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A cycle decomposition conjecture for Eulerian graphs

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Abstract

A classic theorem of Veblen states that a connected graph G has a cycle decomposition if and only if G is Eulerian. The number of odd cycles in a cycle decomposition of an Eulerian graph G is therefore even if and only if G has even size. It is conjectured that if the minimum number of odd cycles in a cycle decomposition of an Eulerian graph G with m edges is a and the maximum number of odd cycles in a cycle decomposition is c, then for every integer b such that $a \leq b \leq c$ and b and m are of the same parity, then there is a cycle decomposition of G with exactly b odd cycles. This conjecture is verified for small powers of cycles and Eulerian complete tripartite graphs.

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1 A Circuit Decomposition Problem

It is well-known that if G is a connected graph containing $2k$ odd vertices for some positive integer k, then G can be decomposed into k open trails but no fewer. In 1973, the following [8] was proved.

Theorem 1.1 *If G is a connected graph containing $2k$ odd vertices for some positive integer k, then G can be decomposed into k open trails, at most one of which has odd length.*

A generalization of Theorem 1.1 was established in [4].

Theorem 1.2 *Let G be a connected graph of size m containing $2k$ odd vertices $(k \geq 1)$. Among all decompositions of G into k open trails, let s be the maximum number of such trails of odd length.*

(*a*) *If m is even, then s is even and for every even integer a such that $0 \leq a \leq s$, there exists a decomposition of G into k open trails, exactly a of which have odd length.*

(*b*) *If m is odd, then s is odd and for every odd integer b such that $1 \leq b \leq s$, there exists a decomposition of G into k open trails, exactly b of which have odd length.*

The *distance between two subgraphs F and H* in a connected graph G is

$$d(F, H) = \min\{d(u, v) : u \in V(F), v \in V(H)\}.$$

Theorem 1.3 *For an Eulerian graph G of size m, let s be the maximum number of circuits of odd length in a circuit decomposition of G.*

(*a*) *If m is even, then s is even and for every even integer a such that $0 \leq a \leq s$, there exists a circuit decomposition of G, exactly a of which have odd length.*

(*b*) *If m is odd, then s is odd and for every odd integer b such that $1 \leq b \leq s$, there exists a circuit decomposition of G, exactly b of which have odd length.*

Proof. We only verify (a) because the proof of (b) is similar. Since the size of G is even, s is even. If $s = 0$, then the result is true trivially. Thus we may assume that $s \geq 2$. It suffices to show that there exists a circuit decomposition of G, exactly $s - 2$ of which have odd length. Among all circuit decompositions of G, consider those circuit decompositions containing exactly s circuits of odd length; and, among those, consider one, say $\mathcal{D} = \{C_1, C_2, \ldots, C_k\}$ for some positive integer k, where the distance between some pair C_i, C_j of circuits of odd length is minimum. We claim that this minimum distance is 0. Assume that this is not the case. Suppose that P

is a path of minimum length connecting a vertex w_i in C_i and a vertex w_j in C_j, and let w_ix be the edge of P incident with w_i (where it is possible that $x = w_j$). Then w_ix belongs to a circuit C_p among $C_1, C_2, \ldots, C_k$. Necessarily, C_p has even length, for otherwise, the distance between C_i and C_p is 0, producing a contradiction. Since C_i and C_p have the vertex w_i in common, C_i and C_p may be replaced by the circuit C' consisting of C_i and C_p (that is, $E(C') = E(C_i) \cup E(C_p)$) and C' has odd length. However then, the circuit decomposition $\mathcal{D}' = (\{C_1, C_2, \ldots, C_k\} - \{C_i, C_p\}) \cup \{C'\}$ has exactly s circuits of odd length and the distance between C_j and C' in $\mathcal{D}'$ is smaller than the distance between C_i and C_j in $\mathcal{D}$, which contradicts the defining property of $\mathcal{D}$. Thus, as claimed, the distance between C_i and C_j is 0 and so C_i and C_j have a vertex in common. Hence the circuit C^* consisting of C_i and C_j has even length. Then $(\{C_1, C_2, \ldots, C_k\} - \{C_i, C_j\}) \cup \{C^*\}$ is a circuit decomposition of G, exactly $s - 2$ of which have odd length. ■

2 The Eulerian Cycle Decomposition Conjecture

The earliest and a major influential book on topology was written by Veblen [17] in 1922 and titled *Analysis Situs*, with a second edition in 1931. The first chapter of this book was titled *Linear Graphs* and dealt with graph theory. In fact, both editions preceded the first book entirely devoted to graph theory, written by König [14] in 1936. In 1736 Euler [9] wrote a paper containing a solution of the famous Königsberg Bridge Problem. This paper essentially contained a characterization of Eulerian graphs as well, although the proof was only completed in 1873 in a paper by Hierholzer [12]. In 1912 Veblen [16] himself obtained a characterization of Eulerian graphs.

