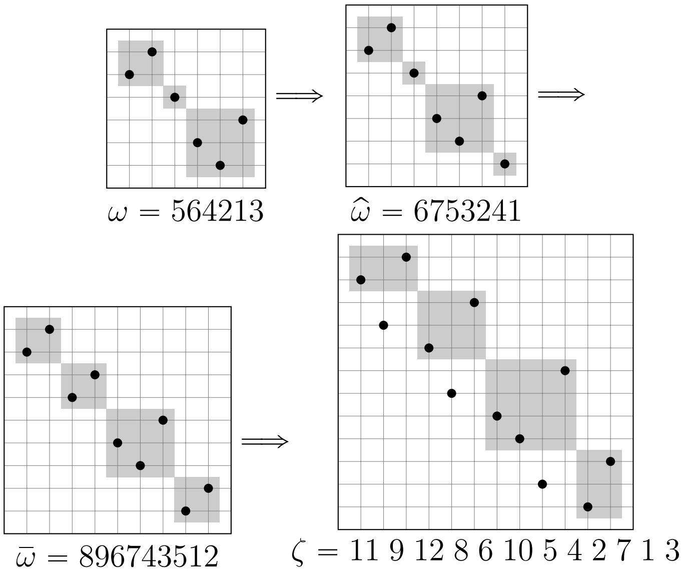


Figure 3: The progression from ω to ζ, when π has a corner point, as described in the proof of Lemma 2.1.

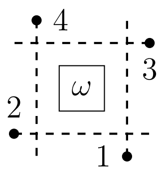


Figure 4: The embedding of ω into ζ in the case where ω does not have a corner point as described in the proof of Lemma 2.1.

and all other cases follow symmetrically. However, since we assumed that π does not have a corner point, the outer point 1 must also be involved. Similarly, 1 would be a corner point unless the outer point 3 were involved, and again 3 would be a corner point unless the outer point 4 were involved. Therefore, the outer points of π form a 2413, a contradiction. □

We can now describe the general strategy that we will follow for proving that a class $\mathrm{Av}(\pi)$ is not deflatable. Recall that an equivalent condition is that, for every $\omega \in \mathrm{Av}(\pi)$ there exists a simple permutation $\sigma \in \mathrm{Av}(\pi)$ with $\omega \leqslant \sigma$. We aim to construct σ by forming a sequence of extensions of ω, all belonging to $\mathrm{Av}(\pi)$ and each one being "closer to simple" than the previous one, in a sense we will make precise shortly. The first such extension is one provided by Lemma 2.1, by means of which we embed ω in an indecomposable permutation. We take this extension as given, and henceforth assume throughout that ω itself is indecomposable. As remarked in Proposition 1.1 this implies that the maximal intervals of ω are disjoint.

In a simple permutation, all the maximal intervals have length 1, so we introduce a statistic, SD, defined for indecomposable permutations:

$$\mathrm{SD}(\omega) = \sum_{\gamma} (|\gamma| - 1)\,,$$

where the summation is over all maximal intervals γ of ω, and $|\gamma|$ denotes the number of elements in γ. We take SD as a measure of "how close" a permutation is to being simple, noting that $\mathrm{SD}(\omega) = 0$ if and only if ω is simple. Note also that SD is invariant under symmetries.

To establish the non-deflatability of $\mathrm{Av}(\pi)$ it is clearly sufficient to show that for any indecomposable $\omega \in \mathrm{Av}(\pi)$ we can find an indecomposable extension $\omega^+ \in \mathrm{Av}(\pi)$ of ω with $\mathrm{SD}(\omega^+) < \mathrm{SD}(\omega)$. In fact, the extensions we form will always use just a single additional element (which we generally call x) and the type of extension will be as described in the next lemma.

Lemma 2.2. *Suppose that ω is an indecomposable permutation with an interval α of maximum length $\ell > 1$. Suppose that ω^+ is an extension of ω by a point x that cuts α and that $\alpha \cup \{x\}$ does not form an interval in ω^+. Then, ω^+ is indecomposable and $\mathrm{SD}(\omega^+) < \mathrm{SD}(\omega)$.*

Proof. It is evident that ω^+ is indecomposable because x cuts α and so the new point x is not a corner point.

We compare the maximal intervals of ω with the maximal intervals of ω^+. Because of the invariance of SD under symmetry we may assume that x cuts α by position so, since $\alpha \cup \{x\}$ does not form an interval in ω^+, α is separated from x by value. Then, x cuts no other maximal interval of ω by position, and cuts at most one maximal interval of ω by value.

The maximal interval μ of ω^+ that contains x is just the singleton $\{x\}$, for if there were another point in this interval, the interval would have to include at least one positional neighbor u in α. Take v to be a point separating x from α by value. Then v must lie in μ since μ contains points of value less than v and greater than

v. Now, $\mu\backslash\{x\}$ would be an interval of ω that contains points from two distinct maximal intervals, a contradiction. Clearly, the maximal interval $\{x\}$ contributes 0 to $\mathrm{SD}(\omega^+)$.

Now consider a maximal interval μ of ω^+ that does not contain x. Then, μ is also an interval of ω and so μ is contained in a maximal interval ν of ω. If x does not cut ν, then ν is also an interval of ω^+ and hence, by the maximality of μ, we have $\mu = \nu$. Thus, such intervals contribute equal amounts to both $\mathrm{SD}(\omega)$ and $\mathrm{SD}(\omega^+)$. However, if x cuts ν (which certainly happens if $\nu = \alpha$), then ν is not an interval of ω^+ and so ν will be a proper union $\mu_1 \cup \cdots \cup \mu_k$ of more than one maximal interval. Since the union is proper,

$$\sum_{i=1}^{k} (|\mu_i| - 1) = |\nu| - k < |\nu| - 1.$$

It follows that $\mathrm{SD}(\omega^+) < \mathrm{SD}(\omega)$, as desired. □

Given a class $\mathcal{C}$ a non-simple but indecomposable permutation $\omega \in \mathcal{C}$ which has an extension of the form described in the lemma above will be called *breakable*. If we also wish to specify which interval is broken, we will say that ω is breakable through α.

Lemma 2.1 and Lemma 2.2 set the stage for all of our proofs that a principal class $\mathrm{Av}(\pi)$ is not deflatable. Call a class, $\mathcal{C}$, *extendible* if every indecomposable non-simple permutation ω of $\mathcal{C}$ is breakable. By our previous remarks, any extendible class is non-deflatable and, extendibility is actually the property that we will establish for all the cases that we consider. For future reference we record the key property of extendibility for our purposes:

Lemma 2.3. *If a class $\mathcal{C}$ is extendible, then it is not deflatable.*

3 Non-Deflatable Permutation Classes

This section is mainly devoted to identifying families of permutations π for which $\mathrm{Av}(\pi)$ is non-deflatable. We have had little success in this endeavour for indecomposable permutations π and so we focus here on decomposable permutations. Then, using the usual symmetries, we may further assume that π is sum-decomposable. We provide here a sequence of results showing that $\mathrm{Av}(\pi)$ is not deflatable if any of the following conditions are met:

- π can be written as a sum of three or more permutations (Theorem 3.2),
- π can be written as the sum of two permutations, each of size greater than 1 (Theorem 3.3), and
- $\pi = 1 \oplus \rho$ where $|\rho| \geqslant 3$ and ρ contains either no decreasing bond, or no increasing bond, where a bond is an interval of size 2 (Theorems 3.4 through 3.8).

