KREIN COVERS OF COMPLETE GRAPHS

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ABSTRACT

Let G be an antipodal distance regular cover of a complete graph with index r. If $r = 2$ then it is known that the neighbourhood of a vertex in G is strongly regular, and the Krein bound is tight for G. We use the theory of spherical designs (due to Delsarte, Goethals and Seidel) to show that if $r > 2$ and the Krein bound is tight for G then the neighbourhood of any vertex in G is strongly regular. Further, if θ denotes the second largest eigenvalue of G then r must divide $\theta + 1$, and if $r = \theta + 1$ then G arises by a standard construction from a generalised quadrangle with a spread. If θ is a prime power and r divides $\theta + 1$, covers of index r for which the Krein bound holds can be constructed and in many cases the neighbourhoods of a vertex in these covers are new strongly regular graphs.

1. ANTIPODAL DISTANCE REGULAR COVERS

We are going to study a class of distance regular covers of complete graphs. Thus we begin by explaining what a cover is, and what it means for one to be distance regular. (However we assume that the reader knows what a distance-regular graph is. A brief introduction will be found in [1], and an extensive treatment in [2]. Unsupported assertions about properties of antipodal distance regular covers of K_n are taken from [8].)

We say that a graph H is *cover* of a graph G if there is a surjection f from $V(H)$ onto $V(G)$ such that f maps edges of H to edges of G, and the restriction of f to the neighbourhood of any vertex is an isomorphism. (If we view G and H

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as topological spaces, this is equivalent to the topologist's definition of a cover, if it helps.) If $v \in V(G)$ then $f^{-1}(v)$ called a *fibre* of the cover. When G is connected it is not hard to show that all fibres must have the same cardinality, which we call the *index* of the cover. In general the cover is not determined just by the graphs H and G — there may be more than one possible local isomorphism f from H to G, and so it may be necessary to explicitly identify the fibres. A cover H of G with diameter d is *antipodal* if two vertices of H are at distance d if and only they are in the same fibre.

Informally, any cover of G with index r can be constructed as follows. For each vertex in G there is a corresponding set of r disjoint vertices in H, and if u and v are adjacent in G then the corresponding r-sets are joined by the edges of an r-matching. The cube is an antipodal cover of K_4 with index two and the line graph of Petersen's graph is an antipodal cover of K_5 with index three. Both these covers have the property that the number of common neighbours of two vertices is determined by the distance between them. Any antipodal cover of K_n with this property is *distance regular*, and must have diameter equal to three. If H is an antipodal distance regular cover of K_n then we denote the number of common neighbours of two adjacent points by a_1, and the number of common neighbours of two points at distance two by c_2. The parameters n, r, a_1 and c_2 are related by

$$n - 2 - a_1 = (r - 1)c_2. \tag{1}$$

Thus a_1 is determined by the other three parameters. An expression such as "H is a cover with parameters (n, r, c_2)" will indicate that H is an distance regular antipodal cover of K_n with index r, and a_1 determined by (1). We note that any distance regular cover of K_n with diameter three must be antipodal.

Let H be an (n, r, c_2) cover. Since H is distance regular with diameter three, its adjacency matrix $A(H)$ has exactly four distinct eigenvalues. Since it is a cover of K_n, two of these eigenvalues are $n - 1$ and -1, with multiplicities 1 and $n - 1$ respectively. The remaining two eigenvalues will be denoted by θ and τ, and are the zeros of the quadratic

$$x^2 - (a_1 - c_2)x - (n - 1). \tag{2}$$

We will refer to θ and τ as the *non-trivial* eigenvalues of G. We assume that $\theta \geq \tau$; from (2) we have $\theta\tau = 1 - n$, whence we deduce that $\theta > 0 > \tau$. If m is the multiplicity of τ as an eigenvalue of G then

$$m = \frac{n(r - 1)\theta}{\theta - \tau}. \tag{3}$$

The fact that m must be an integer is a very useful restriction on the parameter set (n, r, c_2). From (2) we have that

$$\theta + \tau = a_1 - c_2, \quad rc_2 = -(\theta + 1)(\tau + 1).$$

If θ and τ are not integers then they are conjugate roots of (2) and their multiplicities as eigenvalues of $A(G)$ are equal, and hence equal to $n(r-1)/2$. From (3) we then obtain $\tau = -\theta$, and thus that $a_1 = c_2$.