Theorem 2.1 (**Veblen's Theorem**) *A nontrivial connected graph G is Eulerian if and only if G has a decomposition into cycles.*

When it comes to cycle decompositions, the Eulerian graphs that have received the most attention are the complete graphs of odd order and, to a lesser degree, the complete graphs of even order in which (the edges of) a 1-factor has been removed. In 1847, Kirkman [13] proved that the complete graph K_n, where $n \geq 3$ is odd, can be decomposed into 3-cycles if and only if $3 \mid \binom{n}{2}$. At the other extreme, in 1890 Walecki (see [2]) proved that the complete graph K_n, where $n \geq 3$ is odd, can always be decomposed into n-cycles. Consequently, when $n \geq 3$ is an odd integer, the complete graph K_n can be decomposed into m-cycles for $m = 3$ or $m = n$ if and only if $m \mid \binom{n}{2}$. In 2001 Alspach and Gavlas [3] proved for every odd integer $n \geq 3$ and odd integer m with $3 < m < n$ that K_n can be decomposed into m-cycles if and only if $m \mid \binom{n}{2}$. In addition, they proved that for every even integer $n \geq 4$ and even integer m with $3 < m < n$ and for a 1-factor I of K_n, the graph $K_n - I$ can be decomposed into m-cycles if and only if $m \mid (n^2 - 2n)/2$. In 2002, Šajna [15] proved the remaining cases for m-cycle decompositions of K_n and $K_n - I$, namely the cases when m and n are of opposite parity. These results verify special cases of a conjecture made by Alspach [1] in 1981.

Alspach's Conjecture *Suppose that $n \geq 3$ is an odd integer and that $m_1, m_2, \ldots, m_t$ are integers such that $3 \leq m_i \leq n$ for each i $(1 \leq i \leq t)$ and $m_1+m_2+\cdots+m_t = \binom{n}{2}$. Then K_n can be decomposed into the cycles $C_{m_1}, C_{m_2}, \ldots, C_{m_t}$. Furthermore, for every even integer $m \geq 4$ and integers $m_1, m_2, \ldots, m_t$ such that $3 \leq m_i \leq n$ for each i $(1 \leq i \leq t)$ with $m_1 + m_2 + \cdots + m_t = (n^2 - 2n)/2$, there is a decomposition of $K_n - I$ for a 1-factor I of K_n into the cycles $C_{m_1}, C_{m_2}, \ldots, C_{m_t}$.*

Following many years of attempting to establish Alspach's Conjecture by many mathematicians, the conjecture was verified in its entirety by Bryant, Horsley and Pettersson [6] in 2012. We now state another conjecture involving cycle decompositions of Eulerian graphs.

The Eulerian Cycle Decomposition Conjecture (ECDC) *Let G be an Eulerian graph of size m, where a is the minimum number of odd cycles in a cycle decomposition of G and c is the maximum number of odd cycles in a cycle decomposition of G. For every integer b such that $a \leq b \leq c$ and b and m are of the same parity, there exists a cycle decomposition of G containing exactly b odd cycles.*

In the case of the complete graphs of odd order or complete graphs of even order in which a 1-factor has been removed, the maximum number of odd cycles in a cycle decomposition of each such graph is given below. This follows from results of Kirkman [13], Guy [10] and Heinrich, Horák and Rosa [11].

Corollary 2.2 (a) *For an odd integer $n \geq 3$, the maximum number s of odd cycles in a cycle decomposition of K_n is*

$$s = \begin{cases} \frac{n(n-1)}{6} & \text{if } n \equiv 1, 3 \pmod 6 \\ \frac{n(n-1)-8}{6} & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

(b) *For an even integer $n \geq 4$ and a 1-factor I of K_n, the maximum number s of odd cycles in a cycle decomposition of $K_n - I$ is*

$$s = \begin{cases} \frac{n(n-2)}{6} & \text{if } n \equiv 0, 2 \pmod 6 \\ \frac{n(n-2)-8}{6} & \text{if } n \equiv 4 \pmod 6. \end{cases}$$

For complete graphs K_n of odd order $n \geq 3$ and graphs $K_n - I$ where $n \geq 4$ is even and I is a 1-factor of K_n, the ECDC is then a special case of Alspach's Conjecture and therefore is satisfied for these two classes of graphs.

