AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **61(1)** (2015), Pages 130–137

An f-chromatic spanning forest of edge-colored complete bipartite graphs*

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Abstract

A heterochromatic spanning tree is a spanning tree whose edges have distinct colors, where any color appears at most once. In 2001, Brualdi and Hollingsworth proved that a $(2n-1)$-edge-colored balanced complete bipartite graph $K_{n,n}$ with color set $\mathbb{C}$ has a heterochromatic spanning tree, if for any non-empty subset $R \subseteq C$, the number of edges having a color in R is more than $|R|^2/4$. In 2013, Suzuki generalized heterochromatic graphs to f-chromatic graphs, where any color c appears at most $f(c)$ times, and he presented a necessary and sufficient condition for graphs to have an f-chromatic spanning forest with exactly w components. In this paper, using this necessary and sufficient condition, we generalize the Brualdi-Hollingsworth theorem above.

1 Introduction

We consider finite undirected graphs without loops or multiple edges. For a graph G, we denote by $V(G)$ and $E(G)$ its vertex and edge sets, respectively. An *edge-coloring* of a graph G is a mapping $color : E(G) \to \mathbb{C}$, where $\mathbb{C}$ is a set of colors. An *edge-colored graph* $(G, \mathbb{C}, color)$ is a graph G with an edge-coloring *color* on a color set $\mathbb{C}$. We often abbreviate an edge-colored graph $(G, \mathbb{C}, color)$ as G.

1.1 Heterochromatic spanning trees

An edge-colored graph G is said to be *heterochromatic* if no two edges of G have the same color, that is, $color(e_i) \neq color(e_j)$ for any two distinct edges e_i and e_j of G. A heterochromatic graph is also said to be *rainbow*, *multicolored*, *totally multicolored*, *polychromatic*, or *colorful*. Heterochromatic subgraphs of edge-colored graphs have been studied in many papers, as in the survey by Kano and Li [4].

* This work was supported by MEXT. KAKENHI 21740085.

Akbari & Alipour [1], and Suzuki [5] independently presented a necessary and sufficient condition for edge-colored graphs to have a heterochromatic spanning tree, and they proved some results by using the condition. Here, we denote by $\omega(G)$ the number of components of a graph G. Given an edge-colored graph G and a color set R, we define $E_R(G) = \{e \in E(G) \mid color(e) \in R\}$. Similarly, for a color c, we define $E_c(G) = E_{\{c\}}(G)$. We denote the graph $(V(G), E(G) \setminus E_R(G))$ by $G - E_R(G)$.

Theorem 1.1 (Akbari and Alipour [1], Suzuki [5]). *An edge-colored graph G has a heterochromatic spanning tree if and only if*

$$\omega(G - E_R(G)) \le |R| + 1 \qquad \text{for any } R \subseteq \mathbb{C}.$$

Note that if $R = \emptyset$ then the condition is $\omega(G) \le 1$. Thus, the condition of this theorem includes a condition for graphs to have a spanning tree, namely, to be connected. Suzuki [5] proved the following theorem by using Theorem 1.1.

Theorem 1.2 (Suzuki [5]). *An edge-colored complete graph K_n has a heterochromatic spanning tree if $|E_c(G)| \le n/2$ for any color $c \in \mathbb{C}$.*

Jin and Li [3] generalized Theorem 1.1 to the following theorem, from which we can obtain Theorem 1.1 by taking $k = n - 1$.

Theorem 1.3 (Jin and Li [3]). *An edge-colored connected graph G of order n has a spanning tree with at least k $(1 \le k \le n-1)$ colors if and only if*

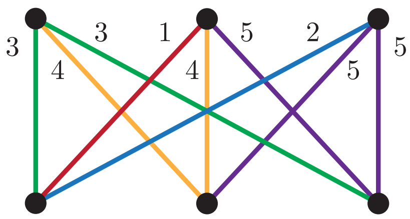
$$\omega(G - E_R(G)) \le n - k + |R| \qquad \text{for any } R \subseteq \mathbb{C}.$$

If an edge-colored connected graph G of order n has a spanning tree with at least k colors, then G has a heterochromatic spanning forest with k edges, that is, G has a heterochromatic spanning forest with exactly $n-k$ components. On the other hand, if an edge-colored connected graph G of order n has a heterochromatic spanning forest with exactly $n-k$ components, then we can construct a spanning tree with at least k colors by adding some $n-k-1$ edges to the forest. Hence, we can rephrase Theorem 1.3 as the following.

