Constructing Hadamard matrices from orthogonal designs

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Abstract

The Hadamard conjecture is that Hadamard matrices exist for all orders $1, 2, 4t$ where $t \geq 1$ is an integer. We have obtained the following results which strongly support the conjecture:

(i) Given any natural number q, there exists an Hadamard matrix of order $2^s q$ for every $s \geq [2\log_2(q-3)]$.

(ii) Given any natural number q, there exists a regular symmetric Hadamard matrix with constant diagonal of order $2^{2s}q^2$ for s as before.

A significant step towards proving the Hadamard conjecture would be proving "Given any natural number q and constant c_0 there exists a Hadamard matrix of order $2^c q$ for some $c < c_0$."

We make steps toward proving the Hadamard conjecture by showing that "If there is an $OD(4p; s_1, s_2, s_3, s_4)$ and a set of T-matrices of order t there is an $OD(16p^2t; 4pts_1, 4pts_2, 4pts_3, 4pts_4)$. In particular, if there is an $OD(4p; p, p, p, p)$ and a set of T-matrices of order t there is an $OD(16p^2t; 4p^2t, 4p^2t, 4p^2t, 4p^2t)$. Further, if there are Williamson matrices of order w there is a Hadamard matrix of $16p^2tw$."

Currently the aforementioned matrices are known for $p, t \in$ {orders of Hadamard matrices, orders of conference matrices, $1 + 2^a 10^b 26^c$, a, b, c non-negative integers, 1,3,...,71,75,77,81,85,87,91,93,95,99} or for all orders of $t \leq 100$ except possibly $t \in \{73, 79, 83, 89, 97\}$ plus other orders, and w for a number of infinite families. New T-sequences for lengths 35, 61, 71, 183 and 671 are given.

This paper gives 36 new orders $< 40,000$ for which Hadamard matrices exist.

The current paper lends support to the belief that $c \leq 5$.

1 Introduction

Let $H = (h_{ij})$ be a matrix of order h with $h_{ij} \in (1, -1)$. H is called an *Hadamard matrix* of order h, if $HH^T = H^TH = hI_h$, where I_h denotes the identity matrix of order h.

An *orthogonal design* A, of order n, and type $(s_1, s_2, \ldots, s_u)$, denoted $OD(n; s_1, s_2, \ldots, s_u)$ on the commuting variables $(\pm x_1, \pm x_2, \ldots, \pm x_u, 0)$ is a square matrix of order n with entries $\pm x_k$ where each x_k occurs s_k times in each row and column such that the distinct rows are pairwise orthogonal.

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In other words

$$AA^T = (s_1x_1^2 + \ldots + s_ux_u^2)I_n.$$

It is known that the maximum number of variables in an orthogonal design is $\rho(n)$, the Radon number, where for $n = 2^a b$, b odd, set $a = 4c + d, 0 \le d < 4$, then $\rho(n) = 8c + 2^d$.

$OD(4t; t, t, t, t)$, otherwise called Baumert-Hall arrays, and $OD(2^s; a, b, 2^s - a - b)$ have been extensively used to construct Hadamard matrices and weighing matrices. For details see Geramita and Seberry [8].

Cooper and J.S.Wallis (=Seberry) first defined T-matrices of order t to construct $OD(4t; t, t, t, t)$ (which at that time they called Hadamard arrays). Four circulant (type 1) matrices T_1, T_2, T_3, T_4 of order t which have entries $0, +1$ or -1 and which are non-zero for each of the t^2 entries for exactly one i, i.e.

$$T_i * T_j = 0 \text{ for } i \neq j,$$

where * is the Hadamard (or element by element) product, and which satisfy

$$\sum_{i=1}^{4} T_i T_i^T = tI_t$$

are called *T-matrices of order t*.

We know that if the row sum (and column sum) of a T-matrix, T_i, of order t is x_i then

$$\sum_{i=1}^{4} x_i^2 = t.$$

Notation. We use $T = (t_{ij})$ given by $t_{ij} = 1$ for $j - i = 1$ and 0 otherwise for the shift matrix.