In addition to the above multiplicity condition, we also have the *Krein condition*,

$$((r-1)^2 - 1)((n-1)^2 + \tau^3) \geq 0.$$

If $r > 2$ then

$$-\tau \leq \theta^2. \tag{4}$$

An antipodal distance regular cover of K_n with index two is equivalent to a "regular 2-graph". These have received considerable attention. (See [12, 13].) It is known [2: Theorem 1.5.3(iii)], and comparatively easy to prove, that the neighbourhood of any vertex in such a cover must be strongly regular. When the index is greater than two, this is not generally true. In Section 4 however, we will prove that if (4) holds with equality then the neighbourhood of any vertex in the cover must be strongly regular.

2. REPRESENTATIONS

Let G be a distance regular graph on n vertices with adjacency matrix A, and let θ be an eigenvalue of A with multiplicity m. Then, since A is symmetric, θ is real and the eigenvectors belonging to θ form a subspace of $\mathbf{R}^m$ with dimension m. Let U be $n \times m$ matrix whose columns form an orthonormal basis for this eigenspace. Then $U^T U = I$ and $AU = \theta U$. If $A = (a_{ij})$ and u_i denotes the i-th row of U then the last equation is equivalent to

$$\theta u_i = \sum_{j \sim i} u_j. \tag{1}$$

(Here we write $j \sim i$ to denote that j is adjacent to i in G.)

The mapping ρ which takes the vertex i of G to the vector u_i will be called a *representation of* G, with eigenvalue θ. We have the following important result, the proof of which may found in [7: Lemma 2.2] (for example).

2.1 LEMMA. *Let G be a distance regular graph and let ρ be a representation of G with eigenvalue θ. If i and j are two vertices of G then the inner product $(\rho(i), \rho(j))$ is determined by the distance between i and j in G.* □

Applying this when $i = j$, we deduce that all the vectors $\rho(i)$ have the same length. Hence the image of $V(G)$ under ρ is contained in a sphere in $\mathbf{R}^m$, with centre at the origin. If i and j are at distance r we use w_r to denote the cosine of the angle between $\rho(i)$ and $\rho(j)$. If G has diameter d, we refer to $(w_0, \ldots, w_d)$ as the *sequence of cosines* belonging to θ. From (1) we have

$$\theta\rho(i) = \sum_{j \sim i} \rho(j).$$

Taking the inner products of both sides of this with $\rho(i)$ we obtain easily that

$$w_1 = \frac{\theta}{k},$$

where k is the valency of G. More complicated expressions can be found for the other cosines. For our purposes the following will suffice.

2.2 LEMMA. *Let G be an antipodal distance regular cover of K_n, with least eigenvalue τ. Then the cosines with respect to τ are:*

$$w_0 = 1, \quad w_1 = \frac{\tau}{n-1}, \quad w_2 = -\frac{\tau}{(r-1)(n-1)}, \quad w_3 = -\frac{1}{r-1}. \;\square$$

2.3 LEMMA. *Let G be an antipodal distance regular graph with fibres of cardinality r and antipodal quotient H. Let θ be an eigenvalue of G which is not also an eigenvalue of H, and let its multiplicity m. Then the image of the neighbourhood of a vertex under the representation associated to θ spans a space of dimension at most $m - r + 2$.*

Proof. Let ρ denote the representation associated to θ, let N be the set of all neighbours of some vertex u in G, and let F be the fibre of G containing u. Suppose that G has diameter d. If v is in N and x and y are distinct vertices in $F \setminus u$ then

$$\operatorname{dist}(v, x) = d - 1 = \operatorname{dist}(v, y)$$

whence $\rho(v)$ must be orthogonal to $\rho(x) - \rho(y)$. Hence the image of N under ρ lies in the subspace of $\mathbf{R}^m$ orthogonal to the space spanned by

$$\{\rho(x) - \rho(y) : x, y \in F \setminus u\}. \tag{2}$$

From [2: Proposition 4.2.3] and the remarks following, the representation ρ is injective on $V(G)$ and the image of any fibre is a regular simplex spanning a subspace of dimension $r-1$. This implies that the vectors in (2) span a subspace of dimension $r-2$, and so the lemma follows. □

3. SPHERICAL DESIGNS

Let X be a finite subset of the unit sphere in $\mathbf{R}^m$ or, more generally, of any sphere centred at the origin. We say that X is a *spherical t-design* if the average value over the points in X of any polynomial in m variables of degree at most t, is equal to its average value over the entire sphere. The concept is due to Delsarte, Goethals and Seidel [6]. We will only be concerned with the cases where $t \leq 3$.