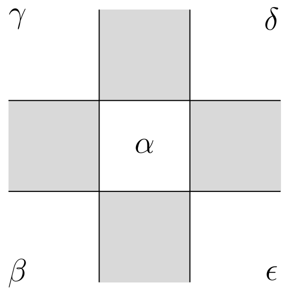


Figure 5: Diagram of an indecomposable ω with a longest maximal interval α. We label the four corner regions as β, γ, δ, and ϵ.

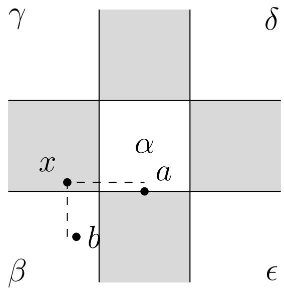


Figure 6: The permutation ω^+ formed by inserting the entry x into ω just to the left of b and just above a.

Taken together, and taking symmetry into account this means that for any decomposable permutation π of length at least four, $\text{Av}(\pi)$ is not deflatable unless possibly when π has exactly two components, one of which has length 1, and the other of which contains both an increasing and a decreasing bond.

For the remainder of this section, with the exception of the final result, we shall be considering sum-decomposable permutations π, and we begin by setting some notation and standing assumptions. We suppose that $\omega \in \text{Av}(\pi)$ is indecomposable and non-simple. Let α be a longest maximal interval of ω. Then, ω can be depicted as in Figure 5, where the shaded regions signify that no entries cut α by either position or value. We will always refer to the regions α, β, γ, δ, and ϵ as shown in Figure 5.

Lemma 3.1. *Suppose that $\pi = \lambda \oplus \mu \oplus \rho$ (where we allow μ to be possibly empty). If $\lambda \leqslant \beta$ or $\rho \leqslant \delta$, then ω is breakable through α.*

Proof. Suppose that $\lambda \leqslant \beta$. Let b be the rightmost point of the leftmost occurrence of λ in β. Let a be the bottommost point of α. Insert a new entry x just to the left of b and just above a to form ω^+, as in Figure 6. It is clear that $\alpha \cup \{x\}$ is not an interval because they are separated by b. Furthermore, suppose that the insertion of x introduced an occurrence of π in ω^+. Then, x itself must be involved, otherwise ω would have contained an occurrence of π. However, x cannot play a role in the λ part of π, for otherwise there would be an occurrence of $\mu \oplus \rho$ above and to the right of x, and hence above and to the right of the occurrence of λ which ends with b. This would imply that ω contained an occurrence of π, a contradiction. Moreover, if x played the role in the $\mu \oplus \rho$ part of π in ω^+, then there would be an occurrence of λ below and to the left of x, contradicting our choice of b. Thus, $\omega^+ \in \text{Av}(\pi)$.

Therefore, if $\lambda \leqslant \beta$, it follows that ω is breakable. A symmetric argument shows that ω is breakable if $\rho \leqslant \delta$. □

The above lemma shows that for $\pi = \lambda \oplus \mu \oplus \rho$ (with μ possibly empty), when trying to show that $\omega \in \text{Av}(\pi)$ is breakable, we may always assume that $\beta \in \text{Av}(\lambda)$

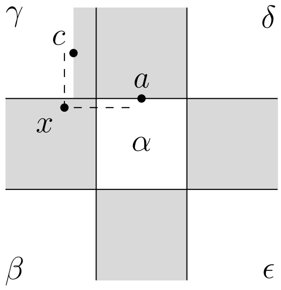


Figure 7: The permutation ω^+ formed by inserting the entry x into ω just to the left of c and just below a, in the proof of Theorem 3.2

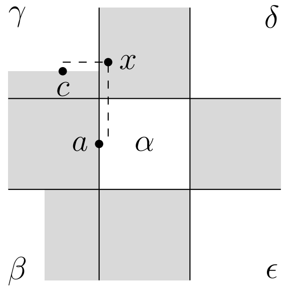


Figure 8: The diagram for Case 1 in the proof of Theorem 3.3.

and $\delta \in \mathrm{Av}(\rho)$. Additionally, the indecomposability of ω implies that γ and ϵ are not both empty.

We can now begin to investigate which decomposable permutations π lead to non-deflatable classes $\mathrm{Av}(\pi)$ as outlined at the beginning of this section.

Theorem 3.2. *Let $\pi = \lambda \oplus \mu \oplus \rho$ with all three summands non-empty. Then,* $\mathrm{Av}(\pi)$ *is not deflatable.*

Proof. Let $\omega \in \mathrm{Av}(\pi)$ be indecomposable and not simple, and choose α to be a maximal non-singleton interval. We can assume that $\beta \in \mathrm{Av}(\lambda)$, $\delta \in \mathrm{Av}(\rho)$ or else ω is breakable through α by the preceding lemma. Since ω is indecomposable, at least one of γ and ϵ is non-empty. Since π satisfies the conditions of the hypothesis if and only if π^{-1} does, we can assume without loss of generality that γ is non-empty. Let c be the rightmost entry of γ and let a be the topmost entry of α. Form ω^+ by inserting an entry x into ω that lies just to the left of c and just below a, as in Figure 7.

Suppose that x is part of an occurrence of π in ω^+. If x plays a role in the ρ part of π, then $\lambda \leqslant \beta$, a contradiction. If x plays a role in the λ part of π, then the $\mu \oplus \rho$ part of the occurrence of π lies among $\{a, c\} \cup \delta$. At least one of a or c must belong to the ρ part of this occurrence, since otherwise we would have $\rho \leqslant \delta$. However, $a \in \rho$ and $c \in \rho$ are each impossible since μ is non-empty and a and c are, respectively, the first and lowest elements of $\{a, c\} \cup \delta$. Hence, x is not part of an occurrence of π, which shows that ω is breakable. Since ω was an arbitrary indecomposable permutation, $\mathrm{Av}(\pi)$ is extendible, and so, by Lemma 2.3, is not deflatable. □

As the above theorem did not require λ, μ, and ρ to be sum-indecomposable, it handles all sum-decomposable permutations except for those of the form $\pi = \lambda \oplus \rho$ with λ and ρ sum-indecomposable. The next theorem begins to handle this case.

Theorem 3.3. *Let $\pi = \lambda \oplus \rho$, with $|\lambda|, |\rho| \geqslant 2$. Then,* $\mathrm{Av}(\pi)$ *is not deflatable.*

Proof. Theorem 3.2 allows us to assume that λ and ρ are sum-indecomposable. Suppose that $\omega \in \mathrm{Av}(\pi)$ is indecomposable. We wish to show that ω is breakable. To this end, choose a largest maximal interval α of ω and let β, γ, δ and ϵ be as in Figure 5. Let c be the smallest entry of γ. By our previous arguments we may also suppose that $\beta \in \mathrm{Av}(\lambda)$ and $\delta \in \mathrm{Av}(\rho)$. We can assume by symmetry that γ is non-empty (for the same reason as in the previous theorem). Furthermore, at least one of β and δ is non-empty, as otherwise ω is skew-decomposable. We consider a division into cases.