Further, we have the following important theorem.

Theorem 1 (Cooper–Seberry–Turyn) . *Suppose there exist T-matrices T_1, T_2, T_3, T_4 of order t (assumed to be circulant or block circulant = type 1). Let a, b, c, d be commuting variables. Then*

$$A = aT_1 + bT_2 + cT_3 + dT_4$$

$$B = -bT_1 + aT_2 + dT_3 - cT_4$$

$$C = -cT_1 - dT_2 + aT_3 + bT_4$$

$$D = -dT_1 + cT_2 - bT_3 + aT_4$$

can be used in the Goethals-Seidel array (or J. Seberry Wallis-Whiteman array for block-circulant i.e. type 1 and 2 matrices)

$$\begin{bmatrix} A & BR & CR & DR \\ -BR & A & D^TR & -C^TR \\ -CR & -D^TR & A & B^TR \\ -DR & C^TR & -B^TR & A \end{bmatrix} \quad (1)$$

where R is the permutation matrix which transforms circulant to back-circulant matrices or type 1 to type 2 matrices, to form an $OD(4t; t, t, t, t)$.

Replacing the variables of Theorem 1 by Williamson type matrices we have:

Method 1 (Cooper–Seberry–Turyn) Suppose there exist T-matrices T_1, T_2, T_3, T_4 of order t (assumed to be circulant). Let A, B, C, D be Williamson type matrices of order m. Then

$$\begin{aligned} X &= T_1 \times A + T_2 \times B + T_3 \times C + T_4 \times D \\ Y &= T_1 \times -B + T_2 \times A + T_3 \times D + T_4 \times -C \\ Z &= T_1 \times -C + T_2 \times -D + T_3 \times A + T_4 \times B \\ W &= T_1 \times -D + T_2 \times C + T_3 \times -B + T_4 \times A \end{aligned}$$

can be used in the Goethals–Seidel array to form an Hadamard matrix of order $4mt$

$$GS = \begin{bmatrix} X & YR & ZR & WR \\ -YR & X & -W^TR & Z^TR \\ -ZR & W^TR & X & -Y^TR \\ -WR & -Z^TR & Y^TR & X \end{bmatrix}$$

Remark 1 The survey of Seberry and Yamada [18] gives most presently known T-sequences and T-matrices. Some new results are given in this paper. For $t = 67$ there are only T-matrices known and not as yet T-sequences. These sequences, using Method 1, are a prolific source of Hadamard matrices and $OD(4t; t, t, t, t)$.

Turyn has also a construction which says that an $OD(4t; t, t, t, t)$ implies the existence of an $OD(20t; 5t, 5t, 5t, 5t)$ and Ono-Sawade-Yamamoto another which gives an $OD(36t; 9t, 9t, 9t, 9t)$ from an $OD(4t; t, t, t, t)$. However neither yields T-matrices and neither is recursive. In addition there are $OD(4t; t, t, t, t)$ whenever $2t$ is the order of a Hadamard matrix [14, 6].

Hammer, Sarvate and Seberry [9] applied Kharaghani's method [11] to $OD(n; s_1, ..., s_u)$ and in particular to $OD(4t; t, t, t, t)$ obtaining $OD(12s^2t; 3s^2t, 3s^2t, 3s^2t, 3s^2t)$ and $OD(20s^2t; 5s^2t, 5s^2t, 5s^2t, 5s^2t)$ where s is the length of T-sequences.

Yang has other important constructions which give long sequences with zero auto correlation function but not orthogonal designs. There are details in [4]. Yang has given powerful theorems reformulated in [13] which yield many new $OD(4t; t, t, t, t)$ and Hadamard matrices of order $4t$ from T-sequences of length t. His construction may be stated as

Method 2 If there are base sequences of lengths $m+p, m+p, m, m$ and y is a Yang number then there are T-sequences of lengths $t = (2m + p)y$.