3 The ECDC and Small Powers of Cycles

In a cycle decomposition of an Eulerian graph G, the number of odd cycles in the decomposition and the size of G are of the same parity. One class of Eulerian graphs consists of the squares C_n^2 of cycles C_n where $n \geq 5$, and more generally the kth power C_n^k of C_n for $k \leq \lfloor n/2 \rfloor$, which is a special class of circulant graphs. For each integer $n \geq 3$ and integers $n_1, n_2, \ldots, n_k$ ($k \geq 1$) such that $1 \leq n_1 < n_2 < \ldots < n_k \leq \lfloor n/2 \rfloor$, the *circulant graph* $\langle \{n_1, n_2, \ldots, n_k\} \rangle_n$ is that graph with n vertices $v_1, v_2, \ldots, v_n$ such that v_i is adjacent to $v_{i \pm n_j \pmod n}$ for each j with $1 \leq j \leq k$. The integers n_i ($1 \leq i \leq k$) are called the *jump sizes* of the circulant. The circulant graph $\langle \{1, 2, \ldots, k\} \rangle_n$ is the *kth power* of C_n and is denoted by C_n^k and in particular, if $k = 1$, then $\langle \{1\} \rangle_n = C_n$. The circulant $\langle \{n_1, n_2, \ldots, n_k\} \rangle_n$ is $2k$-regular if $n_k < n/2$ and $(2k-1)$-regular if $n_k = n/2$ where then n is even. Thus circulant graphs are symmetric classes of regular graphs.

Let G be an Eulerian graph of order n and size m. For a sequence $m_1, m_2, \ldots, m_t$ of positive integers, an $(m_1, m_2, \ldots, m_t)$-*cycle decomposition* of G is a decomposition $\{G_1, G_2, \ldots, G_t\}$ where G_i is an m_i-cycle for $i = 1, 2, \ldots, t$. Obviously, necessary conditions for the existence of an $(m_1, m_2, \ldots, m_t)$-cycle decomposition of G are that $3 \leq m_i \leq n$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = m$. In [7] Bryant and Martin proved the following results for cycle decompositions of C_n^2 and C_n^3.

Theorem 3.1 *Let $n \geq 5$ be an integer and let $m_1, m_2, \ldots, m_t$ be a sequence of integers with $m_i \geq 3$ for $i = 1, 2, \ldots, t$. Then $C_n^2 = \langle \{1, 2\} \rangle_n$ has an $(m_1, m_2, \ldots, m_t)$-cycle decomposition if and only if each of the following conditions hold:*

(1) *$m_i \leq n$ for $i = 1, 2, \ldots, t$;*

(2) *$m_1 + m_2 + \cdots + m_t = 2n$; and*

(3) *either*

  (*i*) *$t = 3$ and $\frac{n}{2} \leq m_1, m_2, m_3 \leq n$ or*

  (*ii*) *there exists a $k \in \{1, 2, \ldots, t\}$ such that $m_k \geq n - t + 1$.*

Theorem 3.2 *Let $n \geq 7$ be an integer and let $m_1, m_2, \ldots, m_t$ be any sequence of integers with $3 \leq m_i \leq 5$ for $i = 1, 2, \ldots, t$ with $m_1 + m_2 + \cdots + m_t = 3n$. Then $C_n^3 = \langle \{1, 2, 3\} \rangle_n$ has an $(m_1, m_2, \ldots, m_t)$-cycle decomposition.*

For $k = 2, 3, 4$, we now determine the maximum number of odd cycles in a cycle decomposition of C_n^k for $n \geq 2k + 1$ and show that the ECDC holds in each case. In a $(m_1, m_2, \ldots, m_t)$-cycle decomposition of a graph G, if $m_i = m_{i+1} = \cdots = m_k$, we will write m_i^{k-i+1} for $m_i, m_{i+1}, \ldots, m_k$ in $(m_1, m_2, \ldots, m_t)$.

Theorem 3.3 *For every integer $n \geq 5$, the graph C_n^2 satisfies the ECDC.*

Proof. Let $n \geq 5$ be an integer. By Theorem 3.1, the following cycle decompositions of C_n^2 exist:

- an $(\frac{n}{2}, \frac{n}{2}, n)$-cycle decomposition if $n \equiv 0 \pmod 4$;

- a $(4^{(n+3)/4}, n-3)$-cycle decomposition if $n \equiv 1 \pmod 4$;
- an $(\frac{n}{2}+1, \frac{n}{2}+1, n-2)$-cycle decomposition if $n \equiv 2 \pmod 4$;
- a $(4^{(n+1)/4}, n-1)$-cycle decomposition if $n \equiv 3 \pmod 4$.

Next, let $s(n)$ be the maximum number of odd cycles in a cycle decomposition of C_n^2. Since C_n^2 has $2n$ edges, it follows that $s(n)$ must be even. By Theorem 3.1, the following cycle decompositions of C_n^2 with exactly $2\lfloor \frac{n+2}{4} \rfloor$ odd cycles exist:

- a $(3^{n/2}, \frac{n}{2})$-cycle decomposition if n is even;
- a $(3^{(n-1)/2}, \frac{n+3}{2})$-cycle decomposition if n is odd.

