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A Bollobás-type theorem for affine subspaces

Gábor Hegedüs

Antal Bejczy Center for Intelligent Robotics
Kiscelli utca 82
Budapest, H-1032
Hungary

Abstract

Let W denote the n-dimensional affine space over the finite field $\mathbb{F}_q$. We prove here a Bollobás-type upper bound in the case of the set of affine subspaces. We give a construction of a pair of families of affine subspaces, which shows that our result is almost sharp.

1 Introduction

First we introduce some notation.

In the following let $q = r^\alpha$ be a fixed prime power, $n \geq 1$ be a nonnegative integer. Let W denote the n-dimensional affine space over the finite field $\mathbb{F}_q$.

Bollobás proved in [2] the following famous result.

Theorem 1.1 *Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be two families of sets such that $A_i \cap B_j = \emptyset$ if and only if $i = j$. Then*

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$

In particular if $|A_i| = r$ and $|B_i| = s$ for each $1 \leq i \leq m$, then

$$m \leq \binom{r+s}{r}.$$

The following strengthening of the uniform version of Bollobás's theorem was proved by Lovász in [4] using tensor product methods.

Theorem 1.2 *If $\mathcal{F} = \{A_1, \ldots, A_m\}$ is an r-uniform family and $\mathcal{G} = \{B_1, \ldots, B_m\}$ is an s-uniform family such that*

$$(a)\ A_i \cap B_i = \emptyset$$

for each $1 \leq i \leq m$ and

$$(b)\ A_i \cap B_j \neq \emptyset$$

if $i < j$ $(1 \le i, j \le m)$*, then*

$$m \le \binom{r+s}{r}.$$

Lovász also proved the following generalization of Bollobás' theorem for subspaces of a vector space in [5]:

Theorem 1.3 *Let* $\mathbb{F}$ *be an arbitrary field and* V *be an* n*-dimensional vector space over the field* $\mathbb{F}$*. Let* $U_1, \dots, U_m$ *denote* r*-dimensional subspaces of* V *and* $V_1, \dots, V_m$ *denote* s*-dimensional subspaces of the vector space* V*. Assume that*

$$(a)\ U_i \cap V_i = \{\underline{0}\}$$

for each $1 \le i \le m$ *and*

$$(b)\ U_i \cap V_j \ne \{\underline{0}\}$$

whenever $i < j$ $(1 \le i, j \le m)$*. Then*

$$m \le \binom{r+s}{r}.$$

In the following we give an affine version of Theorem 1.3.

We say that a pair of families of affine subspaces $(A_i, B_i)_{1 \le i \le m}$ of W is *cross–intersecting* if

$$1.\ A_i \cap B_i = \emptyset,$$

for each $1 \le i \le m$ and

$$2.\ A_i \cap B_j \ne \emptyset$$

whenever $i < j$, $(1 \le i, j \le m)$.

Let $m(n, q)$ denote the maximal size of a cross–intersecting pair of families of affine subspaces $(A_i, B_i)_{1 \le i \le n}$.

Our main result is the following modification of Lovász' Theorem 1.3:

Theorem 1.4 *Let* $A_1, \dots, A_m$ *and* $B_1, \dots, B_m$ *be affine subspaces of an* n*-dimensional affine space* W *over the finite field* $\mathbb{F}_q$*, where* $q \ne 2$*. Assume that* $(A_i, B_i)_{1 \le i \le m}$ *is cross–intersecting. Then*

$$m \le q^n + 1,$$

Remark. Theorem 1.4 means that

$$m(n, q) \le q^n + 1.$$

Remark. Our result is a strengthening of Theorem 1.2 in the case of affine hyperplanes.

In Section 2 we prove Theorem 1.4. In the proof we use the polynomial subspace method (see [1]).

In Section 3 we give a simple construction, which shows that $m(n, q) \ge \frac{q^n - 1}{q - 1}$.

Finally in Section 4 we collect some open problems.

2 The proof of the main result

We use the following obvious observation in our proof.