For more information on the values $(2m+p)$ and y see [12, 13, 18]. For the known values of Williamson type matrices see [16, 19, 18] and the tables in [10, 18].

We find here the following new orders of Hadamard matrices: $4.q$ ($q < 10,000$) where $q =$ 213, 781, 1349, 1491, 1633, 2059, 2627, 2769, 3479, 3763, 4331, 4899, 5467, 5609, 5893, 6177, 6461, 6603, 6887, 7739, 8023, 8591, 9159, 9443, 9727, 9869.

2 Background and Kharaghani type results

Kharaghani (1985) defined $C_k = [h_{ki} \cdot h_{kj}]$ and applying that to Hadamard matrices of order $4p$ obtained $4p$ symmetric matrices of order $4p$, satisfying

$$\left.\begin{array}{ll} C_iC_j = 0 & i \neq j \\ \sum_{i=1}^{4p} C_i^2 = (4p)^2 I_{4p} & \end{array}\right\}. \qquad (2)$$

He then used this to show there are Bush-type (blocks J_{4p} down the diagonal) and Szekeres-type ($h_{ij} = -1 \Rightarrow h_{ji} = 1$ and not necessarily vice versa). By using a symmetric Latin square he could also have shown that regular symmetric Hadamard matrices with constant diagonal of order $(4p)^2$ could be constructed by his method.

The result we now give is motivated by Hammer, Sarvate and Seberry [9] but uses a different technique to obtain more powerful results.

Before proceeding to our main theorem we will illustrate by two examples:

Use Kharaghani's method to form $4p$ matrices of order $4p$ satisfying (2) from a Hadamard matrix of order $4p$.

Use these to form 4 block circulant matrices A, B, C, D with first rows

$A: C_1 C_2 \dots C_{3p}$

$B: C_{3p+1} \dots C_{4p} C_1 \dots C_{2p}$

$C: C_{2p+1} \dots C_{4p} C_1 \dots C_p$

$D: C_{p+1} \dots C_{4p}$

These are now used in a modified Goethals - Seidel - or Seberry(Wallis) - Whiteman array. This gives the theorem:

Theorem 2 *If there is a Hadamard matrix of order* $4p$ *there is a Hadamard matrix of order* $16.3.p^2$.

Use Kharaghani's method to make $4p$ matrices of order $4p$, $C_1, C_2, \dots, C_{4p}$ as in Hammer, Sarvate, Seberry. The matrices now have variable entries

$A: C_1, \dots, C_p, C_{2p+1}, \overline{C}_{2p+1}, \dots, C_{4p}, \overline{C}_{4p}$

$B: C_{p+1}, \dots, C_{2p}, C_{2p+1}, C_{2p+1}, \dots, C_{4p}, C_{4p}$

$C: C_{2p+1}, \dots, C_{3p}, C_1, \overline{C}_1, C_2, \overline{C}_2, \dots, C_{2p}, \overline{C}_{2p}$

$D: C_{3p+1}, \dots, C_{4p}, C_1, C_1, C_2, C_2, \dots, C_{2p}, C_{2p}$

where $\overline{C}_s = -C_s$.

Use these to form block circulant matrices which are used in the Goethals - Seidel array. This gives

Theorem 3 *If there is an* $OD(4p; p, p, p, p)$ *there is an* $OD(80p^2; 20p^2, 20p^2, 20p^2, 20p^2)$ *and an Hadamard matrix of order* $16.5.p^2$.

These examples do not give new Hadamard matrices of small order but do give new families. However, if the method is used starting with an $OD(4p; s_1, s_2, s_3, s_4)$ or $OD(4p; p, p, p, p)$ we can get new OD's and Hadamard matrices.