Hence, $s(n) \geq 2\lfloor \frac{n+2}{4} \rfloor$. It remains to show that $s(n) \leq 2\lfloor \frac{n+2}{4} \rfloor$. First note that if C_n^2 has an $(m_1, m_2, \ldots, m_t)$-cycle decomposition, then, by Theorem 3.1,

$$3(t-1) + n - t + 1 \leq m_1 + m_2 + \cdots + m_t = 2n$$

so that $t \leq \frac{n}{2} + 1$, or in fact, $t \leq \lfloor \frac{n}{2} + 1 \rfloor$. Thus, $s(n) \leq \lfloor \frac{n}{2} + 1 \rfloor$. Note that $2\lfloor \frac{n+2}{4} \rfloor = \lfloor \frac{n}{2} + 1 \rfloor$ if $n \equiv 2, 3 \pmod 4$, and hence $s(n) = 2\lfloor \frac{n+2}{4} \rfloor$ for $n \equiv 2, 3 \pmod 4$. If $n \equiv 0, 1 \pmod 4$, then $\lfloor \frac{n}{2} + 1 \rfloor$ is odd and thus, since $s(n)$ must be even, it follows that $s(n) \leq \lfloor \frac{n}{2} + 1 \rfloor - 1 = 2\lfloor \frac{n+2}{4} \rfloor$. Hence, $s(n) \leq 2\lfloor \frac{n+2}{4} \rfloor$ if $n \equiv 0, 1 \pmod 4$ and therefore $s(n) = 2\lfloor \frac{n+2}{4} \rfloor$ for $n \equiv 0, 1 \pmod 4$ as well.

It remains to find cycle decompositions of C_n^2 with exactly r odd cycles for each even integer r with $2 \leq r \leq 2\lfloor \frac{n+2}{4} \rfloor - 2$. Let $r = 2a$ for some positive integer a, where then $1 \leq a \leq \frac{n-2}{4}$ and $n \geq 4a+2$.

First, let $n = 4a+2$ for some positive integer a. By Theorem 3.1, there is a $(3^{2a}, 2a+4)$-cycle decomposition of C_n^2 since $2a+4 \geq n-(2a+1)+1 = n-2a = 2a+2$. Next, assume that $4a+3 \leq n \leq 6a+4$. Then $2n-6a-4 \geq n-(2a+2)+1$ and $2n-6a-4 \leq n$. Thus, by Theorem 3.1, there is a $(3^{2a}, 4, 2n-6a-4)$-cycle decomposition of C_n^2.

Finally, assume that $n \geq 6a+5$. Let $n = 6a+\ell$ for some integer $\ell \geq 5$ and let $b = \lceil \ell/2 \rceil$. Then $n \leq 6a+2b$ and so $2n-6a-2b \leq n$. It follows by Theorem 3.1 that there is a $(3^{2a}, 2b, 2n-6a-2b)$-cycle decomposition of C_n^2. ■

Theorem 3.4 *For every integer $n \geq 7$, the graph C_n^3 satisfies the ECDC.*

Proof. Let $n \geq 7$ be an integer. By Theorem 3.2, C_n^3 has a (3^n)-cycle decomposition and so the maximum number of odd cycles in a cycle decomposition of C_n^3 is n. It remains to show that for each integer r with $0 \leq r \leq n$ such that r and $3n$ are of the same parity, there is a cycle decomposition of C_n^3 having exactly r odd cycles.

First suppose that n is even. Then $n = 2\ell$ for some integer $\ell \geq 4$. Let $r = 2a$ for some nonnegative integer a for which $a \leq \ell$. First, suppose that $\ell - a$ is even, say $\ell - a = 2p$ for some nonnegative integer p. Then by Theorem 3.2, C_n^3 has an

$(3^{2a}, 4^{3p})$-cycle decomposition since $3(2a) + 4(3p) = 3n$. Next, suppose that $\ell - a$ is odd, say $\ell - a = 2p + 1$ for some nonnegative integer p. Then by Theorem 3.2, C_n^3 has an $(3^{2a-1}, 4^{3p+1}, 5)$-cycle decomposition since $3(2a - 1) + 4(3p + 1) + 5 = 3n$.

Next suppose that n is odd. Then $n = 2\ell+1$ for some integer $\ell \geq 3$. Let $r = 2a+1$ for some nonnegative integer a where $a \leq \ell$. First, if $\ell - a$ is even, say $\ell - a = 2p$ for some nonnegative integer p, then by Theorem 3.2, C_n^3 has an $(3^{2a+1}, 4^{3p})$-cycle decomposition since $3(2a + 1) + 4(3p) = 3n$. Next, if $\ell - a$ is odd, say $\ell - a = 2p + 1$ for some nonnegative integer p, then by Theorem 3.2, C_n^3 has an $(3^{2a}, 4^{3p+1}, 5)$-cycle decomposition since $3(2a) + 4(3p + 1) + 5 = 3n$. ■

Thus, the ECDC holds for C_n^2 when $n \geq 5$ and for C_n^3 when $n \geq 7$. We conclude this section by showing that the ECDC holds as well for C_n^4 for $n \geq 9$. For simplicity, we express a cycle $(u_1, u_2, \ldots, u_k, u_1)$, $k \geq 3$, as $(u_1, u_2, \ldots, u_k)$ in the proof of the following theorem.

