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Coincidence among families of mesh patterns

Anders Claesson*

Department of Computer and Information Sciences
University of Strathclyde, Glasgow
U.K.
anders.claesson@strath.ac.uk

Bridget Eileen Tenner†

Department of Mathematical Sciences
DePaul University, Chicago, IL
U.S.A.
bridget@math.depaul.edu

Henning Ulfarsson*

School of Computer Science
Reykjavik University, Reykjavik
Iceland
henningu@ru.is

Abstract

Two mesh patterns are coincident if they are avoided by the same set of permutations. In this paper, we provide necessary conditions for this coincidence, which include having the same set of enclosed diagonals. This condition is sufficient to prove coincidence of vincular patterns, although it is not enough to guarantee coincidence of bivincular patterns. In addition, we provide a generalization of the Shading Lemma (Hilmarsson et al.), a result that examined when a square could be added to the mesh of a pattern.

1 Introduction

The study of patterns in permutations has grown into a rich subfield of combinatorics. The original notion of classical patterns—a subsequence of symbols having a particular relative order—has been expanded to include variations like vincular

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patterns [1, 6], bivincular patterns [2], Bruhat-restricted patterns [10], and barred patterns [9]. Each of these flavors adds its own nuance to the "relative order" requirement of classical patterns. Mesh patterns, which encompass nearly all of the previously defined flavors were introduced by Brändén and the first author in [3].

The main questions about patterns concern how and when a pattern might be contained (or avoided) by arbitrary permutations. Two patterns π and σ, perhaps having different flavors, are *coincident* if the set of permutations that avoid π is equal to the set of permutations that avoid σ. Note that this is stronger than Wilf-equivalence, which requires that the permutations of size n that avoid π is equinumerous to the permutations of size n that avoid σ, for all n.

It is particularly interesting, and often surprising, when patterns of different types are coincident. The second author analyzed this phenomenon for vincular and barred patterns in [7]. In the current paper, we are concerned with the more general category of mesh patterns, and we analyze how much of a mesh is required to distinguish it from others. Put another way, we examine which portions of a mesh might be superfluous. The Shading Lemma of the third author together with Hilmarsson, Jónsdóttir, Sigurðardóttir, and Viðarsdóttir [5] was the first general result in this direction. It gives sufficient conditions for when a square can be added to a mesh. Here we will give a more powerful version of that lemma, which captures almost all coincidences of small mesh patterns.

Recent work of the second author described when a mesh pattern is coincident to a classical pattern [8]. This phenomenon is characterized by avoidance of a single configuration in the mesh, called an *enclosed diagonal*. In the present work, we show that enclosed diagonals do, in fact, provide a necessary condition for two mesh patterns to be coincident, but this property is not sufficient in general. On the other hand, we do prove that for some families $\mathcal{F}$ of mesh patterns, elements of $\mathcal{F}$ are coincident exactly when their enclosed diagonals coincide.

This paper is organized as follows. In Section 2, we recall the definition of a mesh pattern. In Section 3, previous results are reviewed. In Section 4, we give necessary conditions for mesh pattern coincidence. In Section 5, we show that common enclosed diagonals are sufficient to determine coincidence for vincular patterns. Examples of pattern families for which coincidence is not governed by enclosed diagonals are given in Section 6, including bivincular patterns. In Section 7, we establish the Simultaneous Shading Lemma, and future directions of research are discussed in Section 8.

2 Mesh patterns

The set $\mathfrak{S}_n$ consists of all bijections from $[1, n] = \{1, \ldots, n\}$ to itself. Each of these bijections is a *permutation*, and we can denote $w \in \mathfrak{S}_n$ by the word $w = w(1)w(2)\cdots w(n)$. Two sequences $a_1a_2\cdots a_k$ and $b_1b_2\cdots b_k$ of real numbers are said to be *order isomorphic* if $a_i < a_j$ if and only if $b_i < b_j$ for all $i, j \in [1, k]$.

Definition 2.1. Fix $p \in \mathfrak{S}_k$. The permutation $w \in \mathfrak{S}_n$ *contains* a p-pattern if there exist indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $w(i_1)w(i_2)\cdots w(i_k)$ is order isomorphic to p. The subsequence $w(i_1)w(i_2)\cdots w(i_k)$ is an *occurrence* of p in w. If w does not contain p, then w *avoids* p.

Definition 2.1 is the classical framework for permutation patterns. The following example demonstrates this classical containment and avoidance.

Example 2.2. The permutation 42135 contains five occurrences of the pattern 213, namely 425, 415, 435, 213, and 215. The permutation 42135 avoids the pattern 132.

A permutation $w \in \mathfrak{S}_n$ can also be represented graphically by plotting the points

$$G(w) = \{(i, w(i)) : i \in [1, n]\}.$$

Note that $G(w)$ is a subset of the Cartesian product $[1, n]^2 = [1, n] \times [1, n]$. Classical pattern containment and avoidance can be formulated in terms of these graphs. Namely, w contains a p-pattern if and only if $G(w)$ contains a copy of $G(p)$, as shown in the following examples.

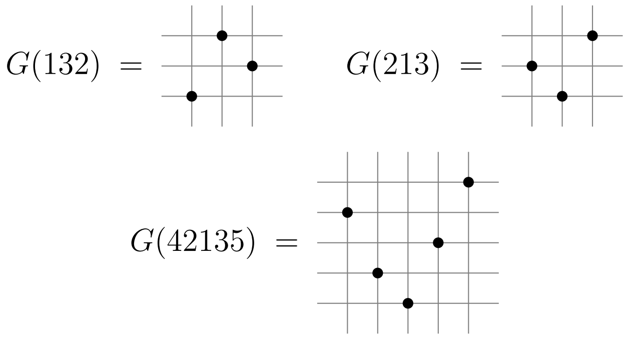


Figure 1: The permutations 132, 213, 42135 and their graphs.