Theorem 4 *Suppose there exists an* $OD(4t; s_1, s_2, s_3, s_4)$*, in particular an* $OD(4t; t, t, t, t)$*, the following* OD*'s exist, the particular case is given in brackets.*

(i) $OD(16t^2; 4ts_1, 4ts_2, 4ts_3, 4ts_4)$, $(OD(16t^2; 4t^2, 4t^2, 4t^2, 4t^2))$;

(ii) $OD(48t^2; 12ts_1, 12ts_2, 12ts_3, 12ts_4)$, $(OD(48t^2; 12t^2, 12t^2, 12t^2, 12t^2))$;

(iii) $OD(80t^2; 20ts_1, 20ts_2, 20ts_3, 20ts_4)$, $(OD(80t^2; 20t^2, 20t^2, 20t^2, 20t^2))$.

Proof. As in Hammer, Sarvate and Seberry, let $S = (a_{ij})$ be the OD. Replace all the variables of S by 1 making a *weighing matrix*, U, of order $4t$ and weight $w = s_1 + s_2 + s_3 + s_4$

(in the particular case $w = 4p$). Write S_k and U_k for the kth rows of S and U respectively. Form

$$C_k = S_k \times U_k^T$$

where $\times$ is the Kronecker product.

Then

$$\begin{aligned} C_k C_j^T &= (S_k \times U_k^T)(S_j \times U_j^T)^T \\ &= S_k S_j^T \times U_k^T U_j \\ &= 0 \end{aligned}$$

if $k \neq j$ because S is an orthogonal design.

Now

$$\begin{aligned} \sum_{k=1}^{4p} C_k C_k^T &= \sum_{k=1}^{4p} (S_k \times U_k^T)(S_k \times U_k^T)^T \\ &= \sum_{k=1}^{4p} S_k S_k^T \times U_k^T U_k \\ &= \left(\sum_{j=1}^{4} s_j x_j^2 w \right) I_{4p} \end{aligned}$$

by the properties of U.

In particular where $s_j = p$, all j, we get

$$\sum_{k=1}^{4p} C_k C_k^T = 4p^2(x_1^2 + x_2^2 + x_3^2 + x_4^2) I_{4p}.$$

The $C_1, \ldots, C_{4t}$ are now used to form first rows for block circulant matrices, as in the examples leading to Theorems 4 and 5 for (ii) and (iii), or with

$$\begin{aligned} A &: C_1 C_2 \ldots C_t \\ B &: C_{t+1} \ldots C_{2t} \\ C &: C_{2t+1} \ldots C_{3t} \\ D &: C_{3t+1} \ldots C_{4t} \end{aligned}$$

for (i). □

The examples above illustrate that we really need $4t$ matrices $P_1, \ldots, P_{4t}$ of order q, with elements $0, +1, -1$ such that in each of the q^2 places one and only one of the P_i has a nonzero element, i.e. $P_i * P_j = 0, i \neq j$

$$\sum_{i=1}^{4t} P_i \text{ is a } (1, -1) \text{ matrix}$$

$$P_i P_i^T = \text{ constant } I.$$

Then

$$\sum_{i=1}^{4p} C_i \times P_i$$

would be an Hadamard matrix of order $4tq$ or an $OD(4tq; tq, tq, tq, tq)$ say.

Note we need no algebraic relation between the P_i, except disjointness, as $C_iC_j = 0$, $i \neq j$.

The remainder of this paper is devoted to finding matrices such as the P_i. We give one method here which is a blending of ideas derived from writings of Turyn and C.H.Yang.

Let h, i, j, k be symbols so that $h^2 = i^2 = j^2 = k^2 = 1$, $xy = 0$, $x \neq y$, $x, y \in \{h, i, j, k\}$. Call a sequence of length m of symbols $\pm h, \pm i, \pm j, \pm k$ which have zero periodic (or non periodic) autocorrelation function an *m, δ – sequence.*

For example, $hi\bar{i}jj$ is a 5,δ-sequence with zero non-periodic (implies also periodic) autocorrelation function because for $hi\bar{i}jj$ we form the matrix

$$\begin{bmatrix} h & i & \bar{i} & j & j \\ 0 & h & i & \bar{i} & j \\ 0 & 0 & h & i & \bar{i} \\ 0 & 0 & 0 & h & i \\ 0 & 0 & 0 & 0 & h \end{bmatrix}$$

and notice the inner product of any row with any other is zero.