Theorem 3.5 *For every integer $n \geq 9$, the graph C_n^4 satisfies the ECDC.*

Proof. Let $n \geq 9$ be an integer and recall that $C_n^4 = \langle\{1, 2, 3, 4\}\rangle_n$ with vertex set $\{v_1, v_2, \ldots, v_n\}$. Consider the set $\mathcal{C}$ of 4-cycles defined by

$$\mathcal{C} = \{(v_i, v_{i+1}, v_{i-1}, v_{i+3}) \mid i = 0, 1, \ldots, n - 1\}$$

where all arithmetic is done modulo n. Since $\mathcal{C}$ is a decomposition of C_n^4 into 4-cycles, there exists a cycle decomposition of C_n^4 with no odd cycles. Note also that in any cycle decomposition of C_n^4, the number of odd cycles must be even since C_n^4 has an even number of edges.

Next, for an integer j with $0 \leq j \leq n - 3$, consider the subgraph H_j of C_n^4 consisting of three consecutive 4-cycles from $\mathcal{C}$, starting at j, that is,

$$H_j = \{(v_i, v_{i+1}, v_{i-1}, v_{i+3}) \mid i = j, j + 1, j + 2\}.$$

Note that H_j can also be decomposed into four 3-cycles, as given by the collection

$$\{(v_j, v_{j+1}, v_{j+4}), (v_{j+1}, v_{j+2}, v_{j+5}), (v_j, v_{j+2}, v_{j+3}), (v_{j+1}, v_{j+3}, v_{j-1})\},$$

or decomposed into two 3-cycles and a 6-cycle, as given by the collection

$$\{(v_j, v_{j+1}, v_{j+2}), (v_{j+1}, v_{j-1}, v_{j+3}), (v_j, v_{j+4}, v_{j+1}, v_{j+5}, v_{j+2}, v_{j+3})\}.$$

We now consider two cases, according to whether n is congruent to 0, 1 modulo 3 or to 2 modulo 3.

Case 1. Let $n \equiv 0, 1 \pmod 3$. Assume first that $n \equiv 0 \pmod 3$, say $n = 3k$ for some positive integer k. Then, since C_n^4 has $4n$ edges and $3 \mid n$, the maximum possible number of odd cycles in a cycle decomposition of C_n^4 is $4n/3 = 4k$. If $n \equiv 1 \pmod 3$, then $n = 3k + 1$ for some positive integer k. In this case, C_n^4 has

$4n = 12k + 4$ edges and since a cycle must have at least 3 edges, it follows that the maximum possible number of odd cycles in a cycle decomposition of C_n^4 is also $4k$. Now, let r be an even integer with $0 \le r \le 4k$. We show that there exists a cycle decomposition of C_n^4 having exactly r odd cycles. Since the case $r = 0$ has already been handled, we may assume that $r > 0$.

First, suppose that $r \equiv 0 \pmod 4$. Then $r = 4\ell$ for some integer ℓ with $0 < r \le k$. For integer j and i, let $A_j = (v_j, v_{j+1}, v_{j+4})$, $B_j = (v_j, v_{j+2}, v_{j+3})$, $D_j = (v_{j+1}, v_{j+3}, v_{j-1})$ and $F_i = (v_i, v_{i+1}, v_{i-1}, v_{i+3})$. Then,

$$\{A_j, A_{j+1}, B_j, D_j :\ j = 0, 3, 6, \ldots, 3(\ell-1)\} \cup \{F_i :\ i = 3\ell, 3\ell+1, \ldots, n-1\}$$

is an $(3^{4\ell}, 4^{n-3\ell})$-cycle decomposition of C_n^4.

Next, suppose that $r \equiv 2 \pmod 4$. Then $r = 4\ell + 2$ for some integer ℓ with $0 < \ell \le k$. Then,

$$\{(v_0, v_1, v_2), (v_1, v_{n-1}, v_3), (v_0, v_4, v_1, v_5, v_2, v_3)\} \cup \\ \{A_j, A_{j+1}, B_j, D_j :\ j = 3, 6, 9, \ldots, 3\ell\} \cup \{F_i :\ i = 3\ell+3, 3\ell+4, \ldots, n-1\}$$

where the second set is empty if $\ell = 0$, is an $(3^{4\ell+2}, 4^{n-(3\ell+3)}, 6)$-cycle decomposition of C_n^4.