Theorem 1.4 (Jin and Li [3]). *An edge-colored connected graph G of order n has a heterochromatic spanning forest with exactly $n-k$ components* $(1 \le k \le n-1)$ *if and only if*

$$\omega(G - E_R(G)) \le n - k + |R| \qquad \text{for any } R \subseteq \mathbb{C}.$$

Brualdi and Hollingsworth [2] presented a sufficient condition for complete bipartite graphs to have a heterochromatic spanning tree.



$$e_1 = 1 > 1^2/4$$
$$e_1 + e_2 = 2 > 2^2/4$$
$$e_1 + e_2 + e_3 = 4 > 3^2/4$$
$$e_1 + e_2 + e_3 + e_4 = 6 > 4^2/4$$
$$e_1 + e_2 + e_3 + e_4 + e_5 = 9 > 5^2/4$$

Fig. 1: An example of Theorem 1.5.

Theorem 1.5 (Brualdi and Hollingsworth [2]). *Let G be an edge-colored balanced complete bipartite graph $K_{n,n}$ with a color set $\mathbb{C} = \{1, 2, 3, \ldots, 2n-1\}$. Let e_c be the number of edges having a color c, namely, $e_c = |E_c(G)|$, and assume that $1 \leq e_1 \leq e_2 \leq \cdots \leq e_{2n-1}$. If $\sum_{i=1}^{r} e_i > r^2/4$ for any color $r \in \mathbb{C}$, then G has a heterochromatic spanning tree.*

Fig. 1 shows an example of Theorem 1.5. The sum of numbers of edges having $1, 2, \ldots,$ or r is more than $r^2/4$ for any color r, thus, this graph has a heterochromatic spanning tree.

1.2 f-Chromatic spanning trees

Heterochromatic means that any color appears at most *once*. Suzuki [6] generalized *once* to a mapping f from a given color set $\mathbb{C}$ to the set of non-negative integers, and introduced the following definition as a generalization of heterochromatic graphs.

Definition 1.6. *Let f be a mapping from a given color set $\mathbb{C}$ to the set of non-negative integers. An edge-colored graph $(G, \mathbb{C}, color)$ is said to be f-chromatic if $|E_c(G)| \leq f(c)$ for any color $c \in \mathbb{C}$.*

Fig. 2 shows an example of an f-chromatic spanning tree of an edge-colored graph. Let $\mathbb{C} = \{1, 2, 3, 4, 5, 6, 7\}$ be a given color set of 7 colors, and a mapping f is given as follows: $f(1) = 3$, $f(2) = 1$, $f(3) = 3$, $f(4) = 0$, $f(5) = 0$, $f(6) = 1$, $f(7) = 2$. Then, the left edge-colored graph in Fig. 2 has the right graph as a subgraph. It is a spanning tree where each color c appears at most $f(c)$ times. Thus, it is an f-chromatic spanning tree.

If $f(c) = 1$ for any color c, then all f-chromatic graphs are heterochromatic and also all heterochromatic graphs are f-chromatic. It is expected many previous studies and results for heterochromatic subgraphs will be generalized.

Suzuki [6] presented the following necessary and sufficient condition for graphs to have an f-chromatic spanning forest with exactly w components. This is a generalization of Theorem 1.1 and Theorem 1.4.

Theorem 1.7 (Suzuki [6]). *Let f be a mapping from a given color set $\mathbb{C}$ to the set of non-negative integers. An edge-colored graph $(G, \mathbb{C}, color)$ of order at least w has*

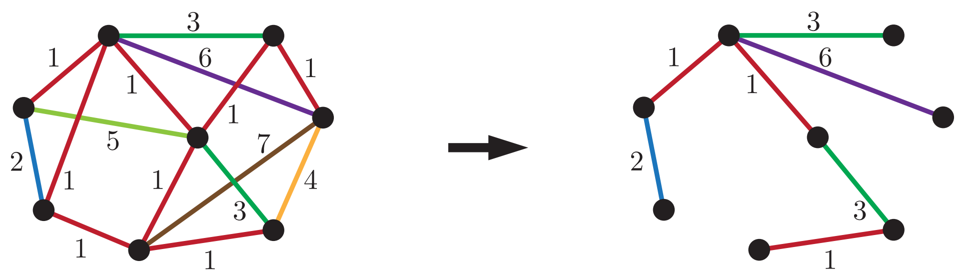


Fig. 2: An f-chromatic spanning tree of an edge-colored graph.