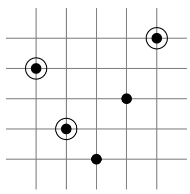


Figure 2: A particular occurrence of 213 in 42135.

Example 2.3. In Figure 1, we show the graphs $G(132)$, $G(213)$, and $G(42135)$. The graph $G(42135)$ contains $G(213)$ in five ways. These correspond to the five occurrences of 213 in 42135, as discussed in Example 2.2. The copy of $G(213)$ that corresponds to the occurrence 425 is marked in Figure 2. The graph $G(42135)$ does not contain $G(132)$, which corresponds to the fact that 42135 avoids the pattern 132.

Definition 2.4. A *mesh pattern* is an ordered pair (p, R) where $p \in \mathfrak{S}_k$ is a permutation, and R is a subset of the $(k+1)^2$ unit squares in $[0, k+1]^2$. The set R is the *mesh*, and elements (squares) of the mesh are indexed by their lower-left corners; that is, $(a, b) \in R$ refers to the square $[a, a+1] \times [b, b+1]$. The mesh pattern (p, R) is depicted graphically by drawing $G(p)$ and shading all squares of the mesh R.

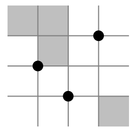
$\big(213, \{(0,3), (1,2), (1,3), (3,0)\}\big) =$ 

Figure 3: A mesh pattern.

To illustrate Definition 2.4, we give a particular mesh pattern in Figure 3.

For a permutation w to contain a mesh pattern (p, R), this w must contain an occurrence of the classical pattern p that does not "interfere" with the mesh. To clarify this interference requirement, we must first explain how the mesh of (p, R) appears in the graph $G(w)$ relative to a particular occurrence of p.

Definition 2.5. Let $w \in \mathfrak{S}_n$ and $p \in \mathfrak{S}_k$. Suppose that w contains a p-pattern in positions $i_1 < i_2 < \cdots < i_k$. By convention, we additionally set $i_0 = p(i_0) = 0$ and $i_{k+1} = p(i_{k+1}) = n+1$. Let $a, b \in [0, k]$. The square $[a, a+1] \times [b, b+1]$ in $G(p)$ *corresponds* to the following rectangle in $G(w)$:

$$\Big[i_a,\, i_{a+1}\Big] \times \Big[w\big(i_{p^{-1}(b)}\big),\, w\big(i_{p^{-1}(b+1)}\big)\Big].$$

Example 2.6. The permutation $w = 42135$ contains a 213-pattern in positions $\{1, 3, 5\}$. The square $[1, 2] \times [3, 4] \subset [0, 4]^2$ in $G(213)$ corresponds to the rectangle $[1, 4] \times [3, 5] \subset [0, 6]^2$ in $G(w)$. In Figure 4 this particular 213-pattern in w is marked by circles around the points, and the rectangle $[1, 4] \times [3, 5]$ has been outlined and shaded.

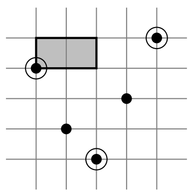


Figure 4: The permutation 42135 with a marked 213-pattern and the rectangle $[1, 4] \times [3, 5]$ outlined and shaded.

Definition 2.7. A permutation w *contains* a mesh pattern (p, R) if w contains an occurrence of p for which the subregion of $[0, n+1]^2$ corresponding to the mesh R contains no points of $G(w)$. If there is no such occurrence of p in w, then w *avoids* the mesh pattern (p, R).

It is important to observe that the subregion mentioned in Definition 2.7 depends on the particular occurrence of p in w.

Example 2.8. The permutation 42135 contains five different 213-patterns. On the other hand, it contains the mesh pattern $(213, \{(0,3), (1,2), (1,3), (3,0)\})$ in only four ways. These are depicted in Figure 5 where thick lines indicate how the four squares of the mesh appear in $G(42135)$.

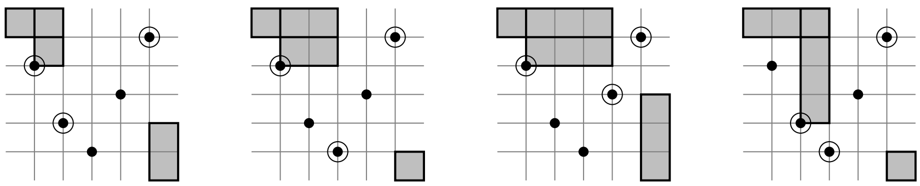


Figure 5: The 4 occurrences, in the permutation 42135, of the mesh pattern $(213, \{(0,3), (1,2), (1,3), (3,0)\})$.

The remaining occurrence of 213 in 42135 does not obey the restrictions of the mesh because the shaded region includes the point $(1,4) \in G(42135)$ as shown in Figure 6.

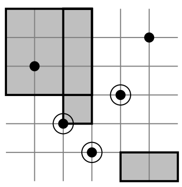


Figure 6: A 213-pattern in the permutation 42135 that is not an occurrence of the mesh pattern $(213, \{(0,3), (1,2), (1,3), (3,0)\})$.

Definition 2.9. For a pattern π of any flavor, let $\mathrm{Av}(\pi)$ be the set of permutations that avoid π and let $\mathrm{Cont}(\pi)$ be the set of permutations that contain π. Similarly, if Π is a collection of patterns, then $\mathrm{Av}(\Pi) = \bigcap_{\pi\in\Pi} \mathrm{Av}(\pi)$ and $\mathrm{Cont}(\Pi) = \bigcup_{\pi\in\Pi} \mathrm{Cont}(\pi) = \left(\bigcup_n \mathfrak{S}_n\right) \setminus \mathrm{Av}(\Pi)$.

The purpose of this article is to examine when the sets of Definition 2.9 coincide for two different mesh patterns. This is captured by the following definition.