Case 2. Let $n \equiv 2 \pmod 3$. Then $n = 3k + 2$ for some positive integer k. Since C_n^4 has $4n = 12k + 8$ edges, C_n^4 could possibly be decomposed into $4k + 1$ 3-cycles and one 5-cycle. Hence, the maximum possible number of odd cycles in a cycle decomposition of C_n^4 is $4k + 2 = 4\lfloor n/3 \rfloor + 1$. We show that there exists a cycle decomposition of C_n^4 with exactly r odd cycles for every even integer r with $0 < r \le 4k+2$ (as the case $r = 0$ has already been settled). As in the previous case, if $r \equiv 0 \pmod 4$, say $r = 4\ell$ for some positive integer ℓ, then

$$\{A_j, A_{j+1}, B_j, D_j :\ j = 0, 3, 6, \ldots, 3(\ell-1)\} \cup \{F_i :\ i = 3\ell, 3\ell+1, \ldots, n-1\}$$

is an $(3^{4\ell}, 4^{n-3\ell})$-cycle decomposition of C_n^4.

Now suppose that $r \equiv 2 \pmod 4$, say $r = 4\ell + 2$ for some nonnegative integer ℓ. Then

$$\{(v_0, v_1, v_2), (v_0, v_4, v_1, v_{n-1}, v_3)\} \cup \{A_j, A_{j+1}, B_j, D_j :\ j = 2, 5, 8, \ldots, 3\ell-1\} \cup \\ \{F_i :\ i = 3\ell+2, 3\ell+3, \ldots, n-1\},$$

where the second set is empty if $\ell = 0$, is an $(3^{4\ell+1}, 4^{n-(3\ell+2)}, 5)$-cycle decomposition of C_n^4. ∎

For an odd integer $n = 2d + 1 \ge 3$, we have $C_n^d = K_n$. Therefore, the maximum number of odd cycles in a cycle decomposition of C_n^k, $1 \le k \le d$, is known for $k \in \{1, 2, 3, 4, d\}$. For an even integer $n = 2d \ge 4$, we have $C_n^{d-1} = K_n - I$, where I is a 1-factor in K_n. Therefore, the maximum number of odd cycles in a cycle decomposition of C_n^k, $1 \le k \le d-1$, is known for $k \in \{1, 2, 3, 4, d-1\}$. Furthermore, we have shown that the ECDC is true for each of these graphs.

4 The ECDC and Complete Tripartite Graphs

We now turn to a class of Eulerian graphs that are not necessarily regular. The complete tripartite graph $K_{r,s,t}$, where $1 \leq r \leq s \leq t$, is Eulerian if and only if r, s, t are all even or all odd. Billington [5] investigated cycle decompositions of these graphs in which every cycle has length 3 or 4. In particular, she obtained the following theorem.

Theorem 4.1 *The complete tripartite graph $K_{r,s,t}$ with $r \leq s \leq t$ can be decomposed into α cycles of length 3 and β cycles of length 4 if and only if*

(*i*) *r, s, t are all even or all odd;*

(*ii*) *if either r is even or if r is odd and $s - r \equiv 0 \pmod 4$, then $\alpha \leq rs$;*

(*iii*) *if r is odd and $s - r \equiv 2 \pmod 4$, then $\alpha \leq rs - 2$;*

(*iv*) *$3\alpha + 4\beta = rs + rt + st$.*

Note that Theorem 4.1 does not show that the complete tripartite graph $K_{r,s,t}$ satisfies the ECDC. Nevertheless the ECDC does hold for this class of graphs.

Theorem 4.2 *The complete tripartite graph $K_{r,s,t}$ where r, s, t are all even or all odd satisfies the ECDC.*

Proof. Let $G = K_{r,s,t}$ where $r \leq s \leq t$ and r, s, t are either all even or all odd. Since every odd cycle in G must contain at least one vertex from each partite set of G, the maximum number of odd cycles in any cycle decomposition of G is at most rs.

Let the partite sets of G be denoted by U, V and W, where $U = \{u_1, u_2, \ldots, u_r\}$, $V = \{v_1, v_2, \ldots, v_s\}$ and $W = \{w_1, w_2, \ldots, w_t\}$. Consider a cycle decomposition of G that contains the rs 3-cycles

$$C_{i,j} = (u_i, v_j, w_{j+i-1}) \ , \ 1 \leq i \leq r, \ 1 \leq j \leq s,$$

where $j + i - 1 \in \{1, 2, \ldots, s\}$ and all arithmetic is performed modulo s.

These rs 3-cycles use all edges incident with the vertices in U. Since the only edges of G not used in these 3-cycles are $s^2 - rs$ edges in $K_{s,s}$ and those edges in $K_{r,t-s}$ and $K_{s,t-s}$, where each of the subgraphs induced by these edges is Eulerian and bipartite, this results in a cycle decomposition of G containing exactly rs odd cycles. Therefore, the maximum number of odd cycles in any cycle decomposition of G is rs.

