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Note on equitable coloring of graphs

T. Karthick

Computer Science Unit
Indian Statistical Institute Chennai Centre
Chennai – 600 113
India
karthick@isichennai.res.in

Abstract

An equitable coloring of a graph G is a proper coloring of the vertices of G such that color classes differ in size by at most one. In this note, we verify the equitable coloring conjecture [W. Meyer, *Amer. Math. Monthly* 80 (1973), 920–922] for some classes of graphs which are defined by forbidden induced subgraphs using known results.

1 Introduction

All of our graphs are simple, finite and undirected, and we follow West [19] for standard notations and terminology. Let $G(V, E)$ be a graph. Let P_t, C_t, K_t respectively denote the path, cycle, complete graph on t vertices. If $\mathcal{F}$ is a family of graphs, a graph G is said to be $\mathcal{F}$*-free* if it contains no induced subgraph isomorphic to any graph in $\mathcal{F}$. If S is a vertex subset of $V(G)$, then $[S]$ denotes the subgraph induced by S. If H is an induced subgraph of G, we write $H \sqsubseteq G$. For a graph G, we denote a partition of $V(G)$ into k (≥ 1) sets by $(V_1, V_2, \ldots, V_k)$.

A *vertex coloring* (or simply *coloring*) of a graph G is an assignment of colors to the vertices of G such that no two adjacent vertices receive the same color. Vertex coloring of graphs has generated a vast literature, and several variations of vertex coloring have been introduced and studied by many researchers; see [10]. Among them, equitable coloring of graphs received much attention.

In a vertex coloring of G, the set of vertices with the same color is called a *color class*. A graph G is said to be *equitably k-colorable* if the vertex set $V(G)$ can be partitioned into k non-empty independent sets $V_1, V_2, \ldots, V_k$ such that $||V_i|-|V_j|| \leq 1$ for every i and j. The smallest integer k for which G is equitably k-colorable is called the *equitable chromatic number* of G, and is denoted by $\chi_e(G)$. This notion was introduced by W. Meyer [15]. For a survey of results on equitable coloring of graphs; see [6, 13]. The *degree* of a vertex in G is the number of vertices adjacent to it. The maximum degree over all vertices in G is denoted by $\Delta(G)$. In 1973, Meyer [15] proposed the following conjecture on equitable coloring.

Conjecture A *[15] (***Equitable Coloring Conjecture (ECC)***) Let G be a connected graph. If G is different from K_n and C_{2n+1} $(n \geq 1)$, then $\chi_e(G) \leq \Delta(G)$.* □

The ECC is still open, and has been verified for bipartite graphs [14], complete r-partite graphs ($r > 1$) [20], and claw-free graphs [5]. A graph could be equitably k-colorable without being equitably $k+1$-colorable. An earlier work of Hajnal and Szemerédi [8] show that a graph G is equitably k-colorable if $k > \Delta(G)$, and see Kierstead and Kostochka [11] for a simpler proof. Further, Kierstead et al. [12] showed that such an equitable coloring using k colors can be obtained in $O(kn^2)$ time, where n is the number of vertices.

In 1994, Chen, Lih and Wu [4] proposed the following.

Conjecture B *[4] (***Equitable Δ-Coloring Conjecture (EΔCC)***) Let G be a connected graph. If G is different from K_n, C_{2n+1}, and $K_{2n+1,2n+1}$ $(n \geq 1)$, then G is equitably $\Delta(G)$-colorable.* □

EΔCC implies the ECC. Conversely, if the ECC holds, so does the EΔCC for non-regular graphs. EΔCC has been verified for bipartite graphs [14], outer planar graphs [21], planar graphs with $\Delta \geq 13$ [22], split graphs (that is, $(2K_2, C_4, C_5)$-free graphs) [3], graphs G with $\Delta(G) \geq \frac{|V(G)|}{2}$ [4] and others.