Definition 2.10. Two patterns π and σ are *coincident* if $\mathrm{Av}(\pi) = \mathrm{Av}(\sigma)$, or, equivalently, if $\mathrm{Cont}(\pi) = \mathrm{Cont}(\sigma)$. This coincidence will be denoted $\pi \asymp \sigma$.

3 Mesh patterns can be coincident — examples and previous results

Mesh pattern coincidence is not a rare phenomenon, as demonstrated in Figure 7.

The property underlying this example is the main result of [8], which we will review in this section. This requires the notion of an enclosed diagonal.

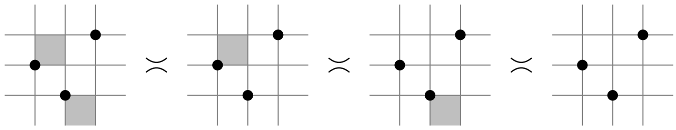


Figure 7: Mesh patterns coincident with a classical pattern.

Definition 3.1. Let (p, R) be a mesh pattern and let a, b, and c be nonnegative integers. Then $D = \big\{(a+i, b+i) : i \in [0, c]\big\} \subseteq R$ is an *enclosed NE-diagonal* if

$$\big\{(a+i, b+i) : i \in [0, c+1]\big\} \cap G(p) = \big\{(a+i, b+i) : i \in [1, c]\big\}$$

Similarly, $D = \big\{(a+i, b-i) : i \in [0, c]\big\} \subseteq R$ is an *enclosed SE-diagonal* if

$$\big\{(a+i, b+1-i) : i \in [0, c+1]\big\} \cap G(p) = \big\{(a+i, b+1-i) : i \in [1, c]\big\}.$$

If D has either of these types, then D can be called, simply, an *enclosed diagonal*. The *length* of an enclosed diagonal is $c+1$, and enclosed diagonals of length greater than one are *proper*. An enclosed diagonal of length 1 is both an enclosed NE-diagonal and an enclosed SE-diagonal, and such a square will be called *pointless* because it intersects no elements of the graph $G(p)$.

Definition 3.1 is illustrated in Figure 8. We note that our definition of enclosed notation differs from that of [8] because in the present setting, it is more natural to associate an enclosed diagonal with the shaded squares of the mesh.

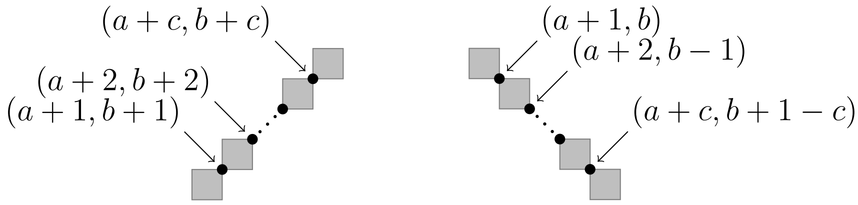
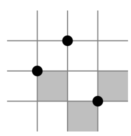


Figure 8: Proper enclosed diagonals in a mesh pattern (p, R). A point is in $G(p)$ if and only if it is marked •.

Example 3.2. The mesh pattern $\big(231, \{(1,1), (2,0), (3,1)\}\big)$ depicted below



has one enclosed diagonal, namely $\{(2,0), (3,1)\} \subset R$. This is an enclosed NE-diagonal and has length 2.

The next lemma follows immediately from the definition of enclosed diagonals and shows that these configurations in a mesh pattern are, in a sense, stable.

Lemma 3.3. *Let (p, R) be a mesh pattern.*

(a) If $R' \subseteq R$, then every enclosed diagonal in the mesh pattern (p, R') is an enclosed diagonal in (p, R).

(b) Each square in R belongs to at most one enclosed diagonal.

It will be useful to look at the collection of enclosed diagonals in a given mesh pattern (p, R), and we will denote this set

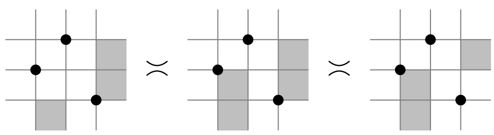
$$\mathsf{enc}(p, R).$$

Previous work by the second author characterized when an entire mesh is superfluous; that is, when a mesh pattern is equivalent to its underlying classical pattern. The coincidences in Figure 7, for example, are explained by this result.

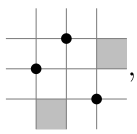
Theorem 3.4 ([8, Theorem 3.5])**.** *A mesh pattern (p, R) is coincident to a classical pattern if and only if $\mathsf{enc}(p, R) = \emptyset$. If this is the case, then $(p, R) \asymp p$.*

Theorem 3.4 suggests that enclosed diagonals are the crux to understanding coincidence between mesh patterns, and that perhaps mesh patterns are coincident if and only if they have the same enclosed diagonals. Unfortunately, as the following example shows, this is not the case.

Example 3.5. The coincidences



follow from the Shading Lemma ([5, Lemma 3.11]), reviewed below. Note, however, that these three mesh patterns in are not coincident to



the mesh pattern containing only the enclosed diagonals. This is because 42513 avoids the first three patterns but contains this last pattern, even though the enclosed diagonals are $\{(1, 0)\}$ and $\{(3, 2)\}$ in each case.

The third author and Hilmarsson, Jónsdóttir, Sigurðardóttir, and Viðarsdóttir have given sufficient conditions for when $(p, R) \asymp (p, R')$ where R and R' differ by only one square.

