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On H-dominating matchings and some number partitions

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Abstract

A dominating set of a graph $G = (V, E)$ is a subset D of V such that every vertex of $V - D$ is adjacent to a vertex in D. In this paper we introduce a generalization of domination as follows. For graphs G and H, an H-matching M of G is a subgraph of G such that all components of M are isomorphic to H. An H-dominating matching of G is a H-matching D of G such that for each $x \in V(G)$ there exists $y \in V(D)$ such that $xy \in E(G)$. We consider P_k-dominating matchings for $k \geq 1$. We generalize recent results by Zwierzchowski [Graph Theory Notes of New York, XLVI, NY Acad. Sc. 2004, 13–19] on the number of dominating sets for the graphs P_n and C_n.

1 Introduction

In general we use the standard terminology and notation of graph theory; see [1]. We consider simple, undirected graphs G with the vertex set $V(G)$ and the edge set $E(G)$. A subset S of vertices of a graph G is an *independent set* of G if no two vertices of S are adjacent in G. A subset $D \subseteq V(G)$ is a *dominating set* of G if any vertex $y \in V(G) - D$ is adjacent to a vertex $x \in D$. We say that x dominates y in G or y is dominated by x in G. Domination in graphs is now well studied in graph theory. Recent articles on domination in graphs can be found in two books by Haynes et al. [2, 4]. There are many variations of domination.

We generalize the concept of domination. For graphs G and H, an *H-matching* M of G is a subgraph of G such that all components of M are isomorphic to H. If M is also an induced subgraph of G, then the H-matching is called induced. Problem of counting H-matchings in some graphs was considered in [6]. An *H-dominating matching* of G is an H-matching D of G such for each $x \in V(G)$ there exists $y \in V(D)$ such that $xy \in E(G)$. It is easy to see that if $H = K_1$, then the

H-dominating matching of G is a dominating set of G. An *H-domination matching number* of G, denoted by $\gamma_H(G)$, is the minimum cardinality of an H-dominating matching of G. We have $\gamma_{P_1}(G) = \gamma(G)$. Another generalization of the domination number $\gamma(G)$ is given in [3]. The authors introduced an H-forming set defined as follows. For graphs G and H, a set $S \subseteq V(G)$ is an *H-forming set* of G if for every $v \in V(G) - S$, there exists a subset $R \subseteq S$, where $|R| = |V(H)| - 1$, such that the subgraph induced by $R \cup \{v\}$ contains H as a subgraph (not necessarily induced). The minimum cardinality of an H-forming set of G is the *H-forming number*.

In this paper we give the numbers of all P_k-dominating matchings for the graphs P_n and C_n and we determine the P_k-domination matching number of these graphs for some k. These results generalize results in [5] and [7].

2 Main results

Let $k \geq 1$. Denote by $\partial_{P_k}(G)$ the number of all P_k-dominating matchings of a graph G. Let P_n denote the path on n vertices numbered in the natural fashion. It is easy to see that $\partial_{P_k}(P_n) = 0$ for $n < k$. Assume that $n \geq 2$ and $x_n \in V(P_n)$. Let D be any P_k-dominating matching of P_n. We denote the family of all P_k-dominating matchings of P_n such that $x_n \in V(D)$ ($x_n \notin V(D)$) by $\mathcal{D}_+$ ($\mathcal{D}_-$), respectively. Let $\partial_+(P_n) = |\mathcal{D}_+|$, $\partial_-(P_n) = |\mathcal{D}_-|$. Then the basic rule for counting of P_k-dominating matchings in the path P_n is as follows

$$\partial_{P_k}(P_n) = \partial_+(P_n) + \partial_-(P_n). \tag{1}$$

Denote by $P(x_i, x_{i+1}, \ldots, x_{i+k})$ a subgraph P_k of P_n with the vertex set $V(P_k) = \{x_i, x_{i+1}, \ldots, x_{i+k}\}$. We will use $P(i, i+1, \ldots, i+k)$ instead of $P(x_i, x_{i+1}, \ldots, x_{i+k})$.