Case 1: β is empty, or the last entry of β precedes the smallest entry of γ

Suppose that all entries of β lie to the left of c (it is permissible that β be empty). Let a be the leftmost entry in α. We will show that α can be split by an entry x placed just above c and just to the right of a, as in Figure 8.

Suppose that the placement of x introduces an occurrence of π. If x were an entry of the λ part of π, then the ρ part of π would lie entirely above it and to its right. This would force ρ to be entirely contained in δ, a contradiction to an earlier assumption. Therefore, x must be an entry in the ρ part of π. It follows that the λ part of π occurs in $\{a, c\} \cup \beta$.

If the occurrence of λ contains the point c, then the occurrence of ρ contains x as its first and least entry, a contradiction to the assumption that ρ is sum-indecomposable. If the occurrence of λ contains the point a (but not the point c), then a is the last and greatest entry of λ, contradicting that λ is sum-indecomposable. Hence the occurrence of λ is contained entirely within β, contradicting that $\beta \in \mathrm{Av}(\lambda)$.

Case 2: The smallest entry of γ precedes the last entry of β

Let b be the rightmost entry of β and let c' be the rightmost entry of γ. We handle two cases: either c' precedes b or b precedes c'.

Case 2a: The last entry of γ precedes the last entry of β

Let a' be the topmost entry of α. Consider a splitting entry x which lies just to the right of c' and just below a', as in Figure 9. If the insertion of x creates an occurrence of π, then x lies in the λ part of π – otherwise the λ part of π lies entirely in β, a contradiction. Hence, the occurrence of the ρ part is contained in $\{a'\} \cup \delta$. Since $\delta \in \mathrm{Av}(\rho)$, the occurrence of ρ must contain the point a', implying that ρ is sum-decomposable. This contradicts our previous assumption. Therefore, x splits α without introducing an occurrence of π. Note that this case did not require that $c \neq c'$ nor $a \neq a'$.

Case 2b: The last entry of β precedes the last entry of γ

Assume now that b precedes c'. Let x be a splitting entry which lies just to the right of a and just above c. (See Figure 10.)

Suppose that the insertion of the entry x creates an occurrence of π. Since x cannot lie in the λ part of π (as δ avoids ρ), x must lie in the ρ part of π. Hence, the λ part of π is contained in $\{a, c\} \cup \beta$. The point c must be part of the occurrence of λ, since otherwise λ is sum-decomposable. In fact, the point a cannot be part of the λ occurrence because this, together with c also lying in the λ part would force x to be both the first and smallest entry of ρ, once again implying that ρ

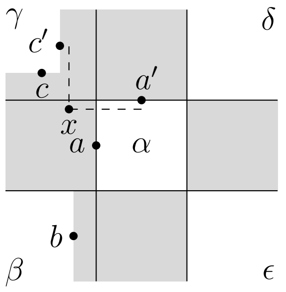


Figure 9: The diagram for Case 2a in the proof of Theorem 3.3.

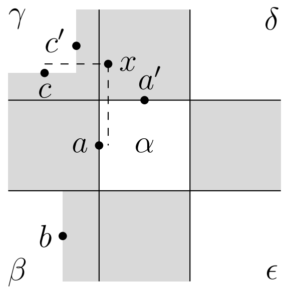


Figure 10: The first diagram for Case 2b in the proof of Theorem 3.3.

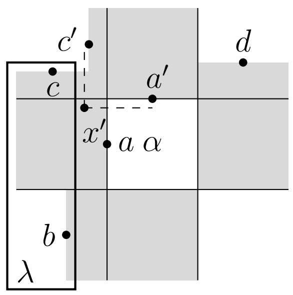


Figure 11: A diagram showing the existence of an occurrence of λ in ω as in Case 2b of Theorem 3.3.

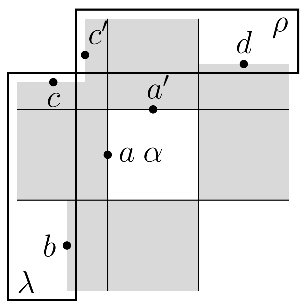


Figure 12: A diagram showing the existence of an occurrence of π in ω as in Case 2b of Theorem 3.3.

is sum-decomposable. Therefore, there is an occurrence of λ within $\{c\} \cup \beta$ that contains c.

Now consider an alternative splitting point x' placed just to the left of c' and just below a'. Let d be the lowest point of δ. If d is lower than c, then we can proceed by an argument symmetrical to Case 2a, and if $d > c'$, then we can proceed by an argument symmetrical to Case 1. Therefore, we may assume that c is lower than d and that d is lower than c'. Figure 11 shows the new splitting point x', along with the occurrence of λ which caused x to fail as a splitting point. Assume also that x' creates an occurrence of π. An argument symmetric to that of the previous paragraph by a reflection over the antidiagonal shows that there must be an occurrence of ρ involving c' and some points of δ, as shown in Figure 12.

Thus $\pi = \lambda \oplus \rho$ is contained in ω, a contradiction. □

For the remainder of this section we concern ourselves largely with the remaining case for decomposable π, namely $\pi = 1 \oplus \rho$, with ρ of length at least three (as we know that Av(132) is deflatable, and Av(123) is not). We will at times need to refer to specific elements of this permutation and may do so either by position, e.g., "the leftmost element of ρ", or by value – here keep in mind that for instance "2" would refer to the least element of ρ. Also, we assume $|\pi| = n$, so n is always the maximum value (and of course occurs somewhere in the ρ part of π).

An *ascent* of a permutation is a pair of entries in consecutive positions that form a 12 pattern, and a *descent* of a permutation is a pair of entries in consecutive positions that form a 21 pattern. An *increasing bond* is an ascent whose entries are also consecutive in value (in other words an interval whose elements form a 12 pattern), and a *decreasing bond* is a descent whose entries are also consecutive in value (an interval of pattern 21). For example, in the permutation 134652 the entries 13 form an ascent but not an increasing bond, the entries 34 form an increasing bond (and an ascent), the entries 65 form a decreasing bond (and a descent), and the entries 52 form a descent but not a decreasing bond.

The presence of bonds in ρ plays an important role in the deflatability of $\pi = 1 \oplus \rho$. In particular, if ρ lacks either an increasing bond or a decreasing bond, then $\mathrm{Av}(\pi)$ is not deflatable. We prove this in two parts, depending on whether ρ starts with an ascent or starts with a descent. In the arguments to follow we will frequently make use of the principle of "substitution" when we are breaking an interval α of ω. We now describe this idea informally. We will add an element x, breaking the interval α. The problem will be to show that in doing so we do not create a copy of $1 \oplus \rho$. If we had created such a copy, then obviously it would need to contain the element x (since there was no such copy in the original permutation). But, the position and value of x will have been chosen in just such a way, that in that case there is some other element of the original permutation (call it c just for the moment) which we can now argue could not occur in the copy (because of some property of ρ), and moreover which sits in the same relative position to the remaining elements of the copy as x does. Then we could "substitute c for x", i.e, remove x in the copy and replace it with c, and thereby obtain a copy of $1 \oplus \rho$ in the original permutation, and hence a contradiction.