Lemma 3.6 (Shading Lemma (northeast) [5, Lemma 3.11])**.** *Let (p, R) be a mesh pattern. Suppose that the following conditions all hold.*

- *Neither the square $(i, p(i))$ nor the square $(i - 1, p(i) - 1)$ is in R.*

- *At most one of the squares* $(i, p(i) - 1)$ *and* $(i - 1, p(i))$ *is in* R*.*
- *For all* $x \notin \{i - 1, i\}$*, if* $(x, p(i) - 1) \in R$ *then* $(x, p(i)) \in R$*.*
- *For all* $y \notin \{p(i) - 1, p(i)\}$*, if* $(i - 1, y) \in R$ *then* $(i, y) \in R$*.*

Then $(p, R) \asymp (p, R \cup \{(i, p(i))\})$*.*

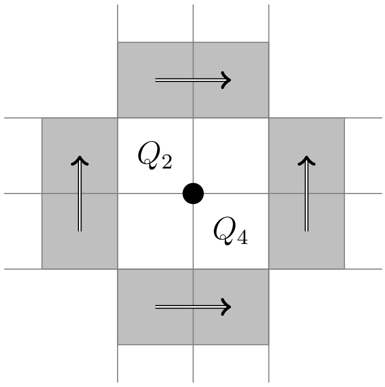


Figure 9: These are the conditions required to apply Lemma 3.6, where at most one of Q_2 or Q_4 is shaded.

Note that there are northwest, southeast, and southwest analogues for Lemma 3.6, which determine whether the other three squares incident to an element of $G(p)$ can be added to R in such a way so as to yield a coincident mesh pattern.

Example 3.5 yields the following somewhat surprising result.

Corollary 3.7. *Fix a mesh pattern* (p, R)*. The set* $\{R' : (p, R) \asymp (p, R')\}$ *need not have a unique smallest element when ordered by set containment of the meshes.*

4 Necessary conditions for mesh pattern coincidence

Consider mesh patterns (p, R) and (p', R'), and suppose that $(p, R) \asymp (p', R')$. Then two conditions must hold – one condition about p and p', and another about R and R'.

Lemma 4.1. *If* $(p, R) \asymp (p', R')$*, then* $p = p'$*.*

Proof. Suppose that p is of length k. Let $\pi = (p, R)$ and $\pi' = (p', R')$. Clearly, $\mathfrak{S}_k \cap \text{Cont}(\pi) = \{p\}$, and $\mathfrak{S}_i \cap \text{Cont}(\pi) = \emptyset$ for all $i < k$. In other words, there is a unique element of $\text{Cont}(\pi)$ using a minimal number of symbols, and that element is p. An analogous statement holds for $\text{Cont}(\pi')$ and p'. Further, if $\pi \asymp \pi'$ then $\text{Cont}(\pi) = \text{Cont}(\pi')$, and thus $p = p'$. □

We now consider coincident mesh patterns $(p, R) \asymp (p, R')$ with the same underlying classical pattern. Example 3.5 shows that having the same enclosed diagonals is not a sufficient condition for coincidence, but the lemma below shows that it is necessary.

Lemma 4.2. *If* $(p, R) \asymp (p, R')$*, then* $\mathsf{enc}(p, R) = \mathsf{enc}(p, R')$*.*

Proof. Fix $p \in \mathfrak{S}_k$ and suppose that $\pi = (p, R)$ has an enclosed diagonal that does not exist in (p, R'). Without loss of generality, let this be a NE-diagonal comprised of $D = \{(a + i, b + i) : 0 \leq i \leq c\} \subseteq R$. Then, as a word,

$$p \;=\; p(1) \cdots p(a)\,(b+1)\,(b+2) \cdots (b+c)\,p(a+c+1) \cdots p(k),$$

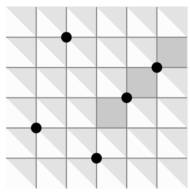
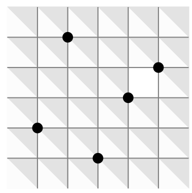
where $p(a) \neq b$ and $p(a + c + 1) \neq b + c + 1$.

Define $q \in \mathfrak{S}_{k+1}$ to be the permutation that is order isomorphic to the word obtained by inserting $b + 1/2$ between $p(a)$ and $p(a + 1) = b + 1$ in the word for p. By construction, the only p-patterns in q are obtained by excluding a single letter $q(x)$, where $a + 1 \leq x \leq a + c + 1$. Thus there is no way to find an occurrence of p that can be drawn in $G(q)$ so that the shaded region corresponding to the mesh of (p, R) does not contain $(x, q(x)) \in G(q)$. Therefore $q \in \mathrm{Av}(\pi)$.

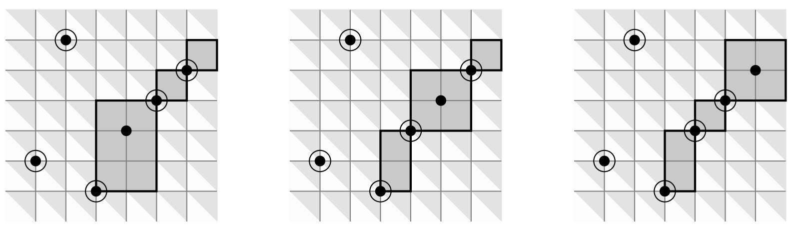
Now consider the mesh pattern $\pi' = (p, R')$, and take t in $[0, c]$ so that $(a+t, b+t) \notin R'$. Then omitting the letter $q(a + t + 1)$ from q yields a p-pattern. For this occurrence of p, the only element of $G(q)$ we need to worry about is $(a+t+1, q(a+t+1))$, but this point does not lie in a shaded region corresponding to the mesh of π' because $(a + t, b + t) \notin R'$. Thus $q \in \mathrm{Cont}(\pi')$, and so $\pi \not\asymp \pi'$. □

It is helpful to see an example illustrating the proof of Lemma 4.2.

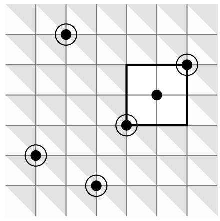
Example 4.3. Let $p = 25134 \in \mathfrak{S}_5$. Suppose that $\{(3, 2), (4, 3), (5, 4)\} \subset R$ and $(4, 3) \notin R'$. Let $\pi = (p, R)$ and $\pi' = (p, R')$. That is,

$\pi \;=$  and $\pi' \;=$ ,

where a half-filled square, ◩, indicates a square whose shading status — that is, whose presence in the mesh — is undeclared. The proof of Lemma 4.2 constructs the permutation $q = 261345 \in \mathfrak{S}_6$, which has three p-patterns: 26145, 26135, and 26134. In each of these occurrences, there is an element of $G(q)$ located in the region corresponding to the mesh from π. This is shown below.



