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Degree diameter problem on triangular networks

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Abstract

The degree diameter problem involves finding the largest graph (in terms of number of vertices) subject to constraints on the degree and the diameter of the graph. Beyond the degree constraint there is no restriction on the number of edges (apart from keeping the graph simple) so the resulting graph may be thought of as being embedded in the complete graph. In a generalisation of this problem, the graph is considered to be embedded in some connected host graph. This article considers embedding the graph in the triangular grid and provides some exact values and some upper and lower bounds for the optimal graphs. Moreover, all the optimal graphs are 2-connected, without this constraints no larger graphs were found.

1 Introduction

In this paper we consider simple undirected graphs only. For definitions and notations not defined here we refer to [1]. Let $G(V, E)$ be a graph with $V(G)$ the vertex set and $E(G)$ the set of edges of G. For $x \in V(G)$, $N_G[x]$ denotes the closed neighbourhood of x in G, i.e., a set of all vertices of G adjacent to x in G together with vertex x

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itself. Analogously, for $A \subset V(G), N_G[A]$ denotes the closed neighbourhood of A, i.e., the set of all vertices of G adjacent to at least one vertex of A in G together with the set A itself. For $x, y \in V(G)$, the distance between x and y in G is denoted by $\text{dist}_G(x, y)$ and the diameter of G is $\max\{\text{dist}_G(x, y)\}$ over all pairs $x, y \in V(G)$. For a given graph G and positive integers Δ and $D, N_G(\Delta, D)$ denotes the number of vertices of a maximal subgraph of G (in terms of number of vertices) with given maximum degree Δ and given diameter D.

Recent articles have studied $N_G(\Delta, D)$ for the multidimensional rectangular mesh [2, 5] and the multidimensional hexagonal grid [3] as host graphs. In this paper we consider the triangular grid as the host graph, partly in response to the motivation from [3] wherein the triangular grid is the dual of the hexagonal grid and partly because the triangular grid is an important structure in its own right. The dual concept can be observed by construction; replace each hexagon in the hexagonal grid by a point and join points with an edge where their corresponding hexagons are adjacent in the hexagonal grid. The resulting structure has $\Delta = 6$ (and 6 regular for the infinite grid) and is traditionally represented as a mesh of equilateral triangles though this is not required. A simple compression along any of the three axes (horizontal for example) would result in a grid of isosceles triangles. Arbitrary compression along all three axes could result in a grid of scalene triangles, though these deformations of the triangular grid are beyond the scope of this paper.

Cellular phone networks are generally represented as a hexagonal grid, especially in areas of dense coverage such as in cities. Analysis within these networks, including maintenance and fault finding is often simplified by considering only the adjacencies in the network so leading to analysis in the triangular environment.

A further feature of triangular networks is that, while the vertices are 6 regular, the faces are 3 (edge) regular. Security schemes proposed for sensitive communication networks such as in military or in confidential industrial research projects often treat a secure cell surrounded by 'protector' cells whose sole aim is to counter intruders or malware.

Many applications of triangulations of the plane involve Delaunay triangulations of Voronoi cells. These differ from the triangular grid by virtue of the triangles assuming shapes other than those given by deformations along the grid axes. The triangular grid is a useful tool for many researches in wireless network theory when considering grid based deployment. In [6] the authors reveal that deployment of wireless sensor networks on a triangular grid shows resilience to horizontal misalignment. In [7] a grid of non overlapping triangles is used to model sensor network security in unattended and hostile environments. The authors in [4] employ the triangular grid to study their reverse carpooling algorithm to minimise energy consumption on a shared wireless grid network. Although this grid may be a simplified version of the real life network structures, it is a useful tool for analysis and the investigation of strategies.

The remainder of the paper is structured as follows. In Section 2 we discuss the case when $\Delta = 6$ and in Section 3 we consider the cases when $\Delta = 3, 4, 5$ separately.

2 Values for $\Delta = 6$

In what follows, any reference to the triangular grid, whether infinite or not, will be taken to mean a grid of equilateral triangles in the Euclidean plane.