an f-chromatic spanning forest with exactly w components if and only if

$$\omega(G - E_R(G)) \leq w + \sum_{c \in R} f(c) \qquad \text{for any } R \subseteq \mathbb{C}.$$

By using Theorem 1.7, he generalized Theorem 1.2, as follows.

Theorem 1.8 (Suzuki [6]). *A g-chromatic graph G of order n with $|E(G)| > \binom{n-w}{2}$ has an f-chromatic spanning forest with exactly w $(1 \leq w \leq n-1)$ components if $g(c) \leq \frac{|E(G)|}{n-w} f(c)$ for any color c.*

In this paper, by using Theorem 1.7, we will generalize the Brualdi-Hollingsworth theorem (Theorem 1.5).

1.3 A generalization of Brualdi-Hollingsworth Theorem

Under the conditions of Theorem 1.5, if $\sum_{i=1}^{r} e_i > r^2/4$ for any color $r \in \mathbb{C}$, then for any non-empty subset $R \subseteq \mathbb{C}$ we have $|E_R(G)| \geq \sum_{i=1}^{|R|} e_i > |R|^2/4$. On the other hand, if $|E_R(G)| > |R|^2/4$ for any non-empty subset $R \subseteq \mathbb{C}$, then for any color r and color subset $Q = \{1, 2, 3, \ldots, r\} \subseteq \mathbb{C}$, we have

$$\sum_{i=1}^{r} e_i = \sum_{i=1}^{r} |E_i(G)| = |E_Q(G)| > |Q|^2/4 = r^2/4.$$

Thus, $\sum_{i=1}^{r} e_i > r^2/4$ for any color $r \in \mathbb{C}$ if and only if $|E_R(G)| > |R|^2/4$ for any non-empty subset $R \subseteq \mathbb{C}$. Hence, we can rephrase Theorem 1.5 as the following.

Theorem 1.9 (Brualdi and Hollingsworth [2]). *Let G be an edge-colored balanced complete bipartite graph $K_{n,n}$ with a color set $\mathbb{C} = \{1, 2, 3, \ldots, 2n-1\}$. If $|E_R(G)| > |R|^2/4$ for any non-empty subset $R \subseteq \mathbb{C}$, then G has a heterochromatic spanning tree.*

In this paper, we generalize this theorem to the following, which is our main theorem.

Theorem 1.10. *Let G be an edge-colored complete bipartite graph $K_{n,m}$ with a color set $\mathbb{C}$. Let w be a positive integer with $1 \leq w \leq n+m$, and f be a function from $\mathbb{C}$ to the set of non-negative integers such that $\sum_{c\in\mathbb{C}} f(c) \geq n+m-w$. If $|E_R(G)| > (n+m-w-\sum_{c\in\mathbb{C}\setminus R} f(c))^2/4$ for any non-empty subset $R \subseteq \mathbb{C}$, then G has an f-chromatic spanning forest with w components.*

Theorem 1.9 is a special case of Theorem 1.10 with $m = n$, $w = 1$, $f(c) = 1$ for any color c, and $|\mathbb{C}| = 2n-1$. The number of edges of a spanning forest with w components of $K_{n,m}$ is $n+m-w$. Thus, in Theorem 1.10, the condition $\sum_{c\in\mathbb{C}} f(c) \geq n+m-w$ is necessary for existence of an f-chromatic spanning forest with w components.

The lower bound of $|E_R(G)|$ in Theorem 1.10 is sharp in the following sense: Let $R \subseteq \mathbb{C}$ be a color subset and $p = n+m-w-\sum_{c\in\mathbb{C}\setminus R} f(c)$. Let G be a complete bipartite graph $K_{n,m}$, and H be a complete bipartite subgraph $K_{\frac{p}{2},\frac{p}{2}}$ of G. Color the edges in $E(H)$ with colors in R, and the edges in $E(G)\setminus E(H)$ with colors in $\mathbb{C}\setminus R$ (Fig. 3). Then, $|E_R(G)| = p^2/4 = (n+m-w-\sum_{c\in\mathbb{C}\setminus R} f(c))^2/4$.