This also says that if $K_{r,s,s}$ has a cycle decomposition with exactly k odd cycles, then so does $K_{r,s,t}$ for every integer $t > s$ (for which s and t have the same parity). Thus, in what follows, we may assume $s = t$.

Suppose now that r and s are even, say $r = 2a$, and $s = 2b$, where $a \leq b$. Let $U_i = \{u_{2i-1}, u_{2i}\}$ for $1 \leq i \leq a$, and $V_i = \{v_{2i-1}, v_{2i}\}$ and $W_i = \{w_{2i-1}, w_{2i}\}$ for $1 \leq i \leq b$. For $1 \leq i \leq a$ and $1 \leq j \leq b$, let $G_{i,j}$ be the induced subgraph of G isomorphic to $K_{2,2,2}$ and having partite sets $U_i, V_j, W_{j+(i-1)}$, where $j + i - 1 \in \{1, 2, \ldots, s\}$ and all arithmetic is performed modulo s.

Since the graph $K_{2,2,2}$ has a $(4, 4, 4)$-, a $(3, 3, 6)$-, and a $(3, 3, 3, 3)$-cycle decomposition, it follows that $K_{2,2,2}$ has a cycle decomposition into $0, 2$ or 4 odd cycles.

Now $K_{r,s,s}$ can be decomposed into the subgraphs $G_{i,j}$ if $r = s$ or into the subgraphs $G_{i,j}$ together with an Eulerian subgraph of $K_{s,s}$ of size $s^2 - rs$ if $s > r$. Since $K_{s,s}$ is bipartite, the only odd cycles in the decomposition are those obtained from the subgraphs $G_{i,j}$. Since each $G_{i,j}$ can be decomposed into 0, 2 or 4 odd cycles, $K_{r,s,s}$ can be decomposed into any even number k of odd cycles, where $0 \leq k \leq 4(ab) = rs$.

Suppose now that r and s are odd, say $r = 2a + 1$ and $s = 2b + 1$ for nonnegative integers a and b with $a \leq b$. In this case, $K_{r,s,s}$ has an odd number of edges and thus in any cycle decomposition of $K_{r,s,s}$, the number of odd cycles must be odd. Now let the partite sets of $K_{r,s,s}$ be denoted by U, V and W, where $U = \{u_0, u_1, \ldots, u_{2a}\}$, $V = \{v_0, v_1, \ldots, v_{2b}\}$ and $W = \{w_0, w_1, \ldots, w_{2b}\}$.

As before, let $U_i = \{u_{2i-1}, u_{2i}\}$ for $1 \leq i \leq a$, and $V_i = \{v_{2i-1}, v_{2i}\}$ and $W_i = \{w_{2i-1}, w_{2i}\}$ for $1 \leq i \leq b$. Also, for $1 \leq i \leq a$ and $1 \leq j \leq b$, let $G_{i,j}$ be the induced subgraph of $K_{r,s,s}$ isomorphic to $K_{2,2,2}$ and having partite sets $U_i, V_j, W_{j+(i-1)}$, where all arithmetic is performed modulo s. Let $H_{i,j}$ be the induced subgraph of $K_{r,s,s}$ having partite sets V_j and $W_{j+(i-1)}$ and note that $H_{i,j}$ is a 4-cycle. Now each $G_{i,j}$ decomposes into 0, 2 or 4 odd cycles. Note also that $\{(u_0, v_{2j-1}, w_0, v_{2j}), (u_0, w_{2j-1}, v_0, w_{2j}) \mid 1 \leq j \leq b\}$ and $\{H_{i,j} \mid a + 1 \leq i \leq b, 1 \leq j \leq b\}$ is a collection of 4-cycles that together with $\{G_{i,j} \mid 1 \leq i \leq a, 1 \leq j \leq b\}$ and the 3-cycle (u_0, v_0, w_0) is a decomposition of $K_{r,s,s}$. Decomposing each $G_{i,j}$ into the required number of odd cycles will yield a cycle decomposition of $K_{r,s,s}$ with exactly k odd cycles for each odd integer k with $1 \leq k \leq 4ab + 1 = (r - 1)(s - 1) + 1$.

It remains to show that for every odd integer k with $(r - 1)(s - 1) + 1 < k < rs$, there exists a cycle decomposition of $K_{r,s,s}$ with exactly k odd cycles. First, observe that the induced subgraph of $K_{r,s,s}$ with vertex set $V \cup W$ is isomorphic to $K_{s,s}$ and has a 1-factorization given by $\{F_j \mid 0 \leq j \leq s - 1\}$ where $F_j = \{\{v_m, w_{m+j}\} \mid 0 \leq m \leq s - 1\}$.