Theorem 1 *Let n, k be integers, $1 \leq k \leq 3$ and $n \geq k$. Then for $n \geq k+3$,*

$$\partial_{P_k}(P_n) = \partial_{P_k}(P_{n-k}) + \partial_{P_k}(P_{n-k-1}) + \partial_{P_k}(P_{n-k-2})$$

with initial conditions $\partial_{P_1}(P_1) = 1$, $\partial_{P_1}(P_2) = 3$, $\partial_{P_1}(P_3) = 5$, $\partial_{P_2}(P_1) = 0$, $\partial_{P_2}(P_2) = 1$, $\partial_{P_2}(P_3) = 2$, $\partial_{P_2}(P_4) = 2$, $\partial_{P_3}(P_1) = \partial_{P_3}(P_2) = 0$, $\partial_{P_3}(P_3) = 1$, $\partial_{P_3}(P_4) = 2$, $\partial_{P_3}(P_5) = 1$.

Proof. Let $1 \leq k \leq 3$ and $n \geq 3$. Assume that vertices of P_n are numbered in the natural fashion. Let D be any P_k-dominating matching of P_n. The initial conditions are obvious. Consider the following cases:

Case 1. $x_n \notin V(D)$
Let $\mathcal{D}'$ be a family of all P_k-dominating matchings D of P_n such that $x_n \notin V(D)$. Then $x_{n-1} \in V(D)$ and $D = D^*$, where D^* is any P_k-dominating matching of the graph $P_n - \{x_n\}$ such that $x_{n-1} \in V(D^*)$. Hence $|\mathcal{D}'| = \partial_+(P_{n-1})$.

Case 2. $x_n \in V(D)$
Let $\mathcal{D}''$ be a family of all P_k-dominating matchings D of P_n such that $x_n \in V(D)$. Consider the following possibilities:

2.1. $x_{n-k} \in V(D)$
Then $D = D^{**} \cup P(n, n-1, n-2, \ldots, n-k+1)$, where D^{**} is any P_k-dominating matching of the graph $P_n - \{x_n, x_{n-1}, \ldots, x_{n-k+1}\}$ containing subgraph $P(n-k, n-k-1, \ldots, n-2k+1)$. Hence we have $\partial_+(P_{n-k})$ P_k-dominating matchings in this subcase.

2.2. $x_{n-k} \notin V(D)$ and $x_{n-k-1} \in V(D)$
Using the same method as in 2.1 we obtain $\partial_-(P_{n-k})$ P_k-dominating matchings in this subcase.

2.3. $x_{n-k} \notin V(D)$ and $x_{n-k-1} \notin V(D)$
Then $x_{n-k-2} \in V(D)$. Using the same method as in 2.1 we obtain $\partial_-(P_{n-k-1})$ P_k-dominating matchings in this subcase.
Thus $|\mathcal{D}''| = \partial_+(P_{n-k}) + \partial_-(P_{n-k}) + \partial_-(P_{n-k-1})$.

Consequently

$$\partial_{P_k}(P_n) = |\mathcal{D}'| + |\mathcal{D}''| = \partial_+(P_{n-1}) + \partial_+(P_{n-k}) + \partial_-(P_{n-k}) + \partial_-(P_{n-k-1}).$$

From Case 1 we obtain

$$\partial_-(P_n) = \partial_+(P_{n-1}). \tag{2}$$

From Case 2 we have

$$\partial_+(P_n) = \partial_+(P_{n-k}) + \partial_-(P_{n-k}) + \partial_-(P_{n-k-1}). \tag{3}$$

Using (3) and (2) we have

$$\begin{aligned}
&\partial_{P_k}(P_n) = \\
&\partial_+(P_{n-k-1}) + \partial_-(P_{n-k-1}) + \partial_-(P_{n-k-2}) + \partial_+(P_{n-k}) + \partial_-(P_{n-k}) + \partial_+(P_{n-k-2}) \\
&= \partial_+(P_{n-k}) + \partial_-(P_{n-k}) + \partial_+(P_{n-k-1}) + \partial_-(P_{n-k-1}) + \partial_+(P_{n-k-2}) + \partial_-(P_{n-k-2}) \\
&= \partial_{P_k}(P_{n-k}) + \partial_{P_k}(P_{n-k-1}) + \partial_{P_k}(P_{n-k-2}).
\end{aligned}$$

The last equation follows from the equality in (1). This ends the proof. □

By the definition of P_k-dominating matching we have

Theorem 2 *Let n, k be integers, $k \geq 4$ and $n \geq k$. Then P_n has a P_k-dominating matching if and only if* $n \bmod k \leq 2(\lfloor \frac{n}{k} \rfloor - 1) + 2$. □