In particular, if T_1, T_2, T_3, T_4 are circulant T-matrices (which can be obtained from T-sequences) of order t then the first row of

$$X = hT_1 + iT_2 + jT_3 + kT_4$$

is a t, δ-sequence because

$$\begin{aligned} XX^T &= h^2T_1T_1^T + i^2T_2T_2^T + j^2T_3T_3^T + k^2T_4T_4^T \\ &= tI_t \end{aligned}$$

using $xy = 0$, $x \neq y$, $x, y \in \{h, i, j, k\}$.

Construction 1 *Suppose we have $4p$ matrices $C_1, \ldots, C_{4p}$ of order $4p$ constructed by Kharaghani's method (as modified by Hammer, Sarvate and Seberry (i.e. with variable entries)) and an m, δ-sequence $m_1, m_2, \ldots, m_m$. To simplify writing write*

$$\begin{aligned} &D_h \text{ for } [C_1 : C_2 : \ldots : C_p] \\ &D_i \text{ for } [C_{p+1} : \ldots : C_{2p}] \\ &D_j \text{ for } [C_{2p+1} : \ldots : C_{3p}] \\ &D_k \text{ for } [C_{3p+1} : \ldots : C_{4p}]. \end{aligned}$$

We now form 4 first rows of A, B, C, D by replacing the elements of the m, δ-sequence. To form A replace h by D_h, $\bar{h}$ by $-D_h$, i by D_i, $\bar{i}$ by $-D_i$, j by D_j, $-j$ by $-D_j$, k by D_k, $-k$ by $-D_k$ respectively and then complete to a block circulant matrix.

A is formed by

$$\pm h \longrightarrow \pm D_h$$

$$\pm i \longrightarrow \pm D_i$$
$$\pm j \longrightarrow \pm D_j$$
$$\pm k \longrightarrow \pm D_k$$

B is formed by

$$\pm h \longrightarrow \pm D_i$$
$$\pm i \longrightarrow \pm D_j$$
$$\pm j \longrightarrow \pm D_k$$
$$\pm k \longrightarrow \pm D_h$$

C is formed by

$$\pm h \longrightarrow \pm D_j$$
$$\pm i \longrightarrow \pm D_k$$
$$\pm j \longrightarrow \pm D_h$$
$$\pm k \longrightarrow \pm D_i$$

D is formed by

$$\pm h \longrightarrow \pm D_k$$
$$\pm i \longrightarrow \pm D_h$$
$$\pm j \longrightarrow \pm D_i$$
$$\pm k \longrightarrow \pm D_j$$

Each is then completed to a block circulant matrix. □

To illustrate we again use the 5,δ-sequence $hi\bar{i}jj$

$$A = \begin{bmatrix} D_h & D_i & \bar{D}_i & D_j & D_j \\ D_j & D_h & D_i & \bar{D}_i & D_j \\ D_j & D_j & D_h & D_i & \bar{D}_i \\ \bar{D}_i & D_j & D_j & D_h & D_i \\ D_i & \bar{D}_i & D_j & D_j & D_h \end{bmatrix}$$

where

$$D_h = \begin{bmatrix} C_1 & C_2 & C_3 & \dots & C_p \\ C_p & C_1 & C_2 & \dots & C_{p-1} \\ \vdots & & & \dots & \vdots \\ C_2 & C_3 & C_4 & \dots & C_1 \end{bmatrix}$$

So

$$D_h D_h^T = I_p \times \sum_{i=1}^{p} C_i^2$$

$D_h D_i^T = 0$ and $D_h D_j^T = 0$ since $C_a C_b = 0$, $a \neq b$.