Proposition 2.1 *Let D be an even positive integer, let T_D be a maximum connected subgraph of the infinite triangular grid of diameter D. Then*

$$|V(T_D)| = \frac{3}{4}D^2 + \frac{3}{2}D + 1.$$

Proof: The maximum subgraph T_D of the infinite triangular grid with (even) diameter D corresponds to an interior of a bounding regular hexagon (including the hexagon) centered on a vertex arbitrarily selected to be the origin (or center), say x. In Figure 1, see the graphs of even diameter and note that the center x is represented by $\otimes$. Consider an infinite triangular grid T. Colour all the horizontal lines of T with colour a, all the lines going from the top left corner to the bottom right corner with colour b and all the remaining lines of T with colour c. Then the corner vertices of the bounding hexagon are the vertices at distance $D/2$ from x, that trace a single colour to the origin.

The number of vertices of T_D is given by

$$\begin{aligned}|V(T_D)| &= 6\frac{D}{2} + 6\frac{D-2}{2} + 6\frac{D-4}{2} + \cdots + 6\frac{2}{2} + 1 \\ &= 3D + 3(D-2) + 3(D-4) + \cdots + 6 + 1 \\ &= \frac{6+3D}{2}\frac{D}{2} + 1 \\ &= \frac{3}{4}D^2 + \frac{3}{2}D + 1.\end{aligned}$$

■

Proposition 2.2 *Let D be an odd positive integer, let T_D be a maximum connected subgraph of the infinite triangular grid of diameter D. Then*

$$|V(T_D)| = \frac{3}{4}D^2 + \frac{3}{2}D + \frac{3}{4}.$$

Proof: The maximum subgraph T_D of the infinite triangular grid with (odd) diameter D corresponds to a bounding non-regular hexagon centered on the centroid of an arbitrarily chosen unit triangle. The vertices of this triangle form the centre of the graph. See, in Figure 1, subgraphs of odd diameter. Colour the 3 edges of the triangle with distinct colours and extend the colours through the grid so that the paths of a single colour form a series of parallel lines through the triangular grid. Then the corner vertices of the bounding hexagon are the vertices, at distance

$(D-1)/2$ from the central triangle that trace a single colour to the nearest central vertex.

The number of vertices of T_D is given by

$$\begin{aligned} |V(T_D)| &= 3\frac{2D}{2} + 3\frac{2(D-2)}{2} + 3\frac{2(D-4)}{2} + \cdots + 3\frac{2}{2} \\ &= 3D + 3(D-2) + 3(D-4) + \cdots + 3 \\ &= \frac{3+3D}{2}\frac{D+1}{2} \\ &= \frac{3}{4}D^2 + \frac{3}{2}D + \frac{3}{4}. \end{aligned}$$

■

In Fig. 1, the graphs T_D are depicted for $D = 3, 4, 5, 6$. The central vertices or triangles are emphasized by $\otimes$.

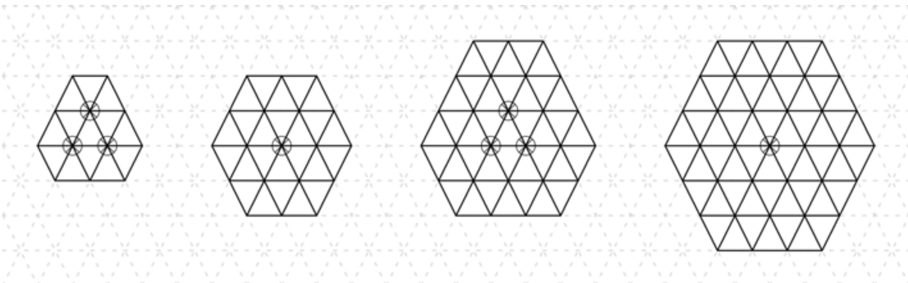


Figure 1: The graphs T_D for $D = 3, 4, 5, 6$.

As an immediate consequence of the previous two propositions we obtain the following statement.

Corollary 2.3 *Let D be a positive integer, let T be the infinite triangular grid. Then*

$$N_T(6, D) = \begin{cases} \frac{3}{4}D^2 + \frac{3}{2}D + \frac{3}{4} & \text{for odd } D, \\ \frac{3}{4}D^2 + \frac{3}{2}D + 1 & \text{for even } D. \end{cases}$$

Note that these values are also trivial upper bounds on $N_T(\Delta, D)$ for $\Delta \le 5$. This also implies that a maximum subgraph G_D of T corresponds to a maximum subgraphs of T_D.

3 Values for $\Delta \le 5$

3.1 Values for $\Delta = 1, 2$

Since for $\Delta \ge \Delta(T)$ we have $N_T(\Delta, D) = |V(T_D)|$, we consider only cases when $\Delta \le 5$. If $\Delta = 1$ then we easily get $N_T(1, D) = 2$, since the maximum subgraph of

T of maximum degree 1 and diameter D is K_2. Now we investigate the case when $\Delta = 2$.