It is easy to see that counting of H-dominating matchings in the graph P_n corresponds to counting of s-elements sequences (a_p) such that $a_i \in \{0, 1, 2, k\}$ for $i = 1, 2, \ldots, s$, $a_1, a_s \in \{1, k\}$ and for $i = 2, 3, \ldots, s-1$ $a_i \in \{0, 1, 2\}$ if $a_{i-1} = k$ and $a_{i+1} = k$. Using the graph interpretation of the number of all P_k-dominating matchings of P_n for $n \geq 3$ and $k \geq 2$ we have the following combinatorial interpretation: $\partial_{P_k}(P_n)$ is the number of ways of writing $n-1$ ($n-1 = |E(P_n)|$) as an ordered sum in which each term is $k-1$ ($k-1 = |E(P_k)|$) or $k-1$ and 1 or 2 or 3 and the first and the last term is either $k-1$ or 1 and the remaining terms 2 or 1 are followed and proceeded by the term $k-1$. For example, by Theorem 1, $\partial_{P_3}(P_{11}) = 8$. There are 8 compositions of 10 satisfied above assumptions. They are presented in the table presented below.

P_3-dominating matching of P_{11}	Composition of 10
$P(1,2,3), P(4,5,6), P(8,9,10)$	$2+1+2+2+2+1$
$P(1,2,3), P(5,6,7), P(8,9,10)$	$2+2+2+1+2+1$
$P(1,2,3), P(6,7,8), P(9,10,11)$	$2+3+2+1+2$
$P(1,2,3), P(4,5,6), P(9,10,11)$	$2+1+2+3+2$
$P(1,2,3), P(5,6,7), P(9,10,11)$	$2+2+2+2+2$
$P(2,3,4), P(5,6,7), P(8,9,10)$	$1+2+1+2+1+2+1$
$P(2,3,4), P(5,6,7), P(9,10,11)$	$1+2+1+2+2+2$
$P(2,3,4), P(6,7,8), P(9,10,11)$	$1+2+2+2+1+2$

Moreover, for $k \geq 5$, the P_k-domination matching number of P_n is the smallest number of terms being $k-1$ in the composition of the number $n-1$.

By the definition of P_k-dominating matching we have the following

Theorem 3 *If P_n has a P_k-dominating matching, then $\gamma_{P_k}(P_n) = \lceil \frac{n}{k+2} \rceil$.* □

Let C_n be a cycle on $n \geq 3$ vertices. Let D be any P_k-dominating matching of C_n. We will use the following notation:
D_{++} is the family of all P_k-dominating matchings of C_n such that $P(1,2,\ldots,k) \subset D$ and $P(n,n-1,\ldots,n-k+1) \subset D$,
D_{+++} is the family of all P_k-dominating matchings of C_n such that there exists path $P \subset D$ with $x_n, x_1 \in V(P)$,
D_{--} is the family of all P_k-dominating matchings of C_n such that $P(1,2,\ldots,k) \not\subset D$ and $P(n,n-1,\ldots,n-k+1) \not\subset D$,
D_{+-} is the family of all P_k-dominating matchings of C_n such that either $P(1,2,\ldots,k) \subset D$ and $P(n,n-1,\ldots,n-k+1) \not\subset D$ or $P(1,2,\ldots,k) \not\subset D$ and $P(n,n-1,\ldots,n-k+1) \subset D$.
Let $\partial_{++}(C_n) = |D_{++}|$, $\partial_{+++}(C_n) = |D_{+++}|$, $\partial_{--}(C_n) = |D_{--}|$, $\partial_{+-}(C_n) = |D_{+-}|$. It is easily seen that

$$\text{for } k=1 \ \partial_{P_k}(C_n) = \partial_{++}(C_n) + \partial_{--}(C_n) + \partial_{+-}(C_n), \tag{4}$$

$$\text{for } k \geq 2 \ \partial_{P_k}(C_n) = \partial_{++}(C_n) + \partial_{--}(C_n) + \partial_{+-}(C_n) + \partial_{+++}(C_n). \tag{5}$$

The following result follows immediately from the definition of P_k-dominating matching.