Note that X is a 1-design if and only if its centre of mass is the origin. Our first result is a special case of Theorem 3.1 from [9].

3.1 LEMMA. *If X is a finite subset of the unit sphere in $\mathbf{R}^m$ then*

$$\frac{1}{|X|^2} \sum_{x,y \in X} (x,y)^2 \geq \frac{1}{m}.$$

If X is a 1-design then equality holds if and only if X is a 2-design. □

If G is a distance regular graph and ρ is a representation of G then $\rho(G)$ is always a 2-design. It is not, in general, a 3-design. (If ρ belongs to the i-th eigenvalue of G then $\rho(G)$ is a 3-design if and only if the i-th Krein parameter $q_{ii}(i)$ is zero.)

A subset X of the unit sphere in $\mathbf{R}^m$ is an *s-distance set* if the distance between distinct elements of X takes on at most s values. Thus, a 1-distance set would be a regular simplex. If α is a real number then $G_\alpha(X)$ will denote the graph with vertex set X, with vectors x and y from X adjacent if and only if $(x,y) = \alpha$. Of course, if α is not chosen appropriately, then $G_\alpha(X)$ is just a complicated notation for the empty graph.

We will need the following result, which is a special case of [6: Theorem 7.4].

3.2 LEMMA. *Let X be a 2-distance set on the unit sphere in $\mathbf{R}^m$ containing two points with inner product α. If X is a 2-design then $G_\alpha(X)$ is strongly regular or complete, and the map taking a vertex of G to the corresponding point in X is a representation of G.* □

4. THE KREIN BOUND

4.1 THEOREM. *Let G be an antipodal distance regular cover of K_n, with non-trivial eigenvalues θ and τ. If $\tau = -\theta^2$ then the neighbourhood of any vertex in G is strongly regular.*

Proof. Let n, r, a_1 and c_2 be the parameters of our cover and let ρ be the representation belonging to τ. Assume that $1 \in V(G)$ and let N be the set of vertices adjacent to 1. If $i \in N$ define

$$u_i = \rho(i) - w_1\rho(1)$$

and let $\hat{u}_i$ be the corresponding unit vector. Then

$$\sum_{i\in N} \rho(i) = \tau\rho(1)$$

and therefore the vectors $\hat{u}_i$ form a 1-design. We aim to apply Lemmas 3.1 and 3.2 to the set $\{u_i : i \in N\}$.

If $\hat{u}_i$ and $\hat{u}_j$ correspond to adjacent vertices in N then

$$\begin{aligned}(\hat{u}_i, \hat{u}_j) &= \frac{w_1 - w_1^2}{1 - w_1^2} \\ &= \frac{w_1}{1 + w_1} \\ &= \frac{\tau}{\tau + n - 1} \\ &= \frac{\tau}{\tau - \theta\tau} \\ &= \frac{1}{1 - \theta}.\end{aligned}$$

A similar computation show that i and j are not adjacent then

$$(\hat{u}_i, \hat{u}_j) = \frac{\frac{\theta}{r-1} - 1}{\theta^2 - 1}.$$

Now fix i in N. Then, using the identity $n - 2 - a_1 = (r-1)c_2$, we find that

$$\sum_{j\in N} (\hat{u}_i, \hat{u}_j)^2 = \frac{(r-1)(\theta+1)^2(\theta^2 - 2\theta + 1 + a_1) + c_2(\theta - r + 1)^2}{(r-1)(\theta^2-1)^2}.$$

As $a_1 - c_2 = \theta + \tau$, we can eliminate a_1 from the above expression. If we also make use of the fact that $rc_2 = -(\theta+1)(\tau+1)$ then the outcome is that the right side equals

$$\frac{\theta^3+\theta\tau}{(\theta^2-1)(\theta-1)}-\frac{\theta^2(\tau+1)}{(r-1)(\theta^2-1)(\theta-1)}.$$

and, if $\tau=-\theta^2$, this reduces to

$$\frac{\theta^2}{(r-1)(\theta-1)}.$$

Let d be the dimension of the space spanned by the vectors u_i, for i in N. Applying Lemma 3.1 with $|X|=-\theta\tau=\theta^3$, we obtain the inequality

