More Mutually Orthogonal Diagonal Latin Squares

B. Du

Department of Mathematics,
Suzhou University
Suzhou 215006 China (FRC)

Abstract

A diagonal Latin square is a Latin square whose main diagonal and back diagonal are both transversals. In this paper we give some constructions of pairwise orthogonal diagonal Latin squares. As an application of such constructions we obtain some new infinite classes of pairwise orthogonal diagonal Latin squares which are useful in the study of pairwise orthogonal diagonal Latin squares.

1 Introduction

A *Latin square* of order n is an $n \times n$ array such that every row and every column is a permutation of an n–set. A *transversal* in a Latin square is a set of positions, one per row and one per column, among which the symbols occur precisely once each. A *symmetric transversal* in a Latin square is a transversal which is a set of symmetric positions. A *transversal Latin square* is a Latin square whose main diagonal is a transversal. A *diagonal Latin square* is a transversal Latin square whose back diagonal also forms a transversal. It is easy to see that the existence of a transversal Latin square with a symmetric transversal implies the existence of a diagonal Latin square.

Two Latin squares of order n are *orthogonal* if each symbol in the first square meets each symbol in the second square exactly once when they are superposed. Also *t pairwise orthogonal diagonal (transversal) Latin squares of* order n, denoted briefly by t PODLS(n) (POILS(n)) are t pairwise orthogonal Latin squares each of which is a diagonal (transversal) Latin square of order n. We let $N(n)(D(n), I(n))$ denote the maximum number of pairwise orthogonal (diagonal, transversal) Latin squares of order n.

Several constructions of pairwise orthogonal diagonal Latin squares have been found; see [1]–[6]. In this paper we give some new constructions of pairwise orthogonal diagonal Latin squares. As an application of such constructions we obtain some new infinite classes of pairwise orthogonal diagonal Latin squares which are useful in the study of pairwise orthogonal diagonal Latin squares.

Australasian Journal of Combinatorics 6(1992), pp.3-6

2 Some Constructions

In order to state our constructions we let Q be a Latin square of order n based on the set $I_n = \{0, 1, \cdots, n-1\}$ and let S, T be transversals of Q. We form a permutation $\sigma_{S,T}$ on I_n as follows: $\sigma_{S,T}(s) = t$ where s and t are the entries of S and T, respectively, occurring in the same row. We denote by $Q(S,T)$ the Latin square obtained by renaming symbols using $\sigma_{S,T}$. Obviously we have:

(a) If U is a transversal of Q then U is also a transversal of $Q(S,T)$;

(b) If V is a Latin square which is orthogonal to Q, then V is also orthogonal to $Q(S,T)$.

Let A, B be Latin squares and let h be a symbol. We denote by A_h the copy of A obtained by replacing each entry x of A with the ordered pair (h, x). Further we denote by (A, B) the Latin square $(A_{b_{ij}})$, where $B = (b_{ij})$.

Theorem 2.1 *For an even integer n, $N(n) \geq t$, let $A_1, A_2, \cdots, A_t$ be t pairwise orthogonal Latin squares of order m which possess n disjoint common transversals $T_1, T_2, \cdots, T_{n-1}$ and the main diagonal D. Then there exist t pairwise orthogonal diagonal Latin squares of order mn.*

Proof. Consider the t pairwise orthogonal Latin squares of order mn,

$$\overline{A}_i = (A_i, B_i) \quad 1 \leq i \leq t,$$

where B_i are t pairwise orthogonal Latin squares of order n $(1 \leq i \leq t)$.

We denote by $\tilde{A}_1$ the Latin square obtained by replacing each subsquare A_1 with $A_1(D, T_j)$ in the j-th block column of $\overline{A}_1, 1 \leq j \leq n-1$. As a result of such replacement the 0-th block column of $\tilde{A}_1$ contains, in the transversals T_j, the mn entries which are just the same as the entries appearing in the main diagonals of blocks in the j-th block column. For each j $(1 \leq j \leq n-1)$ exchange the two entries from the above two sets of entries appearing in the same row of $\tilde{A}_1$. From (a) it follows immediately that the resulting array $\hat{A}_1$ is a transversal Latin square with a symmetric transversal.

Do the same replacement and exchange of entries for $\overline{A}_2, \overline{A}_3, \cdots, \overline{A}_t$, By (b) the resulting squares $\hat{A}_2, \hat{A}_3, \cdots, \hat{A}_t$ together with $\hat{A}_1$ form t POILS with a common symmetric transversal, which consists of the back diagonal in the central block and the main diagonals of other blocks appearing in the block back diagonal. By simultaneously permuting rows and columns we have t PODLS (mn).

Corollary 2.2 *$D(2^r q) \geq 2^r - 1$ for prime power $q > 2^r$.*

Proof. Since for each q there exist $2^r - 1$ pairwise orthogonal Latin squares of order q satisfying the hypotheses of the previous Theorem, we have $D(2^r q) \geq 2^r - 1$.
In particular for $r = 2$, we have $D(4q) \geq 3$ for prime power $q > 4$, so we then have $D(n) \geq 3$ for $n \in \{20, 28, 36, 44, 52\}$, which is undecided in [6].

Theorem 2.3 *For an odd integer n, $N(n) \geq t$, let $A_1, A_2, \cdots, A_t$ be t pairwise orthogonal diagonal Latin squares of order m which possess $n-1$ disjoint common transversals $T_0, T_1, \cdots, T_{[(n-1)/2]-1}, T_{[(n-1)/2]+1}, \cdots, T_{n-1}$ which are disjoint from the main diagonal D and the back diagonal D_1. Then there exist t pairwise orthogonal diagonal Latin squares of order mn.*

Proof. Consider the t pairwise orthogonal Latin squares of order mn,

$$\overline{A}_i = (A_i, B_i)\ 1 \leq i \leq t,$$

where B_i are t pairwise orthogonal Latin squares of order $n(1 \leq i \leq t)$.