Theorem 4 *Let n, k be integers, $k \geq 4$ and $n \geq k$. Then C_n has a P_k-dominating matching if and only if $n \bmod k \leq 2\lfloor \frac{n}{k} \rfloor$.* □

Theorem 5 *Let n, k be integers, $1 \leq k \leq 3$ and $n \geq k$. Then for $n \geq k+5$*

$$\partial_{P_k}(C_n) = \partial_{P_k}(C_{n-k}) + \partial_{P_k}(C_{n-k-1}) + \partial_{P_k}(C_{n-k-2})$$

with initial conditions $\partial_{P_1}(C_3) = 7$, $\partial_{P_1}(C_4) = 11$, $\partial_{P_1}(C_5) = 21$, $\partial_{P_2}(C_3) = 3$, $\partial_{P_2}(C_4) = 6$, $\partial_{P_2}(C_5) = 5$, $\partial_{P_2}(C_6) = 11$, $\partial_{P_3}(C_3) = 1$, $\partial_{P_3}(C_4) = 4$, $\partial_{P_3}(C_5) = 5$, $\partial_{P_3}(C_6) = 3$, $\partial_{P_3}(C_7) = 7$.

Proof. Assume that vertices of the graph C_n are numbered in the natural fashion. The initial conditions are obvious. Assume that $n \geq 4$. Let D be any P_k-dominating matching of C_n. Consider the following cases:

Case 1. $P(1,2,\ldots,k), P(k+1,k+2,\ldots,2k), P(n,n-1,\ldots,n-k+1) \subset D$
Let $\mathcal{D}_1$ be a family of all P_k-dominating matchings D of C_n such that $P(1,2,\ldots,k), P(k+1,k+2,\ldots,2k), P(n,n-1,\ldots,n-k+1) \subset D$. Let D' be any P_k-dominating matching of the graph G_1, such that $V(G_1) = V(C_n) \setminus \{x_1, x_2, \ldots, k\}$ and $E(G_1) = (E(C_n) \setminus \{x_n x_1, x_1 x_2, \ldots, x_k x_{k+1}\}) \cup \{x_{k+1} x_n\}$ and $P(k+1,k+2,\ldots,2k), P(n,n-1,\ldots,n-k+1) \subset D'$. This is easy to see that G_1 is isomorphic to the graph C_{n-k}. Then $D = D' \cup P(1,2,\ldots,k)$. Thus $|\mathcal{D}_1| = \partial_{++}(C_{n-k})$.

Case 2. $P(1,2,\ldots,k), P(n,n-1,\ldots,n-k+1) \subset D$ and $x_{k+1} \notin V(D)$
Let $\mathcal{D}_2$ be a family of all P_k-dominating matchings D of C_n such that $P(1,2,\ldots,k), P(n,n-1,\ldots,n-k+1) \subset D$ and $x_{k+1} \notin V(D)$. Let D' be any P_k-dominating matching of the graph G_2, such that $V(G_2) = V(C_n) \setminus \{x_1, x_2, \ldots, x_k\}$ and $E(G_2) = (E(C_n) \setminus \{x_n x_1, x_1 x_2, \ldots, x_k x_{k+1}\}) \cup \{x_{k+1} x_n\}$ and $P(n,n-1,\ldots,n-k+1) \subset D'$ and $x_{k+1} \notin V(D')$. This is easy to see that G_2 is isomorphic to the graph C_{n-k}. Then $D = D' \cup P(1,2,\ldots,k)$. Thus $|\mathcal{D}_2| = \partial_{+-}(C_{n-k})$.

Case 3. $P(n,n-1,\ldots,n-k+1) \subset D$ and $x_1, x_2 \notin V(D)$
Let $\mathcal{D}_3$ be a family of all P_k-dominating matchings D of C_n such that $P(n,n-1,\ldots,n-k-1) \subset D$ and $x_1, x_2 \notin V(D)$. Then $P(3,4,\ldots,k+2) \subset D$. Let G_3 be the graph with $V(G_3) = V(C_n) \setminus \{x_1, x_2\}$ and $E(G_3) = (E(C_n) \setminus \{x_n x_1, x_1 x_2, x_2 x_3\}) \cup \{x_k x_n\}$. Then $D = D'$, where D' is any P_k-dominating matching of the graph G_3 containing subgraphs $P(n,n-1,\ldots,n-k+1)$ and $P(3,4,\ldots,k+2)$. Thus $|\mathcal{D}_3| = \partial_{++}(C_{n-2})$.