$$(r-1)\theta(\theta-1)\leq d. \tag{1}$$

The multiplicity of τ as an eigenvalue of G is

$$\frac{n(r-1)\theta}{\theta-\tau}=(r-1)(\theta^2-\theta+1).$$

By Lemma 2.3, the span of $\rho(N)$ has dimension at most

$$(r-1)(\theta^2-\theta+1)-r+2=(r-1)(\theta^2-\theta)+1.$$

Since each vector u_i is orthogonal to $\rho(1)$, it follows that they span a space of dimension one less than this, thus

$$d\leq(r-1)(\theta^2-\theta).$$

Hence equality holds in (1)and so, by Lemma 3.1, we find that $\rho(N)$ is a spherical 2-design in $\mathbf{R}^d$. As any two vertices in N are at distance at most two in G, we may now apply Theorem 3.2 to deduce that the subgraph of G induced by N is strongly regular or complete. If the neighbourhoods are complete then G is complete, and does not have diameter three. □

Let us call a distance regular cover of K_n with diameter three and $\tau=-\theta^2$ a *Krein cover.* The argument in the proof above does not determine the parameters of the neighbourhoods. For this purpose it suffices to compute their eigenvalues.

If G is a Krein cover and N is the neighbourhood of a vertex in G, it follows from the proof of Theorem 4.1 that the map taking i in N to $\hat{u}_i$ is a representation of N. If i and j are adjacent in N then $(\hat{u}_i,\hat{u}_j)$ is the first cosine of this representation. We saw that this inner product is equal to $(1-\theta)^{-1}$ and as N has valency a_1, we thus deduce that

$$-\frac{a_1}{\theta-1}$$

is an eigenvalue of N, with multiplicity $(r-1)\theta(\theta-1)$. Since N is regular with valency a_1, we know that a_1 is an eigenvalue with multiplicity at least one. If $\theta - 1 < a_1$ then N is connected, because the only strongly regular graphs which are not connected are the disjoint unions of complete graphs, and they have least eigenvalue -1. Hence a_1 is a simple eigenvalue, and the multiplicity of the remaining eigenvalue is

$$\theta^3 - 1 - r\theta(\theta - 1) = (\theta - 1)((\theta + 1)^2 - r\theta).$$

Using the fact that the sum of the eigenvalues of N is zero, we now deduce that these eigenvalues are:

$$\sigma = \theta - \frac{(\theta+1)^2}{r},$$
$$\nu = \theta - \frac{\theta+1}{r},$$
$$a_1 = (\theta - 1)\left(\frac{(\theta+1)^2}{r} - \theta\right).$$

Since θ is an integer, $(\theta + 1)/r$ is a rational number. As it is equal to the algebraic integer $\nu - \theta$, it is therefore an integer. Hence we deduce the useful restriction:

4.2 LEMMA. *If there is a Krein cover of K_n with index r and second largest eigenvalue θ then r divides $\theta + 1$.* □

This condition eliminates many otherwise feasible parameter sets. The smallest instance of this is a $(65, 3, 25)$ cover. From the expressions given for a_1 and rc_2 at the end of the proof of Theorem 4.1, we also see that $\theta - 1$ divides a_1 and $\theta^2 - 1$ divides c_2.

5. GENERALISED QUADRANGLES

We will now show that Krein covers with $r = \theta + 1$ are equivalent to a class of generalised quadrangles. We assume some familiarity with the theory of generalised quadrangles as described, for example, in the book by Payne and Thas [11]. We use $GQ(s,t)$ to denote a generalised quadrangle with $s + 1$ points on each line, and $t + 1$ lines through each point. A *spread* in a generalised quadrangle is a set of lines which partition its point set. Thus it must contain exactly $st + 1$ lines.