Proposition 3.1 *Let T be the infinite triangular grid, let D be a positive integer. Then $N_T(2, D) = 2D + 1$.*

PROOF: Clearly T_D contains a cycle for each $D > 0$. Thus a maximum graph of diameter D and maximum degree 2 is a cycle of length $2D+1$. For even D, consider a horizontal row of triangles containing the central vertex of T_D. The border cycle of this row contains $2D + 1$ vertices, implying that, for even D, such a subgraph of T_D exists. Similarly for odd D, consider a horizontal row of triangles containing the central triangle of T_D. The border cycle of this row has $2D + 1$ vertices, implying that such a subgraph of T_D exists also for odd D. ■

3.2 Values for $\Delta = 5$

Theorem 3.2 *Let T be the infinite triangular grid and let D be an even positive integer. Then $N_T(5, D) = |V(T_D)| - 1$.*

PROOF: Let G_D denote a maximum induced subgraph of T (which is the same as a maximum subgraph of T_D) of maximum degree 5 and diameter D. First we show that $|V(G_D)| \leq |V(T_D)| - 1$. On the contrary suppose that $|V(G_D)| = |V(T_D)|$ (clearly G_D is a subgraph of T_D and hence $|V(G_D)| \leq |V(T_D)|$). Consider the border hexagon of T_D. We cyclically denote the vertices of this hexagon by $x_1, \ldots, x_6$. Clearly $\text{dist}_G(x_i, x_{i+3}) = D$, $i = 1, 2, 3$ and the only x_i, x_{i+3}-path in T_D of length D is the diagonal x_i, x_{i+3}-path. But this implies that the central vertex of T_D has degree six, a contradiction.

Now we consider a subgraph G_D of T depicted in Fig. 2 (left), where $\square$ denotes the vertex of the closed ball T_D which does not belong to G_D. Note that the central vertex x of G_D corresponds to the central vertex of T_D and is depicted by $\otimes$, and that G_D has a regular rhombic structure except for a subgraph induced by $N_T[x]$. Clearly $\Delta(G_D) = 5$ and $|V(G_D)| = |V(T_D)| - 1$. From the structure of G_D one can see that the distance between any vertex of G_D and the central vertex x is the same in G_D as in T_D except the vertices emphasized by bullets. But for them, the distance from x is at most $\frac{D}{2}$, implying that the diameter of G_D is at most D. ■

Now we focus on odd diameter D.

Theorem 3.3 *Let T be the infinite triangular grid and let D be an odd positive integer. Then $N_T(5, D) = |V(T_D)|$.*

PROOF: Let G_D denote the maximum induced subgraph of T of maximum degree 5 and diameter D as depicted in Fig. 2 (right). Note that the central triangle T_1 of

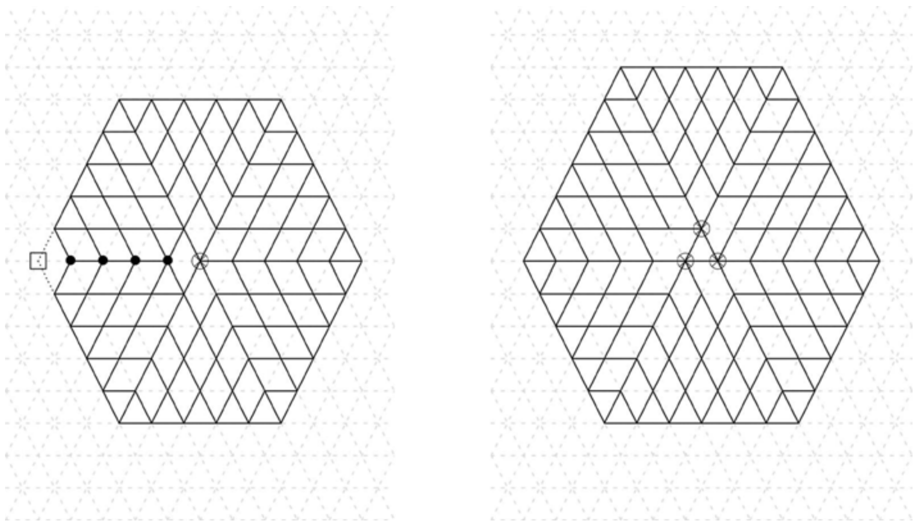


Figure 2: The structures of G_D for $\Delta = 5$

G_D corresponds to the central triangle of a corresponding graph T_D and its vertices are depicted by $\otimes$. Note also that G_D has a regular rhombic structure except for a subgraph induced by $N_T[V(T_1)]$. Clearly $\Delta(G_D) = 5$ and $|V(G_D)| = |V(T_D)|$. From the structure of G_D one can see that the distance between any vertex of G_D and a closest vertex of the central triangle is the same as in the graph T_D. Thus the diameter of G_D is D. ■

3.3 Values for $\Delta = 4$

Now we focus on the case when $\Delta = 4$. Again we need to consider parity of D. We start with even D.