Case 4. $P(2,3,\ldots,k+1), P(n,n-1,\ldots,n-k-1) \subset D$ and $x_1 \notin V(D)$
Let $\mathcal{D}_4$ be a family of all P_k-dominating matchings D of C_n such that $P(2,3,\ldots,k+1), P(n,n-1,\ldots,n-k+1) \subset D$ and $x_1 \notin V(D)$.
Using the same method as in Case 3 we obtain $|\mathcal{D}_4| = \partial_{++}(C_{n-1})$.

Case 5. $P(1,2,\ldots,k), P(n-1,n-2,\ldots,n-k) \subset D$
Let $\mathcal{D}_5$ be a family of all P_k-dominating matchings D of C_n such that $P(1,2,\ldots,k), P(n-1,n-2,\ldots,n-k) \subset D$. Let G_5 be the graph with $V(G_5) = V(C_n) \setminus \{x_n\}$ and $E(G_5) = (E(C_n) \setminus \{x_n x_1, x_n x_{n-1}\}) \cup \{x_1 x_{n-1}\}$. Then $D = D'$, where D' is any P_k-dominating matching of the graph G_5 containing subgraphs $P(1,2,\ldots,k), P(n-1,n-2,\ldots,n-k)$. Thus $|\mathcal{D}_5| = \partial_{++}(C_{n-1})$.

Case 6. $x_n, x_{n-1} \notin V(D)$
Let $\mathcal{D}_6$ be a family of all P_k-dominating matchings D of C_n such that $x_n, x_{n-1} \notin$

$V(D)$. Then $P(1,2,\dots,k), P(n-2,n-3,\dots,n-k-1) \subset D$. Let G_6 be the graph with $V(G_6) = V(C_n)\setminus\{x_n, x_{n-1}\}$ and $E(G_6) = (E(C_n)\setminus\{x_{n-2}x_{n-1}, x_{n-1}x_n, x_nx_1\})\cup\{x_1x_{n-2}\}$. Then $D = D'$, where D' is any P_k-dominating matching of the graph G_6 containing subgraphs $P(1,2,\dots,k), P(n-2,n-3,\dots,n-k-1)$. Thus $|\mathcal{D}_6| = \partial_{++}(C_{n-2})$.

Case 7. $x_1, x_n \notin V(D)$
Let $\mathcal{D}_7$ be a family of all P_k-dominating matchings D of C_n such that $x_1, x_n \notin V(D)$. Then $P(2,3,\dots,k+1), P(n-1,n-2,\dots,n-k) \subset D$. In the same manner as in the Case 6 we obtain that $|\mathcal{D}_7| = \partial_{++}(C_{n-2})$.

For $2 \le k \le 3$ we need the following:
Case 8. There exists a path $P^* \subset D$ such that $x_1, x_n \in V(P^*)$.
It is easily seen that for $k = 3$ there exist two non-isomorphic paths P^* satisfied above assumptions. Let $\mathcal{D}_8$ be a family of all P_k-dominating matchings D of C_n such that $P^* \subset D$. Using the same method as in Cases 1–6 we obtain that $|\mathcal{D}_8| = \partial_{++}(C_{n-k}) + \frac{1}{2}\partial_{+-}(C_{n-k}) + 2\partial_{++}(C_{n-1}) + 2\partial_{++}(C_{n-2})$.
From Cases 1 and 2 we have

$$\partial_{++}(C_n) = \partial_{++}(C_{n-k}) + \frac{1}{2}\partial_{+-}(C_{n-k}). \tag{6}$$

From Cases 3–6 we obtain

$$\partial_{+-}(C_n) = 2\partial_{++}(C_{n-1}) + 2\partial_{++}(C_{n-2}). \tag{7}$$

From Case 7 it follows that

$$\partial_{--}(C_n) = \partial_{++}(C_{n-2}). \tag{8}$$

From Case 8 we obtain for $k = 2$

$$\partial_{+++}(C_n) = \frac{1}{2}\partial_{+-}(C_{n-2}) + 2\partial_{++}(C_{n-1}) + 3\partial_{++}(C_{n-2}) \tag{9}$$

and for $k = 3$

$$\partial_{+++}(C_n) = 2\left(\partial_{++}(C_{n-3}) + \frac{1}{2}\partial_{+-}(C_{n-3}) + 2\partial_{++}(C_{n-1}) + 2\partial_{++}(C_{n-2})\right). \tag{10}$$