In this paper, we are interested in verifying the ECC for some classes of graphs which are defined by forbidden induced subgraphs by making use of the following known results.

(R1) Liu and Wu [14]: If G is a connected bipartite graph different from K_2, then $\chi_e(G) \leq \Delta(G)$.

(R2) Wang and Zhang [18] : If G is a complete multipartite graph different from K_m $(m \geq 1)$, then $\chi_e(G) \leq \Delta(G)$.

(R3) Chen et al. [4]: If G is a connected graph with $\Delta(G) \geq \frac{|V(G)|}{2}$ and different from K_m $(m \geq 1)$, then $\chi_e(G) \leq \Delta(G)$.

(R4) Yap and Zhang [20]: If G is a connected graph different from K_m and C_{2m+1} $(m \geq 1)$ with $\frac{|V(G)|}{2} > \Delta(G) \geq \frac{|V(G)|}{3} + 1$, then $\chi_e(G) \leq \Delta(G)$.

(R5) Chen et al. [3] : If G is a split graph different from K_m $(m \geq 1)$, then $\chi_e(G) \leq \Delta(G)$.

2 Validity of ECC in Certain Graph Classes

A well known and widely studied class of graphs is the class of P_4-free graphs (or *cographs*). Numerous computational problems in group-based cooperation, networking, cluster analysis, scheduling, computational learning, and resource allocation suggested the study of graphs having some local 'density' properties that are equated with the absence of P_4's (see [9] and the references therein). These applications motivated the study of P_4-free graphs, and we refer to [2] for a survey on this class and related ones.

If G is a connected P_4-free graph, then by a result of Seinsche [17], it is easy to see that G is a join of two graphs G_1 and G_2 (that is, every vertex of G_1 is adjacent to every vertex of G_2). For, if G contains a cut vertex v, then v is adjacent to all the vertices of $G - v$. Thus, G is a join of $[\{v\}]$ and $[V(G) \setminus \{v\}]$. So, assume that G is 2-connected. Let S be a minimal cut-set of G. Now, since every $v \in S$ is a cut vertex of $G - (S \setminus \{v\})$, v is adjacent to all the vertices of $G - S$. Hence, G is a join of $[S]$ and $G - S$. So, $\Delta(G) \geq \frac{|V(G)|}{2}$, and hence by (R3), G satisfies EΔCC.

As a natural extension of P_4-free graphs, the class of P_5-free graphs and its subclasses received attention; see [7]. While the ECC for P_4-free graphs follows easily from known results, the ECC is open for P_5-free graphs. Also, ECC is open even for triangle-free graphs. In this note, we verify the ECC for two subclasses of P_5-free graphs using known results, namely $(2K_2, C_4)$-free graphs and (P_5, paw)-free graphs, where 'paw' is a graph on four vertices a, b, c and d, with edges ab, ac, ad and bc. Note that the class of split graphs is a subclass of $(2K_2, C_4)$-free graphs, and the class of (P_5, triangle)-free graphs is a subclass of (P_5, paw)-free graphs.

Before we proceed further, we require the following: If G is a graph, and if S and T are two vertex disjoint subsets of $V(G)$, then $[S, T]$ denotes the set of edges with one end in S and the other in T. The set $[S, T]$ is said to be *complete* if every vertex in S is adjacent with every vertex in T.

Now, we verify the ECC for the class of $(2K_2, C_4)$-free graphs using the following decomposition theorem by Blaszik et al. given in [1].