Theorem 3.4. *Let $\pi = 1 \oplus \rho$, where ρ is sum-indecomposable and starts with an ascent. If ρ lacks either an increasing bond or a decreasing bond, then* $\mathrm{Av}(\pi)$ *is not deflatable.*

Proof. This proof is split into two separate cases in which ρ either has no increasing bond or has no decreasing bond. As always, we assume $\omega \in \mathrm{Av}(\pi)$ is indecomposable, and that α is a largest maximal interval of ω with β, γ, δ and ϵ as in Figure 5. Since $\lambda = 1$, we can assume that β is empty in addition to assuming that $\delta \in \mathrm{Av}(\rho)$. Further δ is non-empty else ω would be skew-decomposable.

Case 1: ρ has no decreasing bond

Let a be the leftmost entry of α and let d be the leftmost entry of δ. Form a one-point extension ω^+ of ω by inserting a point x just to the right of a and just above d as in Figure 13. If ω^+ contains an occurrence of π, then it is immediately clear that x must play a role in the ρ part; otherwise x plays the role of 1 which would force $\rho \leqslant \delta$. Therefore, the 1 of π either lies in γ or is equal to a.

If the 1 of π lies in γ, then no entry in α can play a role in ρ. Hence, d also cannot play a role in ρ, as then it would be the entry immediately following x and

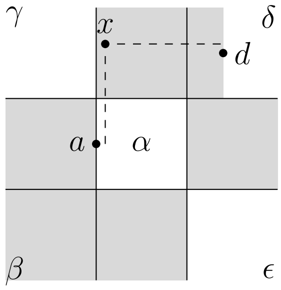


Figure 13: The diagram corresponding to *Case 1* in the proof of Theorem 3.4.

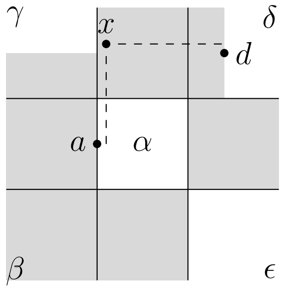


Figure 14: The diagram corresponding to the first part of *Case 2* in the proof of Theorem 3.4.

$\{x, d\}$ would form a decreasing bond. From this it follows that we could substitute d for x and obtain a contradiction.

If the 1 of π lies in α, then a priori it may be possible for other entries of α to play a role in an occurrence of π. However, now x must be the first entry of ρ, and so the presence of any other entry in α (other than a) in ρ would force ρ to start with a descent. So in fact, no other elements of α are used. Hence d again substitutes for x, as again d cannot play a role in the occurrence of π and d and x are split neither by value nor by position by any other entry involved in the occurrence of π.

Case 2: ρ has no increasing bond

Let a be the leftmost entry of α and let d be the leftmost entry of δ. We consider two separate cases: either d is lower in value than all entries that lie in γ, or else there is some entry $c \in \gamma$ which is lower in value than d.

Case 2a: d is lower in value than all entries in γ

Form ω^+ by inserting an entry x just to the right of a and just above d, as in Figure 14. Note that this is the same placement as in *Case 1*. As in the first part, if an occurrence of π in ω^+ contained any entry of α other than a, this would violate the assumption that ρ started with an ascent. Moreover, the 1 of π cannot lie in γ, as there is no entry of γ lower than x. As before, d substitutes for x. This completes the proof under this assumption.

Case 2b: There exists an entry $c \in \gamma$ which is lower in value than d

Here we form ω^+ in a different way, by placing the new entry x just below d instead of just above d. See Figure 15 for a diagram of this placement. Suppose ω^+ contains an occurrence of π. Then, x must play a role in the ρ part of π, otherwise $\rho \leqslant \delta$. We proceed as in *Case 1*. If the 1 of π is in γ, then both x and d cannot be involved as they would form an increasing bond. Since no entry of α can be involved, d substitutes for x. Otherwise, if the 1 of π is a, then x is the second entry in the occurrence of π and the third entry lies above and to the right of d (because ρ starts with an ascent). So, again, d substitutes for x. This completes the proof of *Case 2*.

□

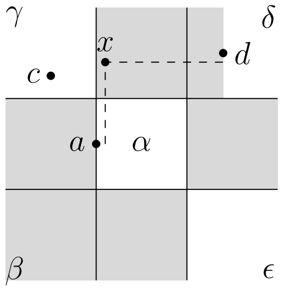


Figure 15: The diagram corresponding to the second part of *Case 2* in the proof of Theorem 3.4.

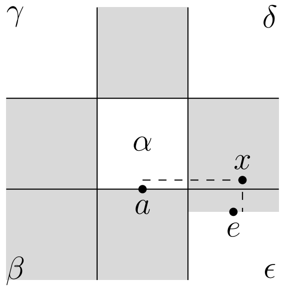


Figure 16: The diagram corresponding to an initial placement of x in Theorem 3.5.

The above theorem handles all cases in which $\pi = 1 \oplus \rho$, where ρ starts with an ascent and does not simultaneously have both kinds of bonds. We next handle the case in which ρ starts with a descent and has no increasing bond. For convenience, we say that π satisfies condition (‡) if:

there is at least one entry to the right of 2 that is less than the leftmost entry of ρ (‡)

In other words π satisfies condition (‡) if, in one line notation, $\pi = 1c \dots 2 \dots b \dots$ for some $b < c$.

Theorem 3.5. *Suppose that $\pi = 1 \oplus \rho$ satisfies condition (‡), starts with a descent and has no increasing bond. Then,* $\mathrm{Av}(\pi)$ *is not deflatable.*

Proof. Note that $\pi^{-1} = 1 \oplus \rho^{-1}$. If ρ^{-1} starts with an ascent, then we can appeal to the previous cases (as $\mathrm{Av}(\pi)$ is deflatable if and only if $\mathrm{Av}(\pi^{-1})$ is). Thus, we can assume that ρ^{-1} starts with a descent. In terms of π, this implies that 3 precedes 2. If 2 is the third entry of π, it follows that the first three entries of π are 132, and since $|\pi| > 3$, this implies that π is a three-component sum, and thus is handled by Theorem 3.2. Therefore, we can assume that π has at least three entries preceding 2 (at least two of which are part of ρ).

Let $\omega \in \mathrm{Av}(\pi)$ be indecomposable, under the same assumptions as in previous proofs. Suppose that ϵ is non-empty. Let e be the topmost entry in ϵ and let a be the bottommost entry in α. Form ω^+ by inserting an entry x just to the right of e and just above a as in Figure 16. Suppose this introduces an occurrence of π. It follows that x plays a role in the ρ part of π. If the 1 of π is in ϵ then the e cannot be involved, otherwise e and x form an increasing bond. This would allow e to substitute for x. So, the 1 must be in α, and in fact the only possibility is that a plays the role of the 1.