Note that $(r - 1)(s - 1) + 1 = (r - 2)s + (s - r + 2)$ and so $k > (r - 2)s$. For fixed integers i and j with $0 \leq i \leq r - 1$ and $0 \leq j \leq s - 1$, the graph $S_{i,j}$ formed by joining u_i to the vertices of F_j can be decomposed into s 3-cycles. Thus, the set $\{S_{i,i} \mid 2 \leq i \leq r - 1\}$ will give rise to $(r - 2)s$ 3-cycles. Let $\ell = k - (r - 2)s$ and note that $\ell \geq 2$ is even, say $\ell = 2t$ for some positive integer t. Consider the graph H formed by $S_{0,0} \cup S_{1,1}$, which is the join of two isolated vertices $\{u_0, u_1\}$ to the cycle $(w_0, v_0, w_1, v_1, \ldots, w_{s-1}, v_{s-1})$ of length $2s$. The cycle decomposition of H given by the collection of $\ell - 1$ 3-cycles

$$\{(u_0, w_i, v_i), (u_1, v_i, w_{i+1}) \mid 0 \leq i \leq t - 2\} \cup \{(u_0, w_{t-1}, v_{t-1})\},$$

the $(2s - \ell + 1)$-cycle

$$(u_1, v_{t-1}, w_t, v_t, w_{t+1}, v_{t+1}, \ldots, w_s, v_s, w_0, u_1),$$

and the collection of 4-cycles

$$\{(u_0, w_i, u_1, v_i) \mid t \leq i \leq s)\}$$

is a cycle decomposition of H with ℓ odd cycles. Since the remaining $s - r$ 1-factors $\{F_i \mid r \leq i \leq s - 1\}$ when taken two at a time form 2-factors of a bipartite graph and hence must consist of even cycles, we have a cycle decomposition of $K_{r,s,s}$ with exactly k odd cycles for each odd integer k with $(r-1)(s-1)+1 < k < rs$. ■

References

[1] B. Alspach, Research problems, Problem 3. *Discrete Math.* **36** (1981), 333.

[2] B. Alspach, The wonderful Walecki construction. *Bull. Inst. Combin. Appl.* **52** (2008), 7–20.

[3] B. Alspach and H. Gavlas, Cycle decompositions of K_n and $K_n - I$, *J. Combin. Theory Ser. B* **81** (2001), 77–99.

[4] E. Andrews, G. Chartrand, C. Lumduanhom and P. Zhang, On Eulerian walks in graphs, *Bull. Inst. Combin. Appl.* (to appear).

[5] E. J. Billington, Decomposing complete tripartite graphs into cycles of lengths 3 and 4, 16th British Combin. Conf. (London, 1997), *Discrete Math.* **197/198** (1999), 123–135.

[6] D. Bryant, D. Horsley and W. Pettersson, Cycle decompositions V: Complete graphs into cycles of arbitrary lengths, *Proc. London Math. Soc.* (to appear).

[7] D. Bryant and G. Martin, Some results on decompositions of low degree circulant graphs, *Australas. J. Combin.* **45** (2009), 251–261.

[8] G. Chartrand, A. D. Polimeni and M. J. Stewart, The existence of 1-factors in line graphs, squares, and total graphs, *Nedrl. Akad. Wetensch. Proc. Ser. A* 76 *Indag. Math.* **35** (1973), 228–232.

[9] L. Euler, Solutio problematis ad geometriam situs pertinentis, *Comment. Academiae Sci. I. Petropolitanae* **8** (1736), 128–140.

[10] R. K. Guy, A problem of Zarankiewicz, *1968 Theory of Graphs* (*Proc. Colloq., Tihany*). Academic Press, New York (1966), 119–150.

[11] K. Heinrich, P. Horák and A. Rosa, On Alspach's conjecture, in: Combinatorial designs—a tribute to Haim Hanani, *Discrete Math.* **77** (1989), 97–121.

[12] C. Hierholzer, Über die Möglichkeit, einen Linienzug ohne Wiederholung und ohne Unterbrechnung zu umfahren, *Math. Ann.* **6** (1873), 30–32.

[13] T. P. Kirkman, On a problem in combinations, *Cambridge and Dublin Math. J.* **2** (1847), 191–204.

[14] D. König, *Theorie der endlichen und unendliehen Graphen*, Akademische Verlagsgesellschaft, Leipzig (1936).

[15] M. Šajna, Cycle Decompositions III: Complete graphs and fixed length cycles, *J. Combin. Des.* **10** (2002), 27–78.

[16] O. Veblen, An application of modular equations in analysis situs, *Ann. Math.* **14** (1912), 86–94.

[17] O. Veblen, *Analysis Situs*, New York: Amer. Math. Soc. Colloquium publications (1922). Second edition (1931).

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