Theorem A *[1] If G is $(2K_2, C_4)$-free, then G is either a split graph or $V(G)$ admits a partition (V_1, V_2, V_3) such that (i) $[V_1] \cong C_5$, (ii) $[V_2]$ is complete, (iii) $[V_3]$ is an edgeless graph, (iv) $[V_1, V_2]$ is complete, and (v) $[V_1, V_3] = \emptyset$.* □

Theorem 1 *If G is a connected $(2K_2, C_4)$-free graph different from C_5 and K_t ($t \geq 1$), then $\chi_e(G) \leq \Delta(G)$.*

Proof: We use the notation as in Theorem A. If G is C_5-free, then G is a split graph, and hence $\chi_e(G) \leq \Delta(G)$, by (R5). So, assume that G contains an induced C_5, and $V(G)$ admits a partition as in Theorem A. Since G is connected and different from C_5, we have $V_2 \neq \emptyset$. Also, note that a maximum degree vertex of G occurs in V_2. Now, consider the graph $G' \cong [V(G) \setminus V_1]$. It is easy to see that $\Delta(G') = \Delta(G) - 5$, and G' is C_5-free and hence G' is a split graph. So, G' can be equitably colored with $t = \Delta(G) - 5$ colors, by (R5). Let $S_1, S_2, \ldots, S_t$ denote the equitable color classes of G' arranged so that their sizes are in non-increasing order. Next, we equitably color G as follows: We use a set of five distinct colors to color the vertices of V_1, and we follow the cyclic order $S_1, S_2, \ldots, S_t, S_1, S_2, \ldots$ to take out one vertex from each $S_i \cap V_3$ sequentially, and combine with the singleton color classes of $[V_1]$ sequentially.

This yields an equitable coloring of G which uses $5 + \Delta(G) - 5$ colors. Hence, $\chi_e(G) \leq \Delta(G)$. □

Next, we verify the ECC for the class of (P_5, paw)-free graphs. To do this, we first derive a structure theorem for the class of (P_5, paw)-free graphs using the following characterization of paw-free graphs by Olariu [16].

Theorem B *[16] Let G be a connected graph. Then G is paw-free if and only if G is triangle-free or complete multipartite.* □

Let G be a graph on n vertices $v_1, v_2, \ldots, v_n$, and let $H_1, H_2, \ldots, H_n$ be any n vertex disjoint edgeless graphs with $|V(H_i)| = m_i$ (≥ 0), for $1 \leq i \leq n$. Then an *independent expansion* $\mathbb{I}[G](m_1, m_2, \ldots, m_n)$ of G is the graph obtained from G by (i) replacing the vertex v_i of G by H_i, $i \in \{1, 2, \ldots, n\}$, and (ii) joining the vertices $x \in H_i$, $y \in H_j$ if and only if v_i and v_j are adjacent in G. If $m_i = m$, for all i, we simply denote $\mathbb{I}[G](m_1, m_2, \ldots, m_n)$ by $\mathbb{I}[G](m)$.

Note that a complete bipartite graph is an independent expansion of K_2, and in general, for $r > 1$, a complete r-partite graph is an independent expansion of K_r. Also, an independent expansion of a bipartite graph is again bipartite.

A consequence of Theorem B is the following.

Theorem 2 *Let G be a connected (P_5, paw)-free graph. Then G is one of the following.*

(i) A complete multipartite graph.

(ii) A P_5-free bipartite graph.

(iii) $\mathbb{I}[C_5](m_1, m_2, m_3, m_4, m_5)$, where $m_i \geq 1$, for all i, $1 \leq i \leq 5$.

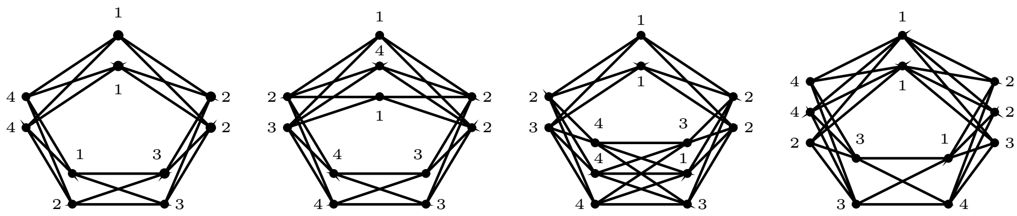
Proof: By Theorem B, assume that G is triangle-free (else, G belongs to (i)). Further if G is C_5-free, then since G is P_5-free, G is C_{2k+1}-free, $k \geq 1$. Hence, G belongs to (ii).