Theorem 3.4 *Let T be the infinite triangular grid, let D be an even positive integer. Then $N_T(4, D) = |V(T_D)| - 2$.*

PROOF: Let G_D denote the maximum subgraph of T of maximum degree 4 (and diameter D) i.e., $N_T(4, D) = V(G_D)$. First we show that $N_T(4, D) \leq |V(T_D)| - 2$. On the contrary suppose that $N_T(4, D) \geq |V(T_D)| - 1$. Consider the border hexagon of T_D. If all of the vertices of the border hexagon of T_D belong to G_D, then (as in the proof of Theorem 3.2) $d_{G_D}(x) = 6$, a contradiction. Thus there is a vertex, say vertex z, of the border hexagon of T_D which does not belong to G_D, and, by the assumption, $\{z\} = V(T_D) \setminus V(G_D)$. Two pairs of diametrically opposed vertices of the border hexagon, which belong to G_D, must be connected by a path going through the central vertex since the diameter of G_D is D. Now we consider, along with z, the remaining corner vertex y of the border hexagon which belongs to G_D. Let z' denote the neighbour of z in T_D lying on the shortest path in T_D connecting y and z. Since the degree of the central vertex x in G_D is 4, $\text{dist}_{G_D}(x, y) > \frac{D}{2}$ and $\text{dist}_{G_D}(x, z') \geq \frac{D}{2}$, implying that $\text{dist}_{G_D}(y, z') > D$, a contradiction. Therefore $N_T(4, D) \leq |V(T_D)| - 2$.

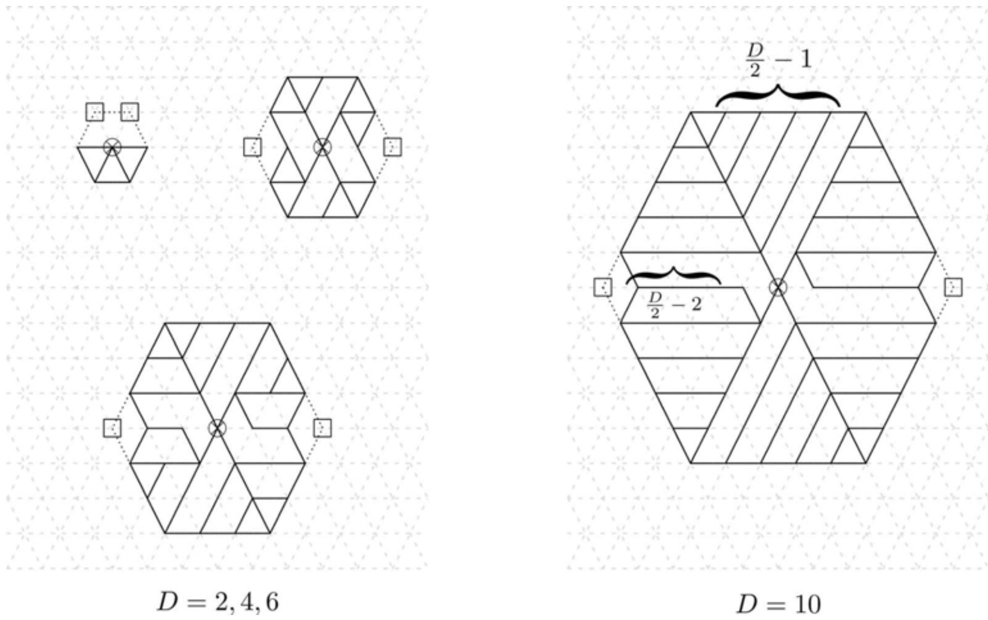


Figure 3: The structures of G_D for $\Delta = 4$ and even D.