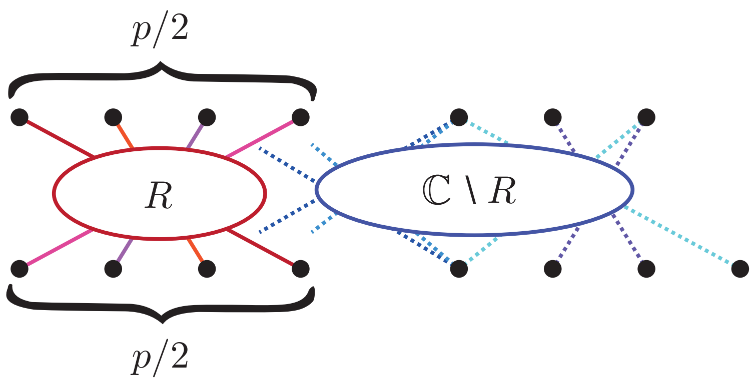


Fig. 3: A graph G and $R \subseteq \mathbb{C}$ with $|E_R(G)| = (n+m-w-\sum_{c\in\mathbb{C}\setminus R} f(c))^2/4$.

Recall that $n+m-w$ is the number of edges of a spanning forest with w components of G, and $\sum_{c\in\mathbb{C}\setminus R} f(c)$ is the maximum number of edges having colors in $\mathbb{C}\setminus R$ of a desired forest. Thus, p is the number of edges having colors in R needed in a desired forest. However, any p edges of H make a cycle because $|V(H)| = p$. Hence, G has no f-chromatic spanning forests with w components, which implies the lower bound of $|E_R(G)|$ is sharp.

In the next section, we will prove Theorem 1.10 by using Theorem 1.7.

2 Proof of Theorem 1.10

In order to prove Theorem 1.10, we need the following lemma.

Lemma 2.1. *Let G be a bipartite graph of order N that consists of s components. Then $|E(G)| \leq (N-(s-1))^2/4$.*

Proof. Take a bipartite graph G^* of order N that consists of s components so that

(1) $|E(G^*)|$ is maximum, and

(2) subject to (1), for the maximum component D_s of G^*, $|V(D_s)|$ is maximum.

By the maximality (1) of G^*, each component of G^* is a complete bipartite graph. Let A_s and B_s be the partite sets of D_s. We assume $|A_s| \leq |B_s|$.

Suppose that some component D except D_s has at least two vertices. Let A and B be the partite sets of D. We assume $|A| \leq |B|$. If $|A| > |B_s|$ then $|A_s| \leq |B_s| < |A| \leq |B|$, which contradicts that D_s is a maximum component of G^*. Thus, we have $|A| \leq |B_s|$. Let x be a vertex of B, where $\deg_G(x) = |A|$. Let $D' = D - \{x\}$, $A' = A$, $B' = B - x$, $A'_s = A_s \cup \{x\}$, $B'_s = B_s$, and $D'_s = (A'_s \cup B'_s, E(D_s) \cup \{xz \mid z \in B'_s\})$. Let G'^* be the resulted graph. Then, we have

$$\begin{aligned} |E(D')| + |E(D'_s)| &= |E(D)| - \deg_G(x) + |E(D_s)| + |B'_s| \\ &= |E(D)| + |E(D_s)| + |B_s| - |A| \\ &\geq |E(D)| + |E(D_s)|, \end{aligned}$$

which implies $|E(G'^*)| = |E(G^*)|$ by the condition (1). However, that contradicts the maximality (2) because $|V(D'_s)| \geq |V(D_s)| + 1$. Hence, every component except D_s has exactly one vertex, which implies that $|V(D_s)| = N - (s-1)$.