So, assume that G contains an induced C_5, say C. Let $C \cong [\{v_1, v_2, v_3, v_4, v_5\}] \sqsubseteq G$.

Claim: Any vertex v in $V(G) \setminus V(C)$ is adjacent to exactly two non-consecutive vertices of C.

Let x be a vertex in $V(G) \setminus V(C)$. Since G is triangle-free, $|N(x) \cap V(C)| \leq 2$. If x is adjacent to exactly one vertex of C, say v_1, then $[x, v_1, v_2, v_3, v_4] \cong P_5 \sqsubseteq G$. And, if x is adjacent to two consecutive vertices of C, say $\{v_1, v_2\}$, then $\{x, v_1, v_2\}$ forms a triangle in G.

So, we conclude that any x in $V(G) \setminus V(C)$ that has a neighbor in C is adjacent to exactly two non-consecutive vertices of C. If there is some vertex in G with no



$\mathbb{I}[C_5](2,2,2,2,2)$ $\mathbb{I}[C_5](3,2,2,2,2)$ $\mathbb{I}[C_5](2,2,3,3,2)$ $\mathbb{I}[C_5](2,3,2,2,3)$

Figure 1: Equitable coloring of $\mathbb{I}[C_5]$'s using 4 colors.

neighbors in C, then, since G is connected, there exist y, z in $V(G) \setminus V(C)$ such that $yz \in E(G)$, y has no neighbors in C and z has at least one neighbor in C. By the above analysis, z is adjacent to exactly two non-consecutive vertices of C and then $P_5 \sqsubseteq G$, a contradiction. Hence, the claim holds.

For $1 \leq i \leq 5$ ($i \mod 5$), define

$$A_i = \{v_i\} \cup \{x \in V(G) \setminus V(C) : N(x) \cap V(C) = \{v_{i-1}, v_{i+1}\}\}.$$

Then $V(G) = V(C) \cup \bigcup_{i=1}^{5} A_i$. For $1 \leq i \leq 5$, $i \mod 5$, we have the following:

(1) A_i is an independent set (else, G contains a triangle).

(2) $[A_i, A_{i+1}]$ is complete (else, if $x \in A_i$ and $y \in A_{i+1}$ are not adjacent, then $[\{x, v_{i-1}, v_{i-2}, v_{i+2}, y\}] \cong P_5 \sqsubseteq G$).

(3) $[A_i, A_{i+2}] = \emptyset$ (else, if $x \in A_i$ and $y \in A_{i+2}$ are adjacent, then $[\{x, v_{i+1}, y\}] \cong K_3 \sqsubseteq G$).

Hence, from (1), (2) and (3), we conclude that $G \cong \mathbb{I}[C_5](m_1, m_2, m_3, m_4, m_5)$, where $m_i = |A_i|$, $1 \leq i \leq 5$. □

In the following, we state some simple observations and derive a lemma which will help us to verify the ECC for (P_5, paw)-free graphs using Theorem 2.

(O1) If G is a graph on n vertices and if G admits a coloring such that each color class contains 2 or 3 vertices, then $\chi_e(G) \leq \lfloor \frac{n}{2} \rfloor$.

(O2) A bipartite graph G admits a coloring such that each color class contains 2 or 3 vertices if and only if G has a bipartition in which neither part consists of just one vertex.

For a fixed integer $m \geq 2$, we say that $G \in \mathcal{G}_1$ if $G \cong \mathbb{I}[C_5](m)$, $G \in \mathcal{G}_2$ if $G \cong \mathbb{I}[C_5](m, m, m, m, m+1)$, $G \in \mathcal{G}_3$ if $G \cong \mathbb{I}[C_5](m, m, m, m+1, m+1)$, and $G \in \mathcal{G}_4$ if $G \cong \mathbb{I}[C_5](m, m+1, m, m, m+1)$.