On the other hand, because $(4, 3) \notin R'$ the occurrence 26135 of p in q does not share this property:



Therefore $q \in \mathrm{Av}(\pi)$ and $q \in \mathrm{Cont}(\pi')$.

Lemmas 4.1 and 4.2 give necessary requirements for coincidence of mesh patterns. In Section 5, we explore circumstances under which these conditions are also sufficient, and in Section 6, we discuss others where they are not.

5 Pattern families that are governed by enclosed diagonals

There are some families of mesh patterns whose coincidence is entirely characterized by the collection of enclosed diagonals.

Definition 5.1. A mesh pattern (p, R) is *vincular* if the mesh R is a union of complete columns. This (p, R) can also be described as the permutation $p \in \mathfrak{S}_k$ together with "bonds" between $p(i)$ and $p(i+1)$ exactly when $\{(i, y) : 0 \leq y \leq k\} \subseteq R$. Thus a permutation w contains this vincular pattern if and only if w has an occurrence of p in which all bonded subwords appear consecutively in w.

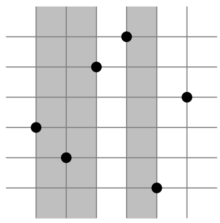


Figure 10: A vincular pattern.

The mesh pattern in Figure 10 is vincular because its mesh is a union of columns. It could also be denoted $\underline{325}\,\underline{61}4$, with brackets indicating the bonds. Vincular patterns have arisen in many places, including [1, 4, 6, 7]. Here, we show that coincidence within this family of mesh patterns behaves exactly as one might hope.

Proposition 5.2. *Let (p, R) and (p, R') be vincular patterns where $p \in \mathfrak{S}_k$. If $\mathsf{enc}(p, R) = \mathsf{enc}(p, R')$, then $(p, R) \asymp (p, R')$. If, moreover, $k > 3$, then $R = R'$ and so the two patterns are the same.*

Proof. Suppose that (p, R) and (p, R') are vincular patterns with $p \in \mathfrak{S}_k$ and $\mathsf{enc}(p, R) = \mathsf{enc}(p, R')$. The result is easy to check for $k \leq 3$, so suppose $k > 3$. Each column in the meshes R and R' contains at least $k + 1 - 4 > 0$ pointless squares, and each of those pointless squares is an enclosed diagonal. Thus we have, in fact, that the meshes R and R' are comprised of exactly the same columns and so $R = R'$. Coincidence is a reflexive relation, completing the proof. □

The example in Figure 11 (see [4, Lemma 2]) shows that when $p \in \mathfrak{S}_k$ and $k \leq 3$, there are indeed pairs of distinct coincidental vincular patterns.

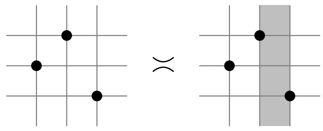


Figure 11: Two distinct but coincidental vincular patterns.

Corollary 5.3. *Two vincular patterns* (p, R) *and* (p', R') *are coincident if and only if* $p = p'$ *and* $\mathsf{enc}(p, R) = \mathsf{enc}(p', R')$. *Moreover, if* $p \in \mathfrak{S}_k$ *for* $k > 3$, *then* $(p, R) \asymp (p', R')$ *if and only if* $(p, R) = (p', R')$.

Proof. This follows from Proposition 5.2 together with Lemmas 4.1 and 4.2. □

Vincularity is a property of the mesh. The next family we examine can also be defined by its mesh, this time requiring that the mesh be suitably meager.

Definition 5.4. A mesh pattern (p, R) is *isolating* if whenever $(i, j) \in R$ is not a pointless square, then the mesh R contains no elements of the form $(i \pm 1, y)$ or $(x, j \pm 1)$.

As with vincular patterns, coincidence among isolating patterns is determined entirely by the set of enclosed diagonals in the isolating pattern.

Proposition 5.5. *If* (p, R) *and* (p, R') *are isolating mesh patterns with* $\mathsf{enc}(p, R) = \mathsf{enc}(p, R')$, *then* $(p, R) \asymp (p, R')$.

Proof. The Shading Lemma of [5] applies directly to show that such isolating patterns are coincident. □

6 Pattern families that are not governed by enclosed diagonals

In this section we look at some families of mesh patterns for which common enclosed diagonals are not enough to determine pattern coincidence. Despite this fact, these families are not entirely unrelated to those discussed in Section 5.

Recall the definition of vincular patterns from Section 5. This placed a requirement on the columns of a mesh but not on the rows, suggesting a certain asymmetry. Indeed, as studied in [2], a more balanced definition yields another interesting class of permutation patterns.

Definition 6.1. *Bivincular* patterns are mesh patterns in which the mesh is a union of complete rows and complete columns.

Given their similarity to vincular patterns, one might hope that coincidence among bivincular patterns can also be fully characterized by enclosed diagonals. This is not the case, however, as demonstrated below.

Example 6.2. Up to symmetry, the smallest non-coincident bivincular patterns having the same enclosed diagonals are

The unique enclosed diagonal in each case is $\{(0,1),(1,0)\}$, but the patterns are not coincident because 132 contains the first pattern and avoids the second. For a larger example see Figure 12. The proper enclosed diagonals in each case are

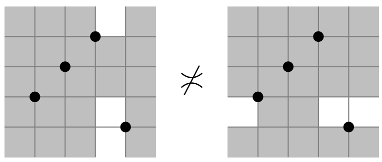


Figure 12: Two non-coincident bivincular patterns with the same enclosed diagonals.