Since a plays the role of the 1, the role of the 2 (the least element of ρ) must be played by x. If an occurrence of π is created, condition (‡) forces all entries of π other than 1 and 2 to be in δ. Pick the leftmost (lexicographically least by position) possibilities for these entries of π. An example is given in Figure 17 with $\pi = 153264$.

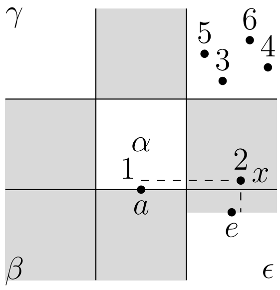


Figure 17: A diagram showing a possible occurrence of π in Theorem 3.5.

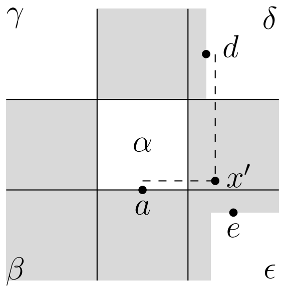


Figure 18: The new placement of a splitting entry in the proof of Theorem 3.5.

In this case, there can be no entry in ϵ which lies to the left of the entry which played the role of the first entry of ρ (in the example above, ϵ has no entry which lies to the left of the entry marked "5"): otherwise that entry would have played the 1 in an occurrence of π in ω which involved e as the 2 and the same remaining entries in δ. Let d be the leftmost entry of δ and place a new splitting entry x' just above a and just to the right d (which may or may not be one of the entries in the occurrence of π), as in Figure 18.

Now, we have forced a to play the role of the 1 in any occurrence of π which involves x'. Again, x' must play the role of the 2 and the remaining entries of π would have to be in δ. However, since we have already shown that three entries of π must precede 2, this is impossible. Therefore, the introduction of x' does not introduce an occurrence of π, and indeed $\omega^+ \in \mathrm{Av}(\pi)$.

If ϵ were actually empty, then the splitting created by x' works for the same reason. □

We continue under the assumption that $\pi = 1 \oplus \rho$ where ρ starts with a descent and has no increasing bond. Assume that π does not satisfy condition (‡); that is, assume that the first entry of ρ is less than every entry to the right of the entry 2. Moreover, we can assume that π^{-1} also fails (‡). In terms of π, this translates to the property that the entry 2 precedes every entry which has value greater than the first entry of ρ.

After a little inspection, one can see if π and π^{-1} both fail condition (‡), then either π is a sum of three or more components or π has the form $1n \cdots 2$ (by this, we do not mean that ρ is decreasing, just that ρ starts with its greates entry and ends with its least entry). The former case is already proved, so we only need to prove the latter.

Theorem 3.6. *Suppose that $\pi = 1 \oplus \rho$ is of the form $1n \cdots 2$ and ρ has no increasing bond. Then* $\mathrm{Av}(\pi)$ *is not deflatable.*

Proof. Let $\omega \in \mathrm{Av}(\pi)$ be indecomposable, under the same assumptions as in previous proofs. Let a be the topmost entry of α, let d be the leftmost entry of δ, and let e

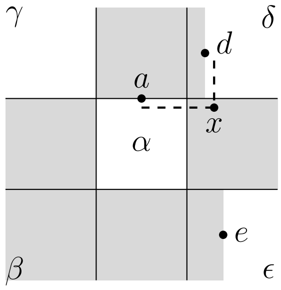


Figure 19: A diagram corresponding to ω^+ in the first case of Theorem 3.6.

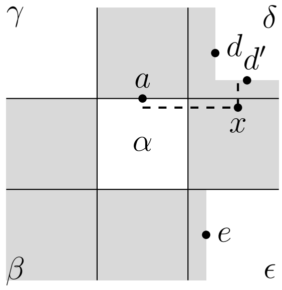


Figure 20: A diagram corresponding to ω^+ in the second case of Theorem 3.6.

be the leftmost entry of ϵ (if it exists). We proceed in two cases. Suppose first that ϵ is empty or that d precedes e. Place a splitting entry x just to the right of d and just below a, as in Figure 19.

In this case, x must play a role in the ρ part of π. Therefore the 1 of any occurrence of π is in α (and is not a). If x is not the 2, then there is no place for the 2. Only a and d can be entries of π other than the 1 or 2. Therefore, $\pi = 132$ (a known case) or $\pi = 1342$ (not of the form $1n \cdots 2$). Hence, this case is complete.

Suppose instead that e precedes d. Let d' be the bottommost entry in δ (it is possible that $d = d'$). Place the splitting entry x just below a and just to the left of d', as in Figure 20. Suppose there is an occurrence of π. Then, x must play a role in the ρ of such an occurrence.

If the 1 of this occurrence is in α and a is not involved, then since x and d' cannot both be involved, d' can substitute for x. Thus a must be involved. If x is not the 2, then there is no place to put the 2. This forces x to be the 2 and a to be the 3. However, unless $\pi = 132$, there is no place now for the biggest entry of π.

If the 1 of this occurrence is in ϵ, the we can substitute d' for x. This completes the second case, and the proof. □

We have now disposed of the case $\pi = 1 \oplus \rho$ where ρ starts with a descent and has no increasing bond. The last case we handle is $\pi = 1 \oplus \rho$ where ρ starts with a descent and has no decreasing bond. As before, we can further assume that ρ^{-1} starts with a descent, i.e., that 3 precedes 2 in π. The two proofs below largely mirror the previous two proofs, with some small changes in the easy cases.

Theorem 3.7. *Suppose that $\pi = 1 \oplus \rho$ satisfies condition (‡), and that ρ starts with a descent and has no decreasing bond. Then* Av(π) *is not deflatable.*

Proof. Let $\omega \in \text{Av}(\pi)$ be indecomposable, under the same assumptions as in previous proofs. If ϵ is empty we can use the same splitting construction as in the proof of Theorem 3.5. So, assume now that ϵ is non-empty. Let a be the bottommost entry of α, let d be the leftmost entry of δ, and let e be the topmost entry of ϵ. We handle two separate cases. First assume that e precedes d. In this case, we must have that e is not also the leftmost entry of ϵ (which we call e'), or else α is not a maximal

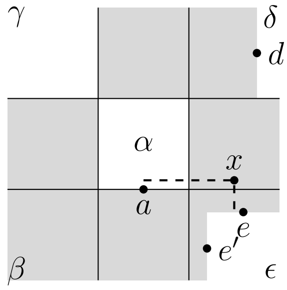


Figure 21: A diagram corresponding to ω^+ in the first case of Theorem 3.7.

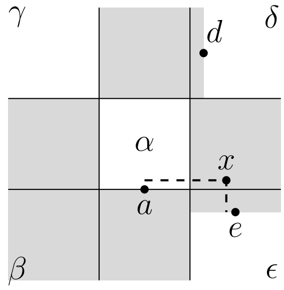


Figure 22: A diagram corresponding to ω^+ in the second case of Theorem 3.7.

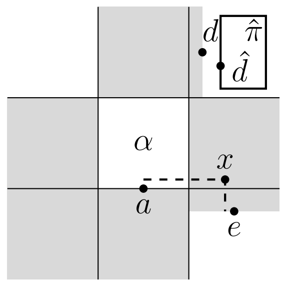


Figure 23: A diagram corresponding to ω^+ in the second case of Theorem 3.7.