We give the proof for $k = 3$; for $k = 1, 2$ we can prove analogously. Since

$$\partial_{P_k}(C_n) = \partial_{++}(C_n) + \partial_{--}(C_n) + \partial_{+-}(C_n) + \partial_{+++}(C_n),$$

we have for $k = 3$

$$\begin{aligned}\partial_{P_k}(C_n) &= \partial_{++}(C_{n-3}) + \frac{1}{2}\partial_{+-}(C_{n-3}) + 2\partial_{++}(C_{n-1}) + 2\partial_{++}(C_{n-2}) + \partial_{++}(C_{n-2})\\ &+ 2\partial_{++}(C_{n-3}) + \partial_{+-}(C_{n-3}) + 4\partial_{++}(C_{n-1}) + 4\partial_{++}(C_{n-2}).\end{aligned}$$

Using three times (6) and (7), we have

$$\begin{aligned}\partial_{P_k}(C_n) &= \partial_{++}(C_{n-3}) + \frac{1}{2}\partial_{+-}(C_{n-3}) + 2\partial_{++}(C_{n-4}) + \partial_{+-}(C_{n-4}) + 2\partial_{++}(C_{n-5}) \\ &+ \partial_{+-}(C_{n-5}) + \partial_{++}(C_{n-5}) + \frac{1}{2}\partial_{+-}(C_{n-5}) + 2\partial_{++}(C_{n-6}) + \partial_{+-}(C_{n-6}) \\ &+ 2\partial_{++}(C_{n-4}) + 2\partial_{++}(C_{n-5}) + 4\partial_{++}(C_{n-4}) + 2\partial_{+-}(C_{n-4}) + 4\partial_{++}(C_{n-5}) \\ &+ 2\partial_{+-}(C_{n-5}). \qquad (11)\end{aligned}$$

By (8) and (7) we obtain for $k = 3$

$$\begin{aligned}\partial_{++}(C_{n-5}) &= \partial_{--}(C_{n-3}), \\ \partial_{++}(C_{n-4}) + \partial_{++}(C_{n-5}) &= \frac{1}{2}\partial_{+-}(C_{n-3}), \\ \frac{1}{2}\partial_{+-}(C_{n-5}) &= \partial_{++}(C_{n-6}) + \partial_{++}(C_{n-7}) = \partial_{--}(C_{n-4}) + \partial_{--}(C_{n-5}).\end{aligned}$$

Using twice, in (11), equalities (6) and (7) and the above results, we have

$$\begin{aligned}\partial_{P_k}(C_n) &= \partial_{++}(C_{n-3}) + \partial_{+-}(C_{n-3}) + \partial_{--}(C_{n-3}) + 2\partial_{++}(C_{n-6}) + \partial_{+-}(C_{n-6}) \\ &+ 4\partial_{++}(C_{n-4}) + 4\partial_{++}(C_{n-5}) + \partial_{++}(C_{n-4}) + \partial_{+-}(C_{n-4}) + \partial_{--}(C_{n-4}) \\ &+ 2\partial_{++}(C_{n-7}) + \partial_{+-}(C_{n-7}) + \partial_{++}(C_{n-5}) + \partial_{+-}(C_{n-5}) + \partial_{--}(C_{n-5}) \\ &+ 2\partial_{++}(C_{n-8}) + \partial_{+-}(C_{n-8}) + 4\partial_{++}(C_{n-5}) + 8\partial_{++}(C_{n-6}) + 4\partial_{++}(C_{n-7}) \\ &= \partial_{++}(C_{n-3}) + \partial_{--}(C_{n-3}) + \partial_{+-}(C_{n-3}) + \partial_{+++}(C_{n-3}) \\ &+ \partial_{++}(C_{n-4}) + \partial_{--}(C_{n-4}) + \partial_{+-}(C_{n-4}) + \partial_{+++}(C_{n-4}) \\ &+ \partial_{++}(C_{n-5}) + \partial_{--}(C_{n-5}) + \partial_{+-}(C_{n-5}) + \partial_{+++}(C_{n-5}) \\ &= \partial_{P_k}(C_{n-3}) + \partial_{P_k}(C_{n-4}) + \partial_{P_k}(C_{n-5}).\end{aligned}$$

The last equation follows from the equality in (5). This ends the proof. □

By the definition of P_k-dominating matching we have

Theorem 6 *Let $n \geq 3$, $k \geq 1$. If C_n has a P_k-dominating matching, then $\gamma_{P_k}(C_n) = \lceil \frac{n}{k+2} \rceil$.* □

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