$\{(0,2),(1,1)\}$, $\{(1,3),(2,2)\}$ and $\{(2,4),(3,3)\}$, and they have the same eleven pointless squares, but the patterns are not coincident because 345162 contains the first pattern and avoids the second.

Recall the definition of isolating patterns from Section 5, and consider the following class of patterns that might, initially, appear to be similarly desolate.

Definition 6.3. A mesh pattern (p,R) is *sparse* if it has at most one shaded square in each row and each column.

Unfortunately, coincidence within this family of mesh patterns is not solely dependent on the enclosed diagonals in the pattern. That is, there are non-coincident sparse mesh patterns that have the same enclosed diagonals.

Example 6.4. The mesh patterns $\pi = (231, \{(3,2)\})$ and $\sigma = (231, \{(1,3),(3,2)\})$ are both sparse. Further, they satisfy $\mathsf{enc}(\pi) = \mathsf{enc}(\sigma) = \{(3,2)\}$, but $\pi \not\asymp \sigma$ because 25314 contains π but avoids σ.

One might hope that nice geometric properties of the mesh would be enough to ensure that enclosed diagonals determine coincidence. Unfortunately, even a basic property like mesh connectivity is insufficient, as demonstrated in Example 6.2 above.

7 The simultaneous shading lemma

We now give an extension of the Shading Lemma (Lemma 3.6 above), the conditions of which are depicted in Figure 13.

Corollary 7.1 (Double Shading Lemma (east)). *Let (p, R) be a mesh pattern. Let i be such that the following conditions all hold.*

- *No square incident to $(i, p(i)) \in G(p)$ is in the mesh R.*
- *For all x, $(x, p(i) - 1) \in R$ if and only if $(x, p(i)) \in R$.*
- *For all y, if $(i - 1, y) \in R$ then $(i, y) \in R$.*

Then $(p, R) \asymp \big(p, R \cup \{(i, p(i)), (i, p(i) - 1)\}\big)$.

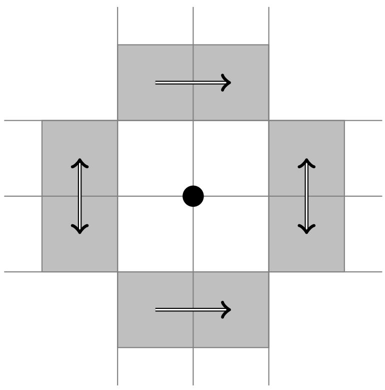


Figure 13: The conditions required to apply Corollary 7.1, the Double Shading Lemma (east).

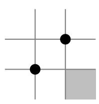
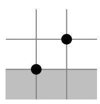
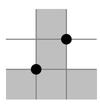
Proof. First use the northeast variant of the Shading Lemma on $(i, p(i))$ to establish that $(p, R) \asymp (p, R \cup \{(i, p(i))\})$. It is easy to see that we can then apply the southeast variant of the Shading Lemma to $(i, p(i) - 1)$ to get

$$\big(p, R \cup \{(i, p(i))\}\big) \asymp \big(p, R \cup \{(i, p(i)), (i, p(i) - 1)\}\big). \qquad \square$$

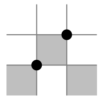
Corollary 7.1 refers to the two squares incident to an element of $G(p)$ on its east side. There are, of course, analogous statements for the north, west, and south sides of elements of $G(p)$.

It is interesting to note that there are mesh pattern coincidences that are not detected by the Shading and Double Shading Lemmas.

Example 7.2. Corollary 7.1 yields

$\pi_1 = (12, R_1) =$  $\asymp$  $\asymp$  $= (12, R_2) = \pi_2.$

It is, however, impossible to use Lemma 3.6 or Corollary 7.1 to prove that any of the above patterns are coincident to

$$\pi_3 = (12, R_3) =$$  .

On the other hand, the mesh inclusions $R_1 \subset R_3 \subset R_2$ imply that $\mathrm{Av}(\pi_1) \subseteq \mathrm{Av}(\pi_3) \subseteq \mathrm{Av}(\pi_2)$. Since $\mathrm{Av}(\pi_1) = \mathrm{Av}(\pi_2)$, we must have that π_3 is, in fact, coincident with all three of the above mesh patterns.

The following lemma records the phenomenon observed in Example 7.2.

Lemma 7.3 (Closure Lemma)**.** *If $(p, R) \asymp (p, R')$ and $R \subseteq S \subseteq R'$, then*

$$(p, R) \asymp (p, S) \asymp (p, R').$$

We now extend the Shading Lemma even further to allow simultaneous addition of multiple squares to a mesh. Before stating this precisely, we illustrate the concept.

Example 7.4. Consider the four mesh patterns in Figure 14. Applying the Shading Lemma (northwest) to $(231, R)$ at $(1, 2) \in G(231)$ gives the coincident pattern $(231, R_1)$. On the other hand, applying the Double Shading Lemma (north) to $(231, R)$ at $(2, 3) \in G(231)$ yields the coincident pattern $(231, R_2)$. It is easy to verify directly that these patterns are all coincident to $(231, R_3)$ even though iterations of the Shading and Double Shading Lemmas will not produce this coincidence. For example, we cannot apply the Double Shading Lemma (north) to $(231, R_1)$ at $(1, 2)$ because of $(0, 3) \in R_1$. Similarly we cannot apply the Shading Lemma (northwest) to $(231, R_2)$ at $(1, 2)$ because of $(1, 3) \in R_2$.

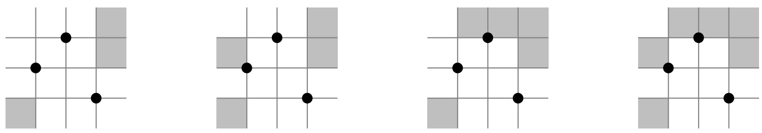


Figure 14: The four mesh patterns of Example 7.4.