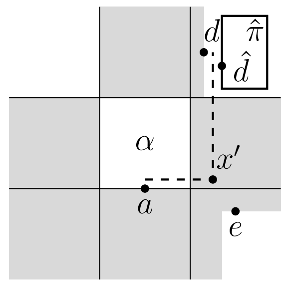


Figure 24: A diagram corresponding to ω^+ in the second case of Theorem 3.7.

interval. Place a splitting entry x just above a and just to the left of e, as shown in Figure 21.

If there is an occurrence of π in ω^+, then the 1 of π either lies in α or ϵ. If the 1 of π lies in ϵ, then we can substitute e for x since both cannot be involved. If the 1 is in α, then a is the 1 of π and x is the 2 of π. Since π satisfies condition (‡), all other entries must be in δ, but then the first two entries π are 12, a contradiction.

Suppose instead that d precedes e. Place a splitting entry x just above a and just to the left of e, as in Figure 22. Again, if the 1 of an occurrence of π is in ϵ, then we can substitute e for x. Thus, the 1 is in α, and a is the 1 of π and x is the 2 of π. Since π satisfies condition (‡), all other entries of π lie in δ. Define $\hat{\pi} = \pi \setminus \{a, x\}$ and let $\hat{d}$ be the leftmost entry of $\hat{\pi}$. It is possible that $\hat{d} = d$. See Figure 23.

If there is an entry z in ϵ that precedes $\hat{d}$, then z and e can together play the same roles as a and x, creating an occurrence of π in ω. If not, then place a splitting entry x' just to the right of d and just above a, as in Figure 24. If x' creates another occurrence of π, then we must have that a is the 1 of π and x' is the 2 of π. Since we know that 3 precedes 2, it follows that π is a three-component sum, and we can appeal to Theorem 3.2. □

Lastly, we consider the case in which π fails condition (‡). As before, we may also assume that π^{-1} fails condition (‡), leaving us only with the case $\pi = 1n \cdots 2$,

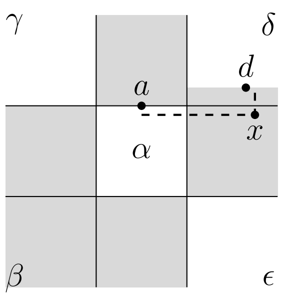


Figure 25: A diagram corresponding to ω^+ in Theorem 3.8.

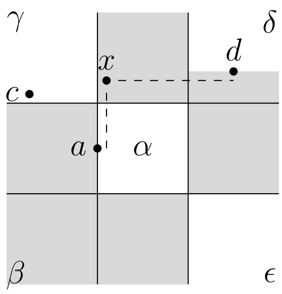


Figure 26: A diagram of the first case in the proof of Theorem 3.9

where π has no decreasing bond.

Theorem 3.8. *Let π have the form $1n\cdots 2$, such that π has no decreasing bond. Then* $\mathrm{Av}(\pi)$ *is not deflatable.*

Proof. Let $\omega \in \mathrm{Av}(\pi)$ be indecomposable, under the same assumptions as in previous proofs. Let a be the topmost entry in α and let d be the bottommost entry of δ. Place a splitting point x just below a and just to the right of d, as in Figure 25.

If there is an occurrence of π and a is not involved, then d can substitute for x, as both cannot be involved simultaneously. Hence, a must be involved, and there is no entry of π in ϵ. Moreover, a cannot be the 1 in an occurrence of π, since x must be involved. Therefore, a must be the n, forcing x to be the $n-1$. It follows there is no allowed location for any other entries which play a role in π. Hence $\pi = 132$. As 132 is known to be deflatable, this is a contradiction. □

Theorems 3.2-3.8 tell us that if a principal class $\mathrm{Av}(\pi)$ is deflatable for sum-decomposable π, then π must have the form $1 \oplus \rho$, where ρ is sum-indecomposable and contains both an increasing and decreasing bond. However, as the next theorem shows, it is possible that π can have these properties and $\mathrm{Av}(\pi)$ still be non-deflatable.

Theorem 3.9. *Let $\pi = 1 \oplus \rho$ for ρ of the form $x \cdots 1$ with $x \neq 2$ and $x \neq |\rho|$, i.e., $\pi = 1z \cdots 2$ with $z \neq 3$ and $z \neq |\pi|$. Then,* $\mathrm{Av}(\pi)$ *is not deflatable.*

Proof. Let $\omega \in \mathrm{Av}(\pi)$ be indecomposable, under the same assumptions as in previous proofs. Let a be the leftmost entry of α and let d be the bottommost entry in δ. We proceed in two separate cases.

First suppose that there is an entry c in γ which lies below d. In this case, insert an entry x just to the right of a and just below d (see Figure 26). Suppose that an occurrence of π is created. If x is the last entry of this π, then d substitutes for x, contradicting the assumption that $\omega \in \mathrm{Av}(\pi)$. So suppose that x is not the last entry of the occurrence of π.

If a is the 1 of π, then x must be the second entry of π, which is not the biggest entry of π. Therefore, π must contain an entry larger than x to the right, and hence

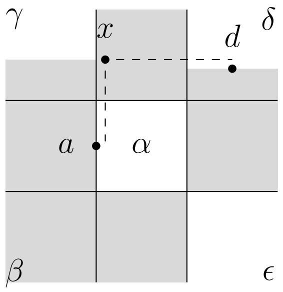


Figure 27: A diagram of the second case in the proof of Theorem 3.9

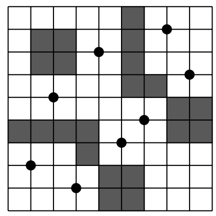


Figure 28: The permutation diagram of the permutation 25173486 ∈ Av(251364).

larger than d and to the right of α. However, the final entry of π must be smaller than x and larger than a, and there is no such entry. If π starts with an entry in γ, then since x is not the last entry of an occurrence of π, there is no place for the last entry of π anywhere. This completes the first case.

Now assume otherwise, that no entry in γ lies below d, as in Figure 27. Place a new entry x just to the right of a and just above d. An occurrence of π cannot involve x as the 1, since otherwise the remainder of the pattern would lie in the upper right quadrant, and so the entry a must play the role of the 1.

Thus, x plays the role of the first entry of ρ, which is neither the biggest nor the smallest entry in ρ. However, the last entry of π is 2, and now we see that there is no place for the 2. □

The theorems above combine to prove the non-deflatability of all classes $\text{Av}(\pi)$ for $|\pi| = 4$ (up to symmetry) with the exception of Av(2413), a special case which we now prove.

Proposition 3.10. *The principal class* Av(2413) *is not deflatable.*

Proof. Let $\omega \in \text{Av}(2413)$ be indecomposable and not simple. Let α be a longest maximal interval.

We would like to make the assumption that the entry immediately following α by position has value greater than all entries in α, i.e, belongs to δ. First we observe that the permutation 2413 is invariant under the four symmetries of the square corresponding to rotations – so to make this assumption without loss of generality it suffices to show that we can find an image of ω under one of those rotations for which it is satisfied.