Note that

$$\begin{aligned} \mathbb{I}[C_5](m_1, m_2, m_3, m_4, m_5) &\cong \mathbb{I}[C_5](m_5, m_1, m_2, m_3, m_4) \\ \cong \mathbb{I}[C_5](m_1, m_5, m_4, m_3, m_2) &\cong \mathbb{I}[C_5](m_3, m_2, m_1, m_5, m_4), \text{etc.} \end{aligned}$$

In general, for all i, $1 \leq i \leq 5$, i mod 5, $\mathbb{I}[C_5](m_i, m_{i+1}, m_{i+2}, m_{i+3}, m_{i+4})$ is isomorphic to $\mathbb{I}[C_5](m_{i+1}, m_{i+2}, m_{i+3}, m_{i+4}, m_{i+5})$.

By (O1), if $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$, then $\chi_e(G) \leq \left\lfloor \frac{5m+2}{2} \right\rfloor$. However, the structure of G enables us to prove that $\chi_e(G) \leq 2m$, and is given in the following:

Lemma 1 *If $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$, then $V(G)$ can be colored equitably with $2m$ colors such that each color class contains 2 or 3 vertices. So, $\chi_e(G) \leq 2m$.*

Proof: Let m be of the form $2t$ or $2t + 1$ ($t \geq 1$) according as m is even or odd respectively. If $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$, then consider a partition of $V(G)$ as follows:

- $\underline{G \in \mathcal{G}_1}$: $V(G) = (V_1, V_2, \ldots, V_t, V_{t'})$, where $[V_i] \cong \mathbb{I}[C_5](2)$, for all i, $1 \leq i \leq t$ and $V_{t'} = \emptyset$, if m is even, and $[V_1] \cong \mathbb{I}[C_5](3, 2, 2, 2, 2)$, $[V_i] \cong \mathbb{I}[C_5](2)$, for all i, $2 \leq i \leq t$, and $[V_{t'}] \cong P_4$, if m is odd.
- $\underline{G \in \mathcal{G}_2}$: $V(G) = (V_1, V_2, \ldots, V_t, V_{t'})$, where $[V_1] \cong \mathbb{I}[C_5](2, 2, 2, 2, 3)$, $[V_i] \cong \mathbb{I}[C_5](2)$, for all i, $2 \leq i \leq t$, and $V_{t'} = \emptyset$, if m is even, and $[V_1] \cong \mathbb{I}[C_5](3, 2, 2, 2, 3)$, $[V_i] \cong \mathbb{I}[C_5](2)$, for all i, $2 \leq i \leq t$, and $[V_{t'}] \cong P_4$, if m is odd.
- $\underline{G \in \mathcal{G}_3}$: $V(G) = (V_1, V_2, \ldots, V_t, V_{t'})$, where $[V_1] \cong \mathbb{I}[C_5](2, 2, 2, 3, 3)$, $[V_i] \cong \mathbb{I}[C_5](2)$, for all i, $2 \leq i \leq t$, and $V_{t'} = \emptyset$, if m is even, and $[V_1] \cong \mathbb{I}[C_5](3, 2, 2, 2, 2)$, $[V_i] \cong \mathbb{I}[C_5](2)$, for all i, $2 \leq i \leq t$, and $[V_{t'}] \cong \mathbb{I}[P_4](2, 2, 1, 1)$, if m is odd.
- $\underline{G \in \mathcal{G}_4}$: $V(G) = (V_1, V_2, \ldots, V_t, V_{t'})$, where $[V_1] \cong \mathbb{I}[C_5](2, 3, 2, 2, 3)$, $[V_i] \cong \mathbb{I}[C_5](2)$, for all i, $2 \leq i \leq t$, and $V_{t'} = \emptyset$, if m is even, and $[V_1] \cong \mathbb{I}[C_5](3, 2, 2, 2, 2)$, $[V_i] \cong \mathbb{I}[C_5](2)$, for all i, $2 \leq i \leq t$, and $[V_{t'}] \cong \mathbb{I}[P_4](2, 1, 1, 2)$, if m is odd.