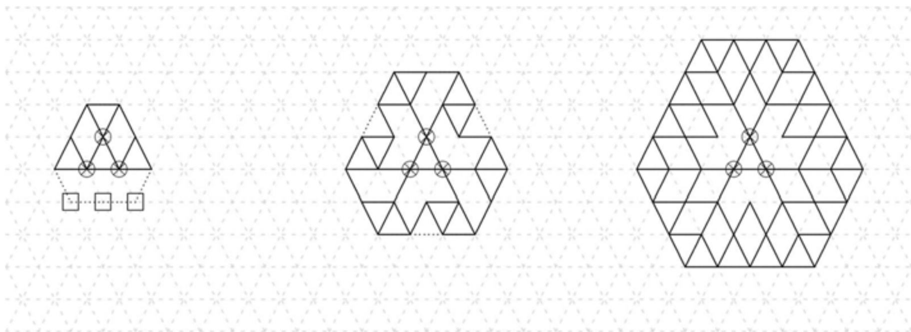
Now we show that $N_T(4, D) \geq |V(T_D)| - 2$. Consider a graph G_D depicted in Fig. 3 right, where $\square$ denotes the vertices of T_D which do not belong to G_D. Note that the central vertex x of G_D corresponds to the central vertex of T_D and is depicted by $\otimes$. Clearly $|V(G_D)| = |V(T_D)| - 2$ and $\Delta(G_D) = 4$. It remains to show that the diameter of G_D is D. We show that the distance between any vertex of G_D and the central vertex x is at most $\frac{D}{2}$. It is easy to see that $\text{dist}_{G_D}(x, y) = \text{dist}_{T_D}(x, y)$ for every vertex of G_D except those vertices belonging to the horizontal diagonal of the hexagon. But clearly, the distance is greater by one in G_D than in T_D, hence we satisfy the condition that $\text{dist}(x, y) \leq \frac{D}{2}$ for every $y \in V(G_D)$. Note that the graph G_D is symmetric about its central vertex x. ■

For odd D and $\Delta = 4$ we can prove the following theorem.

Theorem 3.5 *Let T be the infinite triangular grid and let D be an odd positive integer. Then*

$$\begin{cases} N_T(4, D) \geq 9 & \textit{when } D = 3, \\ N_T(4, D) = |V(T_D)| & \textit{otherwise.} \end{cases}$$

Proof: For $D = 1$, $G_D \simeq T_D$ since T_1 is a triangle. For $D = 3, 5, 7$, a largest subgraph of T with $\Delta = 4$ is depicted in Fig. 4. Clearly the graphs G_1, G_3 and G_5 have the required maximum degree and diameter. For $D \geq 7$, we consider a structure shown in Fig. 5. The whole graph G_D arises from three copies of depicted structure rotated by 0, 120 and 240 degrees with center of rotation in the middle of the central triangle. Note that the central triangle T_1 of G_D corresponds to the central triangle of T_D and its vertices are depicted by $\otimes$. Note also that G_D has a regular rhombic structure except for a subgraph induced by $N_T[V(T_1)]$ and, for $D = 7$, the graphs depicted in Fig. 4 and 5 are the same. Clearly $\Delta(G_D) = 4$ and $|V(G_D)| = |V(T_D)|$. From the structure of G_D one can see that the distance between any vertex of G_D and any vertex of the central triangle is the same in G_D as in T_D except for the vertices emphasized by bullets. However for these vertices, the distance from the central triangle is 3, implying that the diameter of G_D is at most D for $D \geq 7$. ■



$D = 3, 5, 7$

Figure 4: The structures of G_D for $\Delta = 4$ and small odd D.

3.4 Bounds for $\Delta = 3$

Finally we investigate the case when $\Delta = 3$. For even D we prove the following statement.

Theorem 3.6 *Let T be the infinite triangular grid and let D be an even positive integer. Then*

$$N_T(3, D) \geq \begin{cases} 5 & \text{when } D = 2, \\ 11 & \text{when } D = 4, \\ 22 & \text{when } D = 6, \\ 46 & \text{when } D = 8, \\ |V(T_D)| - 6 & \text{otherwise.} \end{cases}$$

Proof: Graphs G_D depicted in Fig. 6 clearly have maximum degree at most 3. One can also check that the distance between any vertex of G_D and its central vertex is at most $\frac{D}{2}$. Hence for $D = 2, 4, 6, 8$ the theorem holds.