Proposition 2.1 *The intersection of a family of affine subspaces is either empty or equal to a translate of the intersection of their corresponding vector subspaces.* □

Recall that our main result was the following:

Theorem 2.2 *Let $A_1, \dots, A_m$ and $B_1, \dots, B_m$ be affine subspaces of an n-dimensional affine space W over the finite field $\mathbb{F}_q$, where $q \neq 2$. Assume that $(A_i, B_i)_{1 \le i \le m}$ is cross–intersecting. Then*

$$m \le q^n + 1,$$

Proof. Let p be an arbitrary, but fixed prime divisor of $q-1$. Since $q \neq 2$, hence $p > 1$. We can assign for each subset $F \subseteq \mathbb{F}_q^n$ its characteristic vector $\underline{v_F} \in \{0,1\}^{q^n} \subseteq \mathbb{F}_p^{q^n}$ such that $\underline{v_F}(s) = 1$ iff $s \in F$. Here $\underline{v_F}(s)$ denotes the s^{th} coordinate of the vector $\underline{v_F}$.

Let $1 \le j \le m$ be fixed. Let $\underline{v_j} = (\underline{v_j}(1), \dots, \underline{v_j}(q^n))$ denote the characteristic vector of the affine subspace A_j and let $\underline{w_j} = (\underline{w_j}(1), \dots, \underline{w_j}(q^n))$ denote the characteristic vector of the affine subspace B_j. Here $\underline{v_j}(i)$ denotes the i^{th} coordinate of the vector $\underline{v_j}$. Similarly, $\underline{w_j}(i)$ denotes the i^{th} coordinate of the vector $\underline{w_j}$.

Consider the polynomials

$$P_i(x_1, \dots, x_{q^n}) := 1 - (\sum_{k=1}^{q^n} \underline{v_j}(k) x_k) \in \mathbb{F}_p[x_1, \dots, x_{q^n}]$$

for each $1 \le i \le m$.

We claim that the polynomials $\{P_i : 1 \le i \le m\}$ are linearly independent functions over $\mathbb{F}_p$. Namely

$$P_i(\underline{w_i}) = 1 - \sum_{k=1}^{q^n} \underline{v_i}(k)\underline{w_i}(k) = 1 - |A_i \cap B_i| = 1$$

and

$$P_i(\underline{w_j}) = 1 - \sum_{k=1}^{q^n} \underline{v_i}(k)\underline{w_j}(k) = 1 - |A_i \cap B_j| = 1 - q^t, \tag{1}$$

where $t \ge 0$, because $(A_i, B_i)_{1 \le i \le m}$ is a cross–intersecting pair of families of affine subspaces and hence we can apply Proposition 2.1. Since

$$q \equiv 1 \pmod{p},$$

thus

$$1 - q^t \equiv 0 \pmod{p}. \tag{2}$$

Consider a linear combination

$$\sum_{r=1}^{m} \lambda_r P_r = 0,$$

where $\lambda_r \in \mathbb{F}_p$. It is easy to prove that $\lambda_r = 0$ for each $1 \leq r \leq m$. Namely for contradiction, suppose that there exists a nontrivial linear relation

$$\sum_{s=1}^{m} \lambda_s P_s = 0. \tag{3}$$

Let s_0 be the smallest s such that $\lambda_s \neq 0$. Substitute $\underline{w_{s_0}}$ for the variable of each side of (3). Then by equations (1) and (2), all but the s_0^{th} term vanish, and what remains is

$$\lambda_{s_0} P_{s_0}(\underline{w_{s_0}}) = 0.$$

But $P_{s_0}(\underline{w_{s_0}}) \neq 0$ implies that $\lambda_{s_0} = 0$, a contradiction. Hence the polynomials $P_1, \ldots, P_m$ are linearly independent functions over $\mathbb{F}_p$.