In all the cases, we color each $[V_i]$ ($1 \leq i \leq t$) with 4 colors such that each color class contains 2 or 3 vertices (see Figure 1), and if $V_{t'} \neq \emptyset$, we color $[V_{t'}]$ with another two colors (by using (O2)). Hence the lemma. □

Theorem 3 *If G is a connected (P_5, paw)-free graph different from C_5 and K_t ($t \geq 1$), then $\chi_e(G) \leq \Delta(G)$.*

Proof: We use Theorem 2, and we use the same notation as in Theorem 2.

(i) If G is a complete multipartite graph, then $\chi_e(G) \leq \Delta(G)$ (by (R2)).

(ii) If G is a P_5-free bipartite graph, then $\chi_e(G) \leq \Delta(G)$ (by (R1)).

(iii) Assume that $G \cong \mathbb{I}[C_5](m_1, m_2, m_3, m_4, m_5)$.

Then $\Delta(G) = \max\{m_{i-1} + m_{i+1}, 1 \leq i \leq 5,\ i \bmod 5\}$. Let $m_1 := \min\limits_{1 \leq i \leq 5} m_i$.

$m_2+m_4-2m_1$	$m_3+m_5-2m_1$	H	$[V(G)\setminus V(H)]$
≥ 2	≥ 2	$\mathbb{I}[C_5](m_1)$	$\mathbb{I}[C_5](0, m_2-m_1, m_3-m_1, m_4-m_1, m_5-m_1)$
≥ 2	$m_3-m_1=1$, $m_5-m_1=0$	$\mathbb{I}[C_5](m_1, m_1, m_1+1, m_1, m_1)$	$\mathbb{I}[C_5](0, m_2-m_1, 0, m_4-m_1, 0)$
≥ 2	$m_3-m_1=0$, $m_5-m_1=1$	$\mathbb{I}[C_5](m_1, m_1, m_1, m_1, m_1+1)$	$\mathbb{I}[C_5](0, m_2-m_1, 0, m_4-m_1, 0)$
≥ 2	0	$\mathbb{I}[C_5](m_1)$	$\mathbb{I}[C_5](0, m_2-m_1, 0, m_4-m_1, 0)$
$m_2-m_1=1$, $m_4-m_1=0$	≥ 2	$\mathbb{I}[C_5](m_1, m_1+1, m_1, m_1, m_1)$	$\mathbb{I}[C_5](0, 0, m_3-m_1, 0, m_5-m_1)$
$m_2-m_1=0$, $m_4-m_1=1$	≥ 2	$\mathbb{I}[C_5](m_1, m_1, m_1, m_1+1, m_1)$	$\mathbb{I}[C_5](0, 0, m_3-m_1, 0, m_5-m_1)$
0	≥ 2	$\mathbb{I}[C_5](m_1)$	$\mathbb{I}[C_5](0, 0, m_3-m_1, 0, m_5-m_1, 0)$

Table 1: Analysis of structure of G in Case 1

Now, we prove the theorem in two cases.

Case 1: $m_1 \geq 2$.

First we delete a subgraph $H \in \mathcal{G}_1 \cup \mathcal{G}_2$ from G according to the numbers $m_2 + m_4 - 2m_1$ and $m_3 + m_5 - 2m_1$, and analyze the remaining graph $[V(G) \setminus V(H)]$. We give the case analysis in Table 1, the graphs H and $[V(G) \setminus V(H)]$ (in all the other cases $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$, and hence the theorem follows by Lemma 1).