Now we consider a structure depicted in Fig. 7. Note that this structure for $D = 10$ is the same as is depicted in Fig. 6 right. The whole subgraph G_D arises from three copies of depicted structure rotated by 0, 120 and 240 degrees with center of rotation at the central vertex x of G_D. In Fig. 7, $\square$ denotes vertices of T_D which do not belong to G_D and dashed edges mean the edges we add to the structure to make the subgraph G_D 2-connected. Note that the central vertex x of G_D corresponds to the central vertex of T_D and is depicted by $\otimes$. Clearly the maximum degree of G_D is

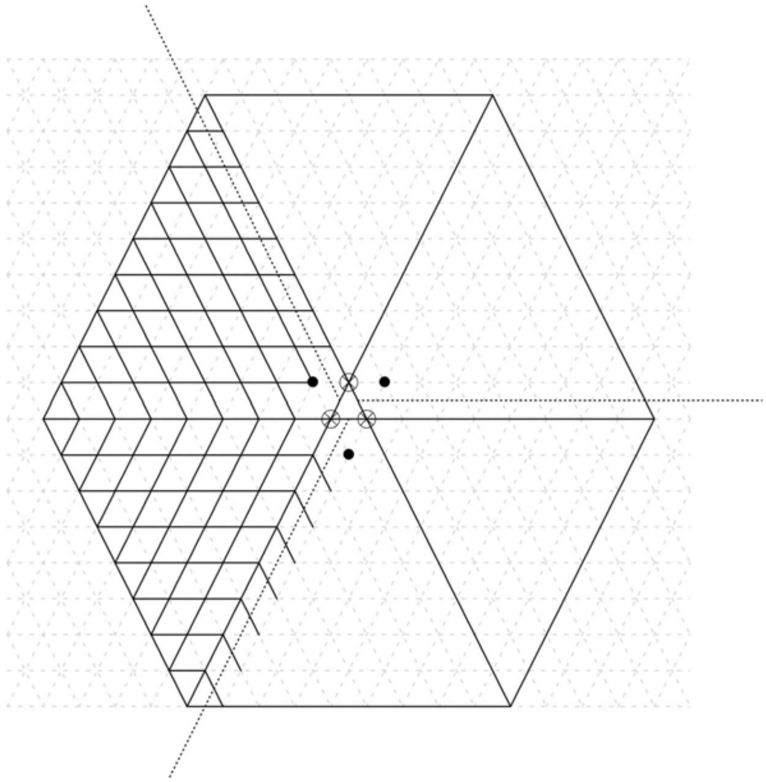


Figure 5: The structures of G_D for $\Delta = 4$ and large odd D.

three. Hence it remains to show that the diameter of G_D is D. We again prove that $\mathrm{dist}_{G_D}(x, y) \leq \frac{D}{2}$ for every vertex y of G_D. As can be seen, all the vertices of G_D, except the vertices emphasized by bullets, have the same distance from x in G_D as in T_D. And, all the vertices emphasized by bullets are at distance at most $\frac{D}{2}$ from x. Hence the diameter of G_D is at most D. ■

For odd D we prove the following statement.

Theorem 3.7 *Let T be the infinite triangular grid and let D be an odd positive integer. Then*

$$N_T(3, D) \geq \begin{cases} 7 & \textit{when } D = 3, \\ 14 & \textit{when } D = 5, \\ 24 & \textit{when } D = 7, \\ 48 & \textit{when } D = 9, \\ 96 & \textit{when } D = 11, \\ |V(T_D)| - 9 & \textit{otherwise.} \end{cases}$$

Proof: Graphs G_D depicted in Fig. 8 clearly have maximum degree at most 3. One can also check that the distance between any vertex of G_D and its central triangle is at most $\frac{D-1}{2}$. Hence for $D = 3, 5, 7, 9, 11$ the theorem holds.