5.1 LEMMA (Brouwer). *Let $\mathcal{Q}$ be a $GQ(s,t)$ where $t > 1$, which contains a spread S. Let G be the graph with the points of $\mathcal{Q}$ as its vertices, and two vertices are adjacent if they are collinear, and the line joining them is not in S. Then G is a $(st + 1, s + 1, t - 1)$ cover of a complete graph.* □

(This is a fairly straightforward consequence of the definitions.) Brouwer also notes that if a cover with these parameters exists, then the graph obtained from the cover by joining any two vertices in a fibre is strongly regular, with the same parameters as the point graph of a generalised quadrangle. (In particular, any cover such that $r = \theta + 1$ arises in this way.) We now observe that if G is a Krein cover of a complete graph with $r = \theta + 1$ then the parameters of G are $(\theta^3 + 1, \theta + 1, \theta^2 - 1)$. Filling in the fibres of G thus provides a strongly regular graph with same parameters as the point graph of a $GQ(\theta, \theta^2)$. But Cameron, Goethals and Seidel [5: Theorem 7.9] have proved that any strongly regular graph with these parameters must be the point graph of a generalised quadrangle. Thus we have:

5.2 LEMMA. *A Krein cover of K_{s^3+1} with second largest eigenvalue s and index $s+1$ is equivalent to a $GQ(s, s^2)$ with a spread.* □

If there is an automorphism of a $GQ(s, s^2)$ with order m which fixes each component of a spread then we may apply [8: Theorem 6.2] to obtain a Krein cover of K_{s^3+1} with index r/m. Whenever q is a prime power, there is a $GQ(q, q^2)$ with a spread (see [11: Chapter 3.4]) and this spread is fixed component-wise by a cyclic automorphism of order $q + 1$. Thus we have the existence of Krein covers of K_{q^3+1} for all indices dividing $q+1$. For the details of this construction we refer the reader to [3], here we note one consequence.

5.3 LEMMA. *If q is a prime power and r divides $q + 1$ then there is a strongly regular graph with parameters*

$$n = q^3,$$
$$k = (q-1)\left(\frac{(q+1)^2}{r} - q\right),$$
$$\lambda = r\left(\frac{q+1}{r} - 1\right)^3 + r - 3,$$
$$\mu = \left(\frac{q+1}{r} - 1\right)\left(\frac{(q+1)^2}{r} - q\right). \square$$

Proof. The eigenvalues of the neighbourhood of a vertex in a Krein cover were determined in Section 4, and after somewhat tedious computations based on these, the above parameters result. □

When $r = (q+1)/2$ the graphs of Lemma 5.3 have the same parameters as the point graphs of a $GQ(q-1, q+1)$. (We do not know that they must be point graphs of generalised quadrangles.) If $r = 2$ we recover the graphs in the family C20 from Hubaut [10]. The corresponding double covers of K_{q^3+1} were found by Taylor. (See [13].) In the remaining cases the parameter sets (and the graphs) seem to be new. (The covers themselves were first obtained by Peter Cameron in [4]; the relationship is discussed at some length in [3].)

6. APPENDIX

6.1 LEMMA. *For every prime power q there is a $GQ(q, q^2)$ with a spread, Σ say, such that the group of automorphisms of the quadrangle which fix each component of Σ is cyclic of order $q+1$.*

Proof. Let V be a vector space of dimension four over $GF(q^2)$ and, if $x, y \in V$, define

$$h(x,y) := x_0^q y_0 + \cdots + x_3^q y_3.$$

Thus h is a non-singular Hermitian form on V, and the lines spanned by the vectors x such that $h(x,x) = 0$ form a generalised quadrangle in projective 3-space over $GF(q^2)$, with q^2+1 points per line, and $q+1$ lines per point. Denote this generalised quadrangle by $\mathcal{H}$.

The intersection of $\mathcal{H}$ with any non-tangent hyperplane is an ovoid [11]. For the Hermitian form we are using we may take $x_0 = 0$ as a non-tangent hyperplane. The matrices

$$\begin{pmatrix} \lambda & 0 \\ 0 & I_3 \end{pmatrix},$$

where λ is an element of $GF(q^2)$ with order dividing $q+1$, represent unitary homologies with axis the hyperplane $x_0 = 0$ and centre $(1,0,0,0)^T$. Since $(1,0,0,0)^T$ is not on $\mathcal{H}$, any such homology acts fixed-point freely on the points of $\mathcal{H}$ not on its axis. Hence they form non-trivial automorphisms of $\mathcal{H}$ fixing each point on an ovoid. It is easy to see that the matrices $D(\lambda)$ form a cyclic group of order $q+1$.

Passing to the generalised quadrangle dual to $\mathcal{H}$, we obtain a $GQ(q, q^2)$ with a spread fixed component-wise by a cyclic group of order $q+1$. □

The above proof is an improvement of the author's original argument, and is due to Andries Brouwer. It can be shown that there is only one 'cyclic' spread in the unitary GQ, up to isomorphism.

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