We denote by $\tilde{A}_1$ the Latin square obtained by replacing each subsquare A_1 with $A_1(D, T_j)$ in the j–th block column of $\overline{A}_1$, $j = 0, 1, \cdots, [(n-1)/2]-1, [(n-1)/2]+1, \cdots, n-1$. As a result of such replacement the $[(n-1)/2]$–th block column of $\tilde{A}_1$ contains in the transversals T_j ($j = 0, 1, \cdots, [(n-1)/2]-1$, $[(n-1)/2]+1, \cdots, n-1$) the mn entries which are just the same as the entries appearing in the main diagonals of blocks in the j–th block column. For each j ($j = 0, 1, \cdots, [(n-1)/2]-1, [(n-1)/2]+1, \cdots, n-1$) exchange the two entries from the above two sets of entries appearing in the same row of $\tilde{A}_1$. From (a) it follows immediately that the resulting array $\hat{A}_1$ is a transversal Latin square with a symmetric transversal.

Do the same replacement and exchange of entries for $\overline{A}_2, \overline{A}_3, \cdots, \overline{A}_t$. By (b) the resulting squares $\hat{A}_2, \hat{A}_3, \cdots, \hat{A}_t$ together with $\hat{A}_1$ form t POILS with a common symmetric transversal, which consists of the back diagonal in the central block and the main diagonals of other blocks appearing in the block back diagonal. By simultaneously permuting rows and columns we have t PODLS (mn).

Corollary 2.4 *$D(2^r q) \geq q-1$ for odd prime power $q < 2^r$.*

Proof. Since for each $2^r > q$, there exist $q-1$ pairwise orthogonal diagonal Latin squares of order 2^r satisfying the hypotheses of the previous Theorem, we have $D(2^r q) \geq q-1$.

In particular for $q = 5$, we have $D(5 \times 2^r) \geq 4$ for $2^r > 5$, so we then have $D(n) \geq 4$ for $n \epsilon \{40, 80\}$, which is undecided in [5].

Theorem 2.5 *For an odd integer n, $I(n) \geq t$, let $A_1, A_2, \cdots, A_t$ be t pairwise orthogonal diagonal Latin squares of order m which possess three disjoint common transversals T_1, T_2 and the main diagonal D. If the positions of T_1, T_2 are symmetric about the main diagonal, then there exist t pairwise orthogonal diagonal Latin squares of order mn.*

Proof. Consider the t pairwise orthogonal Latin squares of order mn,

$$\overline{A}_i = (A_i, B_i)\ 1 \leq i \leq t,$$

where B_i are t pairwise orthogonal transversal Latin squares of order n $(1 \leq i \leq t)$.

Notice that the set $E_i^j(k)$ of the entries of the $(km+j)$-th column of $\overline{A}_1$ which lie on the transversal T_i coincides with the set $\overline{E}_i^j(k)$ of the entries of the $[(n-k-1)m+j]$-th column of $\overline{A}_1$ lying on the transversals T_i, $k = 0, 1, \cdots, \frac{n-3}{2}$. For each $i = 1, 2$, $j = 1, 2, \cdots, m$, exchange in $\overline{A}_1$ the elements of $E_i^j(k)$ and $\overline{E}_i^j(k)$ appearing on the same row, $k = 0, 1, \cdots, \frac{n-3}{2}$. It is clear that the resulting array $\hat{A}_1$ is a transversal Latin square with a transversal which consists of an element in the central cell and a set of elements in symmetric positions.

Do the same exchange of entries for $\overline{A}_2, \overline{A}_3, \cdots, \overline{A}_t$. It is easy to see that the resulting squares $\hat{A}_2, \hat{A}_3, \cdots, \hat{A}_t$ together with $\hat{A}_1$ form t POILS with a common transversal, which consists of the back diagonal in the central block and the T_1 in the upper right blocks and the T_2 in the lower left blocks of the block back diagonal. By simultaneously permuting rows and columns we have t PODLS(mn).

Corollary 2.6 *$D(pq) \geq p - 2$ for odd prime powers p, q, $p < q$.*

Proof. Since for each $q > p$ there exist $p - 2$ pairwise orthogonal Latin squares of order q satisfying the hypotheses of the previous Theorem, we have $D(pq) \geq p - 2$.

In particular for $p = 5$, we have $D(5q) \geq 3$ for odd prime power $q > 5$, we then have $D(n) \geq 3$ for $n \epsilon \{35, 45\}$, which is undecided in [6].

Acknowledgement

The author is thankful to Professor L. Zhu for introducing him to this topic.

References

1. E. Gergely, A remark on doubly diagonalized orthogonal Latin squares, Discrete Math. **10** (1974), 185–188.

2. K. Heinrich and A.J.W. Hilton, Doubly diagonal orthogonal Latin squares, Discrete Math. **46** (1983), 173–182.

3. A.J.W. Hilton, Some simple constructions for double diagonal Latin squares, Colloq. Math. Soc. J. Bolyai **10** (1973), 887–904.

4. C.C. Lindner, Construction of doubly diagonalized orthogonal Latin squares, Discrete Math. **5** (1973), 79–86.

5. W.D. Wallis and L. Zhu, Some bounds for pairwise orthogonal diagonal Latin squares, Ars Combinatoria **17A** (1984), 353–366.

6. L. Zhu, Three pairwise orthogonal diagonal Latin squares, Journal of Combinatorial Mathematics and Combinatorial Computing, **5** (1989), 27–40.

(Received 7/10/91)