Thus

$$AA^T = I_5 \times (D_h D_h^T + 2D_i D_i^T + 2D_j D_j^T) + (T + T^4) \times (-D_i D_i^T + D_j D_j^T)$$

Similarly

$$BB^T = I_5 \times (D_iD_i^T + 2D_jD_j^T + 2D_kD_k^T) + (T + T^4) \times (-D_jD_j^T + D_kD_k^T)$$

$$CC^T = I_5 \times (D_jD_j^T + 2D_kD_k^T + 2D_hD_h^T) + (T + T^4) \times (-D_kD_k^T + D_hD_h^T)$$

$$DD^T = I_5 \times (D_kD_k^T + 2D_hD_h^T + 2D_iD_i^T) + (T + T^4) \times (-D_hD_h^T + D_iD_i^T)$$

So

$$AA^T + BB^T + CC^T + DD^T = 5I_{5p} \times \sum_{i=1}^{4p} C_i^2$$

$$= 20p^2(x_1^2 + x_2^2 + x_3^2 + x_4^2)I_{20p^2}$$

We now use A, B, C, D in the modified GS array to form an $OD(80p^2; 20p^2, 20p^2, 20p^2, 20p^2)$.

Using this method we can establish

Theorem 5 *Suppose an $OD(4p; s_1, s_2, s_3, s_4)$ exists. Suppose there are T-matrices of order t. Then there is an $OD(16pt; 4ts_1, 4ts_2, 4ts_3, 4ts_4)$, an $OD(16p^2t; 4pts_1, 4pts_2, 4pts_3, 4pts_4)$ and an Hadamard matrix of order $16pt$ and $16p^2t$.*

Proof. The matrix of order $16pt$ follows by putting the $OD(4p; s_1, s_2, s_3, s_4)$ in place of the variables of the $OD(4t; t, t, t, t)$ constructed via the T-matrices.

The matrix of order $16p^2t$ is constructed via the construction just given. □

Corollary 1 *Suppose an $OD(4p; s_1, s_2, s_3, s_4)$ exists. Then there is an $OD(16pt; 4ts_1, 4ts_2, 4ts_3, 4ts_4)$ and an $OD(16p^2t; 4pts_1, 4pts_2, 4pts_3, 4pts_4)$ for all the orders of T-matrices listed above and in particular for all orders of $t \leq 100$ except possibly $t \in \{73, 79, 83, 89, 97\}$.*

We give these sequences for odd lengths (Corollary 4.107 is in [7]):
3: hij
5: $h\bar{h}iij$
7: $hh\bar{h}ijik$
9: $hiii\bar{i}jj\bar{j}j$
11: $h\bar{h}\bar{h}ih\bar{i}h\bar{i}\bar{i}ij$
13: Corollary 4.107 $hhi\bar{h}ihij\bar{j}\bar{j}j\bar{j}\bar{j}$
15: $ihjhkkk\bar{h}\bar{k}kkh\bar{k}k\bar{k}$
17: Golay
19: Corollary 4.107 — $hh\bar{h}hhiii\bar{i}jj\bar{j}\bar{j}k\bar{k}kk\bar{k}$
21: Golay
23: Corollary 4.107
25: Corollary 4.107
27: Golay
29: Corollary 4.107
31: Corollary 4.107
33: Golay
35: Seberry - Sproul
37: Williamson

39: Yang
41: Golay
43: Williamson
45: Yang
47: Turyn
49: Yang
51: Corollary 4.107
53: Golay
55: Turyn (5× construction), Yang.
57: Yang
59: Corollary 4.107
61: Hunt (T-matrix not sequence): T-sequences given below.
63: Yang
65: Golay
67: Sawade (T-matrix not sequence)
69: Yang
71: Koukouvinos, Kounias, Seberry, C.H. Yang and J. Yang
73:

3 New Hadamard matrices

We now give three new T-sequences of lengths $2s+1 = 35$, 61 and 71. Each set of sequences is equivalent to a set of base sequences of lengths $s+1$, $s+1$, s, s.
The following are T-sequences (T-matrices) of length $35 = 5^2 + 3^2 + 0^2 + 1^2$.