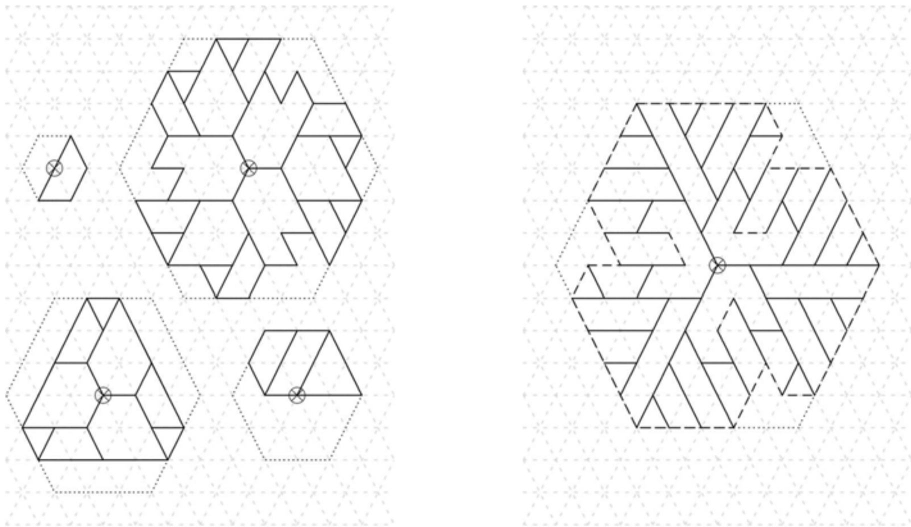


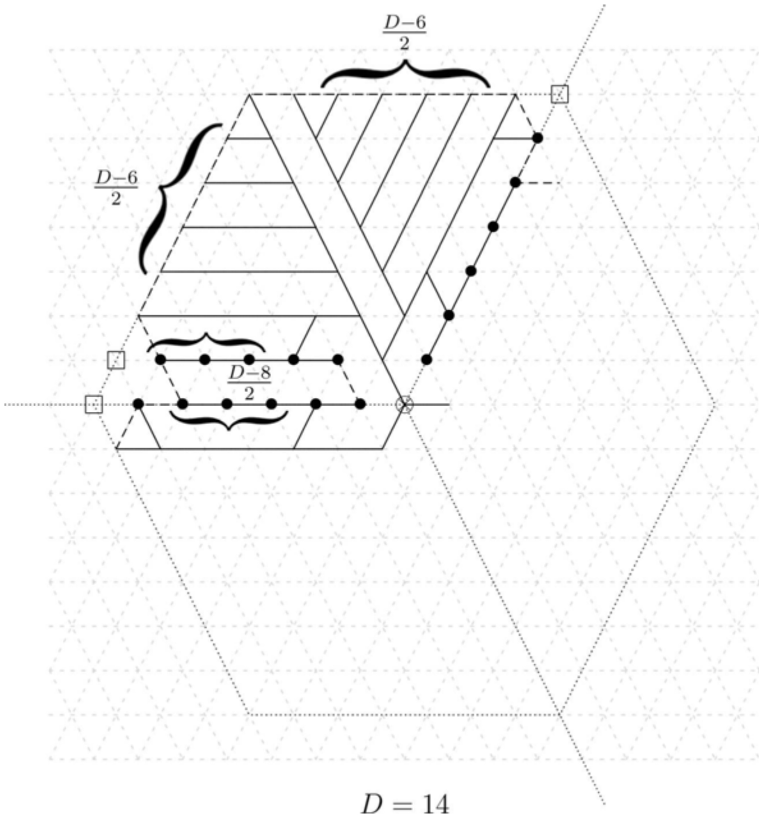
Figure 6: The structures of G_D for $\Delta = 3$ and small even D.

Now we consider a structure depicted in Fig. 9. The whole subgraph G_D arises from three copies of depicted structure rotated by 0, 120 and 240 degrees with center of rotation in the middle of the central triangle of G_D. In Fig. 9, $\square$ denotes vertices of T_D which do not belong to G_D, and dashed edges depict the edges added to the structure to make G_D 2-connected. Note that the central triangle of G_D corresponds to the central triangle of T_D and its vertices are depicted by $\otimes$. Clearly the maximum degree of G_D is three. Hence it remains to show that the diameter of G_D is D. As can be seen, all the vertices of G_D, except those vertices depicted by bullets, have the same distance from the central triangle in G_D as in T_D. And, all the remaining vertices are at distance at most $\frac{D-1}{2}$ from the central triangle in G_D. Hence the diameter of G_D is at most D. ■

As an immediate consequence of Theorems 3.6, 3.7 and of the fact that $N_G(\Delta - 1, D) \leq N_G(\Delta, D)$ for any graph G, we obtain the following statement.