Definition 7.5. Let (p, R) be a mesh pattern. A square that is incident to $g \in G(p)$ but not in R is *shadeable from* g if it satisfies the conditions of one of the Shading Lemma variants. A pair of adjacent squares that are both incident to $g \in G(p)$ and both not in R are *shadeable from* g if they satisfy the conditions of one of the Double Shading Lemma variants.

Lemma 7.6 (Simultaneous Shading Lemma)**.** *Let (p, R) be a mesh pattern with $p \in \mathfrak{S}_k$. Fix $G \subseteq G(p)$ and let s_g be a square or pair of adjacent squares that are shadeable from $g \in G$. Then $(p, R) \asymp (p, R \cup S)$, where $S = \bigcup_{g \in G} s_g$.*

Before giving a proof of Lemma 7.6 we illustrate its idea in the following example.

Example 7.7. Let the two mesh patterns (p, R) and (p, R') be as in Figure 15. Let $G = \{g_1, g_2, g_3\} \subset G(p)$ where $g_i = (i, p(i))$. If we let $s_1 = \{(0,0),(1,0)\}$, $s_2 = \{(1,4),(2,4)\}$, and $s_3 = \{(3,1)\}$, then $R' = R \cup s_1 \cup s_2 \cup s_3$. Note that, for $i \in \{1,2,3\}$, the square s_i is shadeable from g_i. The Simultaneous Shading Lemma claims that $(p, R) \asymp (p, R')$.

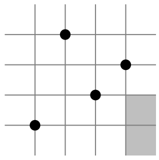
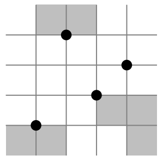
$(p, R) =$  $(p, R') =$ 

Figure 15: The two coincident mesh patterns of Example 7.7.

We wish to show that any permutation containing (p, R) also contains (p, R'); since $R \subset R'$ we already know that $\mathrm{Cont}(p, R) \supseteq \mathrm{Cont}(p, R')$. Let $w = 482951(10)376 \in \mathfrak{S}_{10}$ and consider the occurrence 4857 of (p, R) in w. For this to be an occurrence of (p, R') in w, we would need to be able to shade the regions corresponding to $s_1 \cup s_2 \cup s_3$ without intersecting elements of $G(w)$. Unfortunately $(4, 9) \in G(w)$ intersects the region $[1, 5] \times [8, 11]$ corresponding to s_2. If we consider, instead, the occurrence 4957 of p in w, then we would need to worry about $(3, 2) \in G(w)$ intersecting the region $[0, 0] \times [4, 4]$ corresponding to s_1. So let us, instead, consider the occurrence 2957 of p in w. Now we must worry about $(7, 3) \in G(w)$ intersecting the region $[5, 9] \times [2, 5]$ corresponding to s_3. We might then consider the occurrence 2937 of p in w, and so on. This procedure is illustrated in Figure 16, where dashed rectangles indicate the regions corresponding to the s_i and arrows indicate the ensuing re-selection of the pattern occurrence. The last image in the figure depicts an occurrence of (p, R') in w.

Example 7.7 and Figure 16 describe the strategy to the proof of Lemma 7.6, but we need to show that the procedure terminates.

Proof of Lemma 7.6. Let w be a permutation containing the mesh pattern (p, R). We want to show that w also contains an occurrence of (p, R'). Fix an occurrence of (p, R) in w. Each $g \in G \subseteq G(p)$ corresponds to some $(i_g, w(i_g)) \in G(w)$ in the occurrence of (p, R) in w. Together with s_g, this point will determine

- the horizontal line $y = w(i_g)$ if s_g is a pair of horizontally adjacent squares,
- the vertical line $x = i_g$ if s_g is a pair of vertically adjacent squares, or
- the pair of lines $x = i_g$ and $y = w(i_g)$ if s_g is a single square.

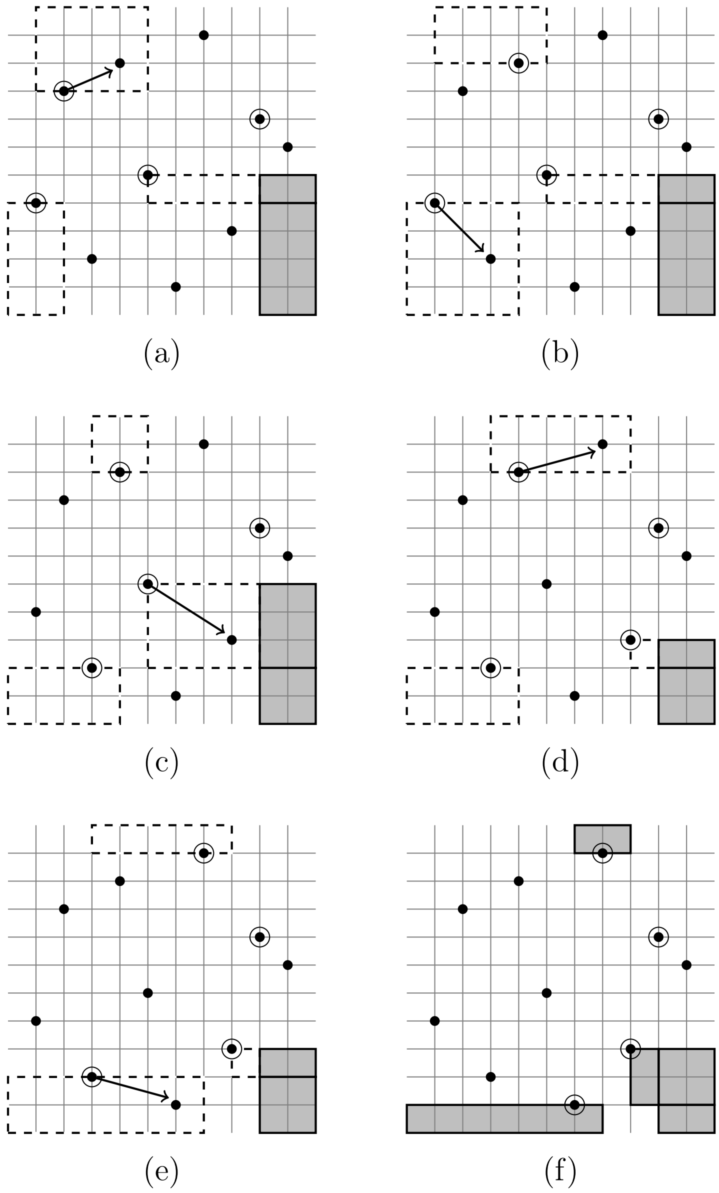


Figure 16: Maintaining the notation of Example 7.7, these figures depict the discovery of an occurrence of (p, R') in w.