Suppose that $|B_s| - |A_s| \geq 2$. Let x be a vertex of B_s, where $\deg_G(x) = |A_s|$. Let $D'_s = (V(D_s), E(D_s - x) \cup \{xz \mid z \in B_s - x\})$. Then, D'_s is a complete bipartite graph, and we have

$$\begin{aligned} |E(D'_s)| &= |E(D_s)| - \deg_G(x) + |B_s - x| \\ &= |E(D_s)| - |A_s| + |B_s| - 1 \\ &\geq |E(D_s)| + 1, \end{aligned}$$

which contradicts the maximality (1). Hence, $|B_s| - |A_s| \leq 1$.

Therefore,

$$\begin{aligned} |E(G)| &\leq |E(G^*)| = |E(D_s)| = |A_s||B_s| \\ &= \lfloor (N-(s-1))/2 \rfloor \lceil (N-(s-1))/2 \rceil \\ &\leq (N-(s-1))^2/4. \end{aligned}$$

■

Then, we shall prove Theorem 1.10 by using Theorem 1.7 and Lemma 2.1. Suppose that G has no f-chromatic spanning forests with w components. By Theorem 1.7, there exists a color set $R \subseteq \mathbb{C}$ such that

$$\omega(G - E_R(G)) > w + \sum_{c \in R} f(c). \tag{1}$$

Let $s = \omega(G - E_R(G))$. Let $D_1, D_2, \ldots, D_s$ be the components of $G - E_R(G)$, and q be the number of edges of G between these distinct components. Note that, the colors of these q edges are only in R, that is, $q \leq |E_R(G)|$.

If $R = \mathbb{C}$ then

$$s = \omega(G - E_{\mathbb{C}}(G)) > w + \sum_{c \in \mathbb{C}} f(c) \geq w + n + m - w = n + m = |V(G)|,$$

by the assumption of Theorem 1.10. This contradicts that $s \leq |V(G)|$. Thus, we can assume $R \neq \mathbb{C}$, namely, $\mathbb{C} \setminus R \neq \emptyset$. Hence, by the assumption of Theorem 1.10,

$$|E_{\mathbb{C}\setminus R}(G)| > (n + m - w - \sum_{c \in \mathbb{C}\setminus(\mathbb{C}\setminus R)} f(c))^2/4 = (n + m - w - \sum_{c \in R} f(c))^2/4.$$

Therefore, we have

$$q \leq |E_R(G)| = |E(G)| - |E_{\mathbb{C}\setminus R}(G)| < |E(G)| - (n + m - w - \sum_{c \in R} f(c))^2/4. \quad (2)$$

On the other hand,

$$q = |E(G)| - |E(D_1) \cup E(D_2) \cup \cdots \cup E(D_s)|.$$

By Lemma 2.1, $|E(D_1) \cup E(D_2) \cup \cdots \cup E(D_s)| \leq (n + m - (s-1))^2/4$. Thus, since $s = \omega(G - E_R(G)) \geq w + 1 + \sum_{c \in R} f(c)$ by (1), we have

$$\begin{aligned} q &= |E(G)| - |E(D_1) \cup E(D_2) \cup \cdots \cup E(D_s)| \\ &\geq |E(G)| - (n + m - (s-1))^2/4 \\ &\geq |E(G)| - (n + m - (w + 1 + \sum_{c \in R} f(c) - 1))^2/4 \\ &= |E(G)| - (n + m - w - \sum_{c \in R} f(c))^2/4, \end{aligned}$$

which contradicts (2). Consequently, the graph G has an f-chromatic spanning forest with w components.

References

[1] S. Akbari and A. Alipour, Multicolored trees in complete graphs. *J. Graph Theory* **54** (2006), 221–232.

[2] R.A. Brualdii and S. Hollingsworth, Multicolored forests in complete bipartite graphs. *Discrete Math.* **240** (2001), 239–245.

[3] Z. Jin and X. Li, Spanning trees with many colors in edge-colored graphs. *J. Xinjiang University* **23** (Supp.) (2006), 12–14.

[4] M. Kano and X. Li, Monochromatic and heterochromatic subgraphs in edge-colored graphs - a survey. *Graphs Combin.* **24** (2008), 237–263.

[5] K. Suzuki, A necessary and sufficient condition for the existence of a heterochromatic spanning tree in a graph. *Graphs Combin.* **22** (2006), 261–269.

[6] K. Suzuki, A Generalization of Heterochromatic Graphs and f-Chromatic Spanning Forests. *Graphs Combin.* **29** (3) (2013), 715–727.

(Received 12 Aug 2014)