Note that in all the above cases, since $H \in \mathcal{G}_1 \cup \mathcal{G}_2$, H can be colored equitably with $2m_1$ colors such that each color class contains 2 or 3 vertices, by Lemma 1. Also, it is easy to see that $[V(G) \setminus V(H)]$ is an independent expansion of a P_4, and hence a bipartite graph with $m_2 + m_4 - 2m_1$ or 0 vertices in one partite set, and with $m_3 + m_5 - 2m_1$ or 0 vertices in the other partite set. So, in non-trivial cases, by (O1) and (O2), $[V(G) \setminus V(H)]$ can be colored equitably with at most $(m_2 + m_3 + m_4 + m_5 - 4m_1)/2$ colors such that each color class contains 2 or 3 vertices. Hence,

$$\chi_e(G) \leq 2m_1 + \frac{m_2 + m_3 + m_4 + m_5 - 4m_1}{2} = \frac{m_2 + m_3 + m_4 + m_5}{2}.$$

We prove that $\chi_e(G) \leq \Delta(G)$ in three cases as follows (the other symmetric cases can be verified in a similar manner):

(1) If $\Delta(G) = m_2 + m_5$, then $m_3 \leq m_2$ and $m_4 \leq m_5$. Hence $\chi_e(G) \leq m_2 + m_5 = \Delta(G)$.

(2) If $\Delta(G) = m_1 + m_4$, then $m_2 = m_1$ (since m_1 is minimum) and $m_3 + m_5 \leq m_1 + m_4$. Hence $\chi_e(G) \leq m_1 + m_4 = \Delta(G)$.

(3) If $\Delta(G) = m_3 + m_5$, then $m_2 + m_4 \leq m_3 + m_5$. Hence $\chi_e(G) \leq m_3 + m_5 = \Delta(G)$.

Case 2: Assume that $m_1 = 1$.

Case 2.1: Assume that $\Delta(G) = m_2 + m_5$.

Note that $m_2 \geq m_3$ and $m_5 \geq m_4$. If $m_2+m_5 \geq m_3+m_4+1$, then $\chi_e(G) \leq \Delta(G)$, by (R3). So, $m_2 + m_5 < m_3 + m_4 + 1$. Hence, $m_2 + m_5 = m_3 + m_4$. Now, if $\Delta(G) \geq \frac{|V(G)|}{3} + 1$, then $\chi_e(G) \leq \Delta(G)$, by (R4). Else, we have $m_2 + m_5 \leq 3$. Since $m_2 + m_5 \geq m_3 + 1$, and $m_2 + m_5 \geq m_4 + 1$, it is easy to verify that $\chi_e(G) \leq \Delta(G)$.

Case 2.2: Assume that $\Delta(G) = m_4 + m_1 = m_4 + 1$.

Note that $m_2 = 1$ and $m_4+1 \geq m_3+m_5$. If $m_4 \geq m_3+m_5$, then $\chi_e(G) \leq \Delta(G)$, by (R3). So, $m_4 < m_3 + m_5$. Hence, $m_3 + m_5 = m_4 + 1$. Now, if $\Delta(G) \geq \frac{|V(G)|}{3} + 1$, then $\chi_e(G) \leq \Delta(G)$, by (R4). Else, we have $m_4 \leq 2$. Now, it is easy to see that $\chi_e(G) \leq \Delta(G)$.

The case $\Delta(G) = m_3 + m_1 = m_3 + 1$ can be verified using similar arguments.

Case 2.3: Assume that $\Delta(G) = m_3 + m_5$.

Note that $m_3 + m_5 \geq m_2 + m_4$. If $m_3 + m_5 \geq m_2 + m_4 + 1$, then $\chi_e(G) \leq \Delta(G)$, by (R3). So, $m_3 + m_5 = m_2 + m_4$. If $\Delta(G) \geq \frac{|V(G)|}{3} + 1$, then $\chi_e(G) \leq \Delta(G)$, by (R4). Else, we have $m_3 + m_5 \leq 3$. Now, it is easy to see that $\chi_e(G) \leq \Delta(G)$.

The case $\Delta(G) = m_2 + m_4$ can be verified using similar arguments. □

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