We infer that the linearly independent polynomials $\{P_1, \ldots, P_m\}$ are in the $\mathbb{F}_p$-space spanned by the monomials

$$\{x^u \in \mathbb{F}_p[x_1, \ldots, x_{q^n}] : \ \deg(x^u) \leq 1\}.$$

Clearly

$$|\{x^u : \ \deg(x^u) \leq 1\}| \leq q^n + 1,$$

hence

$$m \leq q^n + 1,$$

which was to be proved. □

3 A simple construction

We use in our contruction the following simple proposition.

Proposition 3.1 *Let F_j be arbitrary affine subspaces for each $1 \leq j \leq m$. Let $G_j := \underline{\alpha_j} + F_j$, where $\underline{\alpha_j} \notin F_j$. Then $F_i \cap G_j \neq \emptyset$ iff $\underline{\alpha_j} \in F_i - F_j$.*

Proof. First suppose that $\underline{\alpha_j} \in F_i - F_j$. Then we can write $\underline{\alpha_j}$ into the form

$$\underline{\alpha_j} = \underline{f_i} - \underline{f_j},$$

where $\underline{f_i} \in F_i$ and $\underline{f_j} \in F_j$. Hence $\underline{f_i} = \underline{\alpha_j} + \underline{f_j} \in \underline{\alpha_j} + F_j = G_j$.

On the other hand, suppose that $F_i \cap G_j \neq \emptyset$. Let $\underline{v} \in F_i \cap G_j$, i.e., $\underline{v} \in F_i$ and $\underline{v} \in \underline{\alpha_j} + F_j$. Then there exists $\underline{f_j} \in F_j$ such that $\underline{v} = \underline{\alpha_j} + \underline{f_j}$by definition. Hence $\underline{\alpha_j} = \underline{v} - \underline{f_j} \in F_i - F_j$. □

Proposition 3.2 *Let $n \geq 1$ and q be an arbitrary prime power. Then*

$$m(n, q) \geq \frac{q^n - 1}{q - 1}.$$

Proof. Let $m = \frac{q^n-1}{q-1}$. We give a concrete cross–intersecting pair of families of affine subspaces $\{A_1, \ldots, A_m\}$ and $\{B_1, \ldots, B_m\}$ of an n-dimensional affine space W over the finite field $\mathbb{F}_q$. Let

$$\mathcal{H} = \{H_1, \ldots, H_m\}$$

denote an enumeration of the set of hyperplanes of the vector space $\mathbb{F}_q^n$. It is easy to see that $m = \frac{q^n-1}{q-1}$. For each $1 \leq i \leq m$ we fix a vector $\underline{\beta_i} \in \mathbb{F}_q^n \setminus H_i$. Define

$$A_i := H_i,$$

and

$$B_i := H_i + \underline{\beta_i}.$$

Clearly A_i, B_i are affine subspaces of W for each $1 \leq i \leq m$.

Since $\underline{\beta_i} \notin H_i$ for each $1 \leq i \leq m$, it follows that $A_i \cap B_i = \emptyset$ by the definition of A_i and B_i.

On the other hand, since $\underline{\beta_i} \in H_i - H_j = \mathbb{F}_q^n$, it follows from Proposition 3.1 that $A_i \cap B_j \neq \emptyset$ for each $1 \leq i < j \leq m$. □

4 Open problems

Here we collect some interesting open problems.

Open problem 1: What can we say about $m(n, 2)$?

Open problem 2: What is the precise value of $m(n, q)$, if $q > 2$?

Finally we conjecture the following projective version of Theorem 1.4:

Conjecture 1 *Let $\mathbb{F}$ be an arbitrary field. Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be projective subspaces of an n-dimensional projective space W over the field $\mathbb{F}$. Assume that $(A_i, B_i)_{1 \leq i \leq m}$ is cross-intersecting (i.e. $A_i \cap B_i = \emptyset$ for each $1 \leq i \leq m$ and $A_i \cap B_j \neq \emptyset$ whenever $1 \leq i < j \leq m$). Then*

$$m \leq 2^{n+1} - 2.$$

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