$$\begin{aligned}
T_1 &= \{1,2,4,5,9,-10,14,-15,17\} \\
T_2 &= \{3,-6,-7,8,11,-12,-13,-16,-18\} \\
T_3 &= \{19,-21,23,-25,-26,-28,29,31,33,-35\} \\
T_4 &= \{-20,-22,24,-27,30,32,34\}
\end{aligned}$$

The following are T-sequences (T-matrices) of length $61 = 2^2 + 5^2 + 4^2 + 4^2$. Since these sequences are equivalent to base sequences of lengths 31, 31, 30, 30 they yield, using Yang multipliers, new T-sequences of lengths 183 and 671.

$$\begin{aligned}
T_1 &= \{1,-2,-4,-6,-8,-10,12,-14,-16,18,20,22,-24,26,-28,30\} \\
T_2 &= \{3,5,7,9,-11,-13,15,-17,19,21,23,25,-27,-29,31\} \\
T_3 &= \{-32,-33,-36,37,38,40,-42,43,44,46,-47,49,50,51,53,-55,-56,57, \\
&\quad -60,61\} \\
T_4 &= \{34,-35,39,41,-45,48,-52,54,58,59\}
\end{aligned}$$

The following are T-sequences (T-matrices) of length $71 = 6^2 + 5^2 + 3^2 + 1^2$.

$$\begin{aligned}
T_1 = \{&1,-2,-3,4,5,6,-7,8,9,10,-11,-12,-13,-14,15, \\
&16,-17,18,19,-20,21,22,23,24\}
\end{aligned}$$

q			Method
213	=	3×71	1
781	=	11×71	1
1349	=	19×71	1
1491	=	21×71	1
1633	=	27×71	1
2059	=	29×71	1
2627	=	37×71	1
2769	=	39×71	1
3479	=	49×71	1
3763	=	53×71	1
4331	=	61×71	1
4899	=	69×71	1
5467	=	$7 \times 11 \times 71$	2
5609	=	79×71	1
5893	=	83×71	1
6177	=	87×71	1
6461	=	91×71	1
6603	=	93×71	1
6887	=	97×71	1
7739	=	71×109	1
8023	=	113×71	1
8591	=	121×71	1
9159	=	129×71	1
9443	=	$7 \times 19 \times 71$	2
9727	=	137×71	1
9869	=	139×71	1

Table 1New Hadamard matrices

$$\begin{aligned} T_2 &= \{25, 26, 27, 28, -29, 30, 31, -32, 33, 34, 35, -36, 37, -38, \\ &\quad 39, -40, 41, -42, -43, -44, -45, 46, 47\} \\ T_3 &= \{48, 49, 50, 51, -52, -56, 57, 58, 60, -64, 65, -66, -71\} \\ T_4 &= \{-53, -54, 55, -59, 61, -62, 63, 67, 68, -69, -70\} \end{aligned}$$

The new Hadamard matrices may now be constructed as in Table 1.

Method 3 Seberry and Yamada [18] gave the following definition:

Definition 1 We call k a Koukouvinos–Kounias number, or KK number, if $k = g_1 + g_2$ where g_1 and g_2 are both the lengths of Golay sequences.

Then we have

Lemma 1 *Let k be a KK number and y be a Yang number. Then there are T-sequences of length t and $OD(4t; t, t, t, t)$ for $t = yk$.*

q	t	t'
917	3	4
1703	3	4
2227	3	4
2489	3	4
4061	3	4
5109	3	4
6419	3	4
6623	4	10
6943	3	4
9563	3	4

Table 2: New Hadamard matrices of order $2^s q$, $t \leq s < t'$

Example. This gives T-sequences of lengths 2.101, 2.109, 2.113, 8.127, 2.129, 2.131, 8.151, 8.157, 16.163, 2.173, 4.179, 4.185, 4.193, 2.201, 2.205, 2.209, 2.213, 2.221, 2.257, 2.261, 2.269.

With the application of this method we find new orders of Hadamard matrices which are given in Table 2.
(Note: t' is given in Jenkins, Koukouvinos and Seberry [10, Table 6].)

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