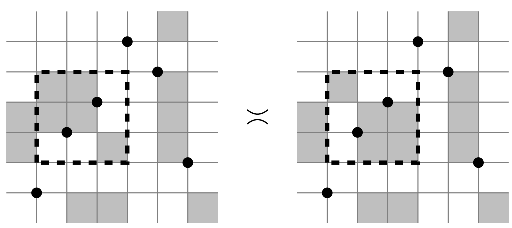


Figure 17: Two mesh patterns whose coincidence is due to the coincidence of two smaller embedded mesh patterns.

For each $g \in G$, let $\widetilde{s}_g$ be the region in w that corresponds to s_g, as defined by the given occurrence of (p, R) in w. If each $\widetilde{s}_g$ is empty then we have already found an occurrence of (p, R') in w. Otherwise, fix $h \in G$ so that $\widetilde{s}_h$ is nonempty. There are essentially two different options for s_h:

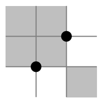
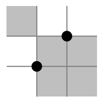
1. s_h is a pair of adjacent squares;
2. s_h is a single square.

For case (1) suppose that s_h is a pair of horizontally adjacent squares that are north of and incident to $h \in G(p)$. Now replace $(i_h, w(i_h))$ in the occurrence of (p, R) by the element of $(i'_h, w(i'_h)) \in \widetilde{s}_h \cap G(w)$ having the largest y-coordinate. Because s_h was shadeable from h this will produce a new occurrence of (p, R) in w. Also, note that s_h now determines the line $y = w(i'_h)$, which is above the previously identified line $y = w(i_h)$.

For case (2) suppose s_h is a single square northeast of and incident to $h \in G(p)$. Now replace $(i_h, w(i_h))$ in the occurrence of (p, R) by the element of $(i'_h, w(i'_h)) \in \widetilde{s}_h \cap G(w)$ having the largest y-coordinate. (Here we could equally well have chosen the largest x-coordinate.) Because s_h was shadeable from h this will produce a new occurrence of (p, R) in w. Also, note that s_h now determines the lines $x = i'_h$ and $y = w(i'_h)$, which are to the right of and above, respectively, the previously identified lines.

Thus at each step, we shift at most one horizontal line and at most one vertical line, and each such line can only move in a fixed direction. The graph $G(w)$ is finite, and so the shifting will have to terminate, at which time $\{\widetilde{s}_g : g \in G\} \cap G(w)$ will be empty. Thus the procedure ends by producing an occurrence of (p, R') in w. □

The Simultaneous Shading Lemma explains, among other things, the coincidences discussed in Example 7.4. The Simultaneous Shading Lemma together with the Closure Lemma are powerful enough to explain every coincidence among mesh patterns of 1 or 2 letters, *except* for the coincidence

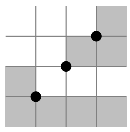
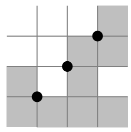
$\gamma_1 =$  $\asymp$  $= \gamma_2,$

and its symmetries. It is easy to verify that a permutation w contains either of γ_1 and γ_2 if and only if w can be written as a direct sum $w = u \oplus v$ of two nonempty permutations u and v. Under certain conditions the coincidence between γ_1 and γ_2 implies coincidences of larger patterns, where γ_1 and γ_2 appear as embedded patterns. An example of this is given in Figure 17 above.

8 Future research

The tools we derived in this paper go some way toward characterizing coincidence of mesh patterns, but they are not a complete answer. We have written tests to apply our results to all mesh patterns of small lengths, partitioning them into coincident classes, and not all coincidences can be detected by the methods described in this paper. For example, the following coincidence cannot be proved by the Simultaneous Shading Lemma, nor by the additional use of some of the other techniques addressed above.

Example 8.1. Consider the following two mesh patterns below.

$\pi = (123, R) =$  $\asymp$  $= (123, R') = \pi'$

Because $R \subset R'$, we have that $\text{Av}(\pi) \subseteq \text{Av}(\pi')$. Now suppose w has an occurrence of π indexed by $i_1 < i_2 < i_3$. If the region corresponding to the square $(2, 1)$ does not intersect $G(w)$ for this occurrence, then it is an occurrence of π' as well. If that region does intersect $G(w)$, then let $g \in G(w)$ be the rightmost element of that intersection. If there are no elements of $G(w)$ northeast of g and also south of $w(i_2)$ (given the choice of g, such points would necessarily be in the region corresponding to $(3, 1)$ for this occurrence), then $\{(i_1, w(i_1)), g, (i_3, w(i_3))\}$ is an occurrence of π' in w. On the other hand, suppose there are such points. If they are in descending order and h is the leftmost (greatest) among them, then $\{(i_1, w(i_1)), g, h\}$ is an occurrence of π' in w. If they are not in descending order, then we can apply the Simultaneous Shading Lemma to an ascent among them, producing the occurrence of π' that we seek.

We are left, then, with two questions. Under what conditions are two arbitrary mesh patterns coincident? Is it possible to describe the maximal set of mesh patterns for which coincidence depends only on enclosed diagonals?

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