If any of β, γ, δ and ϵ are empty, then by rotation we can ensure that ϵ is empty and δ is not. If all four regions are occupied, then if it were never the case that for any one of the four rotations the element immediately following (the image of) α belonged to δ then those four elements of ω whose images did immediately follow the image of α under one of those rotations would form a copy of 2413, a contradiction.

So we may assume that the entry immediately following α by position, which we denote by d, has value greater than all entries in α. Define d' to be the rightmost

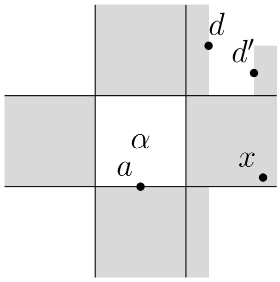


Figure 29: The permutation diagram of ω^+ in Proposition 3.10.

entry of ω which is to the right of d and separates α from d (by value). If there is no such entry, set $d' = d$. Let a be the least entry of α.

Form ω^+ by inserting an entry x into ω that lies just above a and just to the right of d', as in Figure 29. We need to show that $\omega^+ \in \mathrm{Av}(2413)$, so suppose toward a contradiction that the entry x plays a role in an occurrence of π.

If x played the role of the 2 in an occurrence of 2413, then a substitutes for x. If x played the role of the 4 in an occurrence of 2413, then d' substitutes for x. If x plays the role of the 3 in an occurrence of 2413, then the 1 must lie to the left of α (otherwise it acts as the 1 in an occurrence of 2413 using a, d, and d'), and so a substitutes for x.

Therefore, x must play the role of the 1 in some occurrence of 2413. The entry d cannot play the role of the 4, because then there are no entries that can play the role of the 3. The role of 4 also cannot be played by any entry to the left of α, because then a could substitute for x. Therefore, the role of 4 must be played by an entry, say y, above d and positionally between d and d'. However, this would imply that the role of 3 was played by an entry, say z above d and to the right of x. This in turn implies that d, y, d', and z form a copy of 2413, a contradiction. □

4 Deflatable Permutation Classes

Given the results of the previous section, one may wonder whether any principal classes are deflatable other than Av(12), Av(231) and their symmetries. For the larger group of finitely-based classes, the answer is clear: any class with finitely many simple permutations (and infinitely many permutations) must be deflatable, and there are infinitely many such classes. Moreover, the recent successes referred to in the introduction make use of the fact that many classes $\mathrm{Av}(\alpha, \beta)$ with $|\alpha| = |\beta| = 4$ turn out to be deflatable. In this section, we first provide a criterion by which we may prove deflatability of $\mathrm{Av}(\pi)$. We use this criterion to show examples of deflatable classes $\mathrm{Av}(\pi)$ for which π is decomposable, simple, or neither.

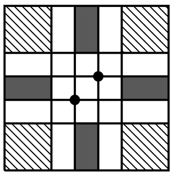
In any deflatable class $\mathcal{C}$, there are permutations τ which cannot be extended to a simple permutation, i.e., there exists no simple $\sigma \in \mathcal{C}$ such that $\sigma \geqslant \tau$. Therefore, if we find a τ with this property in a class $\mathcal{C}$, it follows that $\mathcal{C}$ is deflatable. We call such a τ a *witness of deflatability*.

In this section, we represent permutations by their diagrams, as shown in Section 1 and as produced by PermLab [1]. A square in a permutation diagram is shaded gray if inserting an entry in that square would create a forbidden pattern.

The lemma we now prove aids in finding witnesses of deflatability.

Lemma 4.1. *Let $\omega \in \mathcal{C}$ contain a bond which is not breakable. Then, ω cannot be extended to a simple permutation in $\mathcal{C}$.*

Proof. The figure below gives an example of the configuration ω in question, where the diagonally shaded quadrants may contain entries. The result is not entirely trivial for, though the bond cannot be broken by a single additional element it is not immediately clear that one might not place a sequence of elements splitting it, but not separated from it (e.g., in the square immediately above it) leading eventually to a configuration in which the interval they define could be broken.



Let $\omega \in \mathcal{C}$ be as in the statement of the lemma. Assume without loss of generality that the bond of interest is an increasing bond; the proof follows, mutatis mutandis, when the bond is a decreasing bond. Let $\widehat{\omega} \in \mathcal{C}$ contain ω, and fix an occurrence of ω in $\widehat{\omega}$. In the rest of the proof we refer to this occurrence as ω. Let ν be the maximal box in $\widehat{\omega}$ which contains the two points from the bond of ω, and which is cut by no other point of ω. Note that we could replace these two points by any pair of points that form the pattern 12 inside ν and still have an occurrence of ω.

Since ν has at least one pair of increasing entries, we know that its skew-decomposition (which may have only one summand) has at least one non-trivial skew-indecomposable summand, which we will call θ. Note that it is possible that $\theta = \nu$. We show that $\widehat{\omega}$ is not simple by showing that θ is an interval of $\widehat{\omega}$

Suppose toward a contradiction that $\widehat{\omega}$ has an entry x which cuts θ. Then, x does not lie in ν because θ was chosen as an interval of ν. So x must lie in one of the four regions adjacent to ν, and separated from ν by an element of ω; without loss of generality, we assume that x lies in the region above ν. It follows that every entry of θ which lies to the left of x is greater in value than every entry of θ which lies to the right of x; otherwise, x would lie in the forbidden region defined by the embedded occurrence of ω. This contradicts the assumption that θ is skew-indecomposable. Hence, θ is an interval of length greater than 1 and thus $\widehat{\omega}$ is not simple. □

We can now proceed to identify a number of deflatable principal classes. For example, consider the diagram of the permutation 25173486 ∈ Av(251364), as shown in Figure 28. By Lemma 4.1, the permutation 25173486 is a witness of deflatability for the class Av(251364), proving that Av(251364) is deflatable. We list below a sporadic collection of deflatable classes and witnesses which prove their deflatability. These witnesses were found through a mixture of computer search and "by hand" construction.

Permutation Class	Witness of Deflatability
Av(134652)	6 8 9 3 4 1 10 14 7 13 5 12 11 2
Av(246135)	4 7 2 9 11 5 6 1 10 3 8
Av(246513)	5 9 3 11 8 2 10 6 7 1 4
Av(251364)	2 5 1 7 3 4 8 6
Av(251463)	2 6 1 8 4 3 7 9 5
Av(254613)	5 9 3 11 2 8 10 6 7 1 4
Av(256413)	4 7 9 2 10 8 5 6 1 3
Av(1523764)	11 18 14 16 8 19 6 7 22 13 1 10 5 24 2 3 9 17 23 4 21 20 15 12
Av(2613475)	2 6 1 3 9 4 5 7 10 8
Av(2631574)	2 6 3 1 9 5 4 8 10 7

One should first note that the classes Av(134652) and Av(1523764) are listed in the above table. That these classes are deflatable proves that, in fact, not all classes of the form Av(π) for decomposable π are non-deflatable. But of course the basis elements of both classes have both increasing and decreasing bonds.