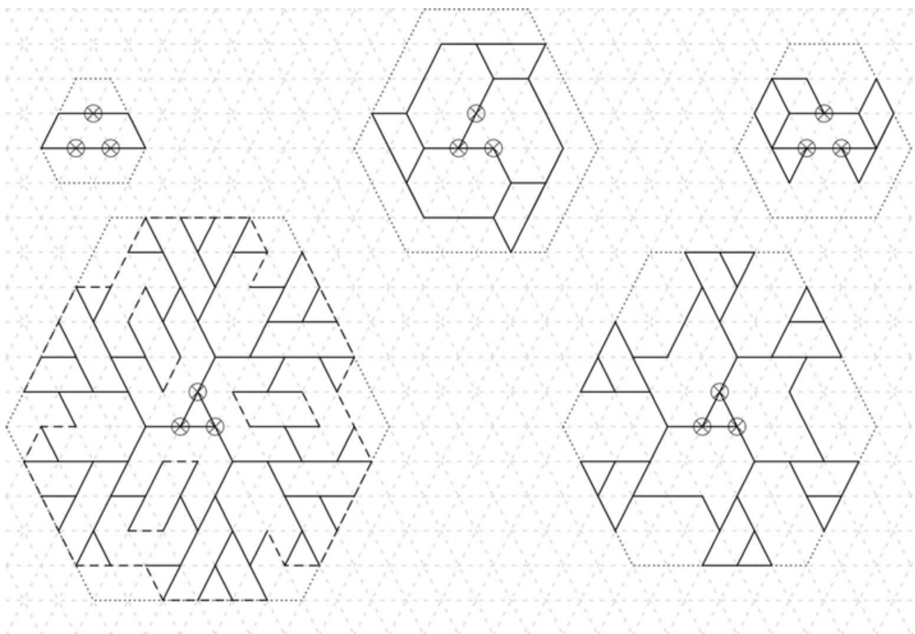
Corollary 3.8 *Let T be the infinite triangular grid and let $D \geq 12$ be a positive integer. Then*

$$\begin{array}{ll} |V(T_D)| - 6 \leq N_T(3, D) \leq |V(T_D)| - 2 & \textit{when } D \textit{ is even,} \\ |V(T_D)| - 9 \leq N_T(3, D) \leq |V(T_D)| & \textit{when } D \textit{ is odd.} \end{array}$$



$D = 14$

Figure 7: The structures of G_D for $\Delta = 3$ and large even D.



$D = 3, 5, 7, 9, 11$

Figure 8: The structures of G_D for $\Delta = 3$ and small odd D.

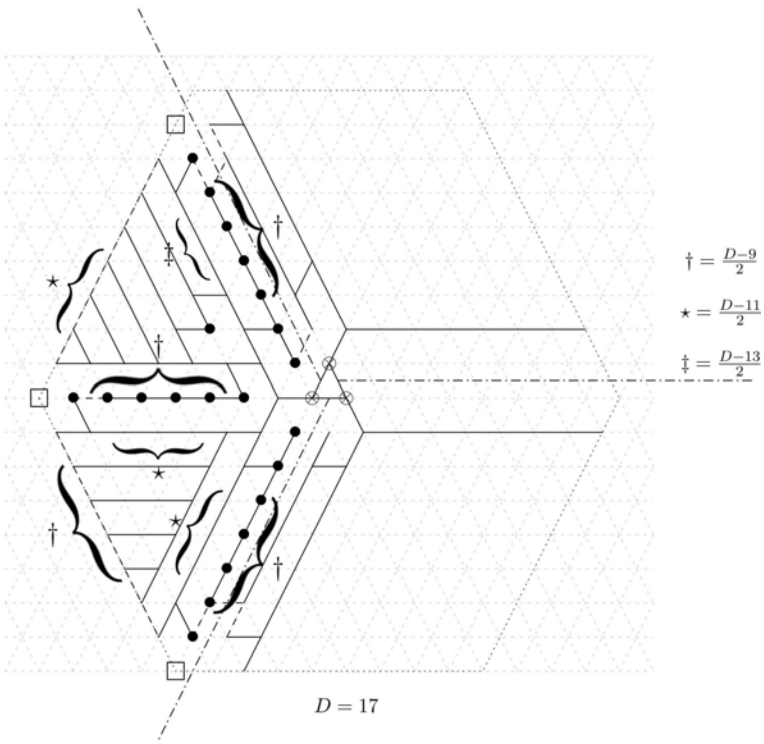


Figure 9: The structures of G_D for $\Delta = 3$ and large odd D.

Remarks

As it was mentioned, all the optimal subgraphs for $\Delta = 2, 3, 4, 5$ are 2-connected. The prescribed structures in Figures 7 and 9 contain maximal subgraphs of given diameter and $\Delta = 3$ and dashed edges which make the structures 2-connected. Without restriction on 2-connectivity we are not able to find larger subgraphs.

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