Many of the other basis elements of classes in the list are simple. It is of particular interest that the class Av(246135) is deflatable, as it is a special type of simple permutation: a *parallel alternation*, i.e., a permutation of the form $246\cdots(2n)135\cdots(2n-1)$ or a symmetry of such a permutation. In fact, the classes Av(2 4 6 8 1 3 5 7), Av(2 4 6 8 10 1 3 5 7 9), Av(2 4 6 8 10 12 1 3 5 7 9 11), and Av(2 4 6 8 10 12 14 1 3 5 7 9 11 13) are also deflatable, as shown by the witnesses

5 8 11 2 13 4 14 16 18 9 10 6 1 15 17 3 7 12,
2 7 10 13 4 16 9 18 6 20 8 22 24 14 15 11 1 19 21 3 23 5 12 17,
3 8 13 16 5 19 9 12 21 2 7 23 11 25 27 29 17 18 14 1 22 24 4 26 6 28 10 15 20, and
3 8 12 16 20 5 23 9 13 18 25 2 7 27 11 29 14 31 33 35 21 22 17 1 26 28 4 30 6 32 10 15 34 19 24,

respectively. This leads to the following conjecture.

Conjecture 4.2. *Let π be a parallel alternation with $|\pi| \geqslant 6$. Then,* Av(π) *is deflatable.*

There is one parallel alternation (up to symmetry) of length less than 6: the permutation 2413. We showed in Section 3 that Av(2413) is not deflatable.

We conclude this section by generalizing the deflatable class Av(251364) to an infinite family of deflatable classes. Set $\pi = 251364$ and consider the inflation $\pi^* = \pi[1, \theta, 1, 1, 1, 1]$ for any permutation θ. Set $\omega = 25173486$ (the witness of deflatability for the class Av(π)) and further define $\omega^* = \omega[1, \theta, 1, \theta, 1, 1, 1, 1]$ for the same θ as before. Both π^* and ω^* are shown in Figure 30.

It is fairly straight-forward to see that $\omega^* \in$ Av(π^*), and it is routine to check that any one-point extension of ω^* by an entry x which splits the interval formed by the entries 3 and 4 (without becoming a part of this interval) contains π^*. Hence, Av(π^*) is deflatable by Lemma 4.1, proving the following theorem.

Theorem 4.3. *There are infinitely many deflatable principal classes.*

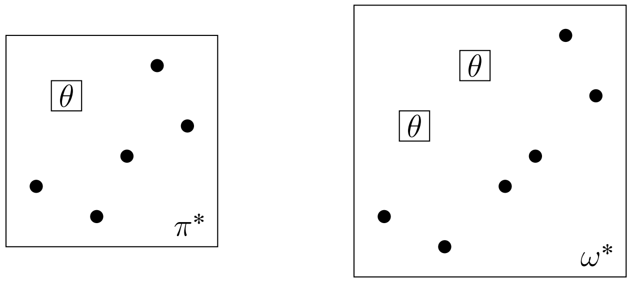


Figure 30: The permutations π^* (on the left) and ω^* (on the right).

5 Open Questions

Although we have shown that there are both infinitely many deflatable principal classes and infinitely many non-deflatable principal classes, the task of classifying exactly which principal classes are deflatable, to say nothing of non-principal classes, remains unfinished.

The results of Section 3 prove the non-deflatability of all classes $\mathrm{Av}(\pi)$ for $|\pi| = 4$. Of the classes $\mathrm{Av}(\pi)$ for $|\pi| = 5$, we have shown that $\mathrm{Av}(\pi)$ is not deflatable for all decomposable π. The remaining classes $\mathrm{Av}(\pi)$ to be checked are Av(25314), Av(24153), Av(23514), and Av(24513). Note that the former two bases consist of simple permutations while the latter two consist of inflations of 2413. This raises the following question:

Question 5.1. *Are the classes* Av(25314)*,* Av(24153)*,* Av(23514)*, and* Av(24513) *deflatable?*

It is already known that $\pi = 134652$ is a minimal length decomposable π such that $\mathrm{Av}(\pi)$ is deflatable. The resolution to Question 5.1 would determine whether or not π is a minimal length such π among all permutations. Moreover, there are three other length 6 decomposable permutations π (up to symmetry) such that the deflatability of $\mathrm{Av}(\pi)$ is unknown. The answer to the Question 5.2 might be helpful in determining a broader classification of deflatable and non-deflatable classes.

Question 5.2. *Are the classes* Av(146523)*,* Av(154623)*, and* Av(164532) *deflatable?*

The reader may have noticed that, despite proving that, for many principal classes, their set of simple permutations is actually contained in a proper subclass, we have not once specified what that proper subclass is. This is not out of neglect; rather, the only way currently known to conjecture the smallest proper subclass $\mathcal{D} \subset \mathcal{C}$ for which $\mathcal{C} \subseteq \langle \mathcal{D} \rangle$ is by direct calculation. For the time being, computational power is not sufficient to perform this calculation for the classes in question.

Acknowledgments

The authors are grateful to Vince Vatter for participating in discussions which furthered this research. In particular he was in part responsible for the original proof of Proposition 3.10 which convinced us that "except in trivial cases principal classes aren't deflatable" was perhaps not as obvious or as easy as one might initially think—and indeed of course we now know it to be false. Additionally, Cheyne Homberger and Jay Pantone wish to thank Michael Albert and Mike Atkinson for their hospitality at the University of Otago in March and April of 2014.

References

[1] M. H. Albert, PermLab: Software for permutation patterns, http://www.cs.otago.ac.nz/PermLab/, 2014.

[2] M. H. Albert and M. D. Atkinson, Simple permutations and pattern restricted permutations, *Discrete Math.* 300 (1-3) (2005), 1–15.

[3] M. H. Albert, M. D. Atkinson and R. Brignall, The enumeration of permutations avoiding 2143 and 4231, *Pure Math. Appl. (PU.M.A.)* 22(2) (2011), 87–98.

[4] M. H. Albert, M. D. Atkinson and R. Brignall, The enumeration of three pattern classes using monotone grid classes, *Electron. J. Combin.* 19(3) (2012), #P20.

[5] M. H. Albert, M. D. Atkinson, R. Brignall, N. Ruškuc, R. Smith and J. West, newblock Growth rates for subclasses of Av(321), *Electron. J. Combin.* 17(1) (2010), #P141.

[6] M. H. Albert, M. D. Atkinson and M. Klazar, The enumeration of simple permutations, *J. Integer Seq.* 6 (2003), Article 03.4.4.

[7] M. H. Albert, M. D. Atkinson and V. Vatter, Inflations of geometric grid classes: three case studies, *Australas. J. Combin.* 58 (2014), 27–47.

[8] M. D. Atkinson, B. E. Sagan and V. Vatter, Counting (3+1)-avoiding permutations, *European J. Combin.* 33(1), 49–61.

[9] M. Bóna, C. Homberger, J. Pantone and V. Vatter, Pattern-Avoiding Involutions: Exact and Asymptotic Enumeration, *arXiv:1310.7003* (2013).

[10] J. Pantone, The Enumeration of Permutations Avoiding 3124 and 4312, *arXiv:1309.0832* (2013).

[11] R. Simion and F. W. Schmidt, Restricted permutations, *European J. Combin.* 6(4) (1985), 383–406.

(Received 14 Oct 2014; revised 31 Mar 2015)