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On connected k-domination in graphs

Karima Attalah Mustapha Chellali*

LAMDA-RO Laboratory, Department of Mathematics
University of Blida
B.P. 270, Blida
Algeria
k_attalah@yahoo.com m_chellali@yahoo.com

Abstract

Let $G = (V(G), E(G))$ be a simple connected graph, and let k be a positive integer. A subset $D \subseteq V(G)$ is a connected k-dominating set of G if its induced subgraph is connected and every vertex of $V(G) - D$ is adjacent to at least k vertices of D. The connected k-domination number $\gamma_k^c(G)$ is the minimum cardinality among the connected k-dominating sets of G. In this paper, we give some properties of connected graphs G with $\gamma_k^c(G) = n - 2$. Then we provide a complete characterization of connected cubic graphs G with $\gamma_2^c(G) = n - 2$ and connected 4-regular claw-free graphs with $\gamma_3^c(G) = n - 2$.

1 Introduction

We consider finite, undirected and simple graphs $G = (V(G), E(G))$ with vertex set $V(G)$ and edge set $E(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order* and is denoted by $n = n(G)$. The *open neighborhood* $N(v) = N_G(v)$ of a vertex v consists of the vertices adjacent to v and $d_G(v) = |N(v)|$ is the *degree* of v. The *closed neighborhood* of a vertex v is defined by $N[v] = N_G[v] = N(v) \cup \{v\}$. If S is a subset of $V(G)$, then $N(S) = \cup_{x \in S} N(x)$, $N[S] = \cup_{x \in S} N[x]$, and the subgraph induced by S in G is denoted by $G[S]$. We may write $G - X$ instead of $G[V(G) - X]$ for any $X \subseteq V(G)$. We denote the *minimum degree* and the *maximum degree* of a graph G by $\delta(G)$ and $\Delta(G)$, respectively.

We write K_n for the *complete graph* of order n, and $K_{s,t}$ for the *complete bipartite graph* with bipartition X, Y such that $|X| = s$ and $|Y| = t$. A *k-regular graph* or regular graph of degree k is a graph whose vertices are all of degree k. A 3-regular graph is called a *cubic graph*. The *claw* is the star $K_{1,3}$. A graph G is *claw-free* if it does not have any induced subgraph isomorphic to $K_{1,3}$. A *clique* of a graph G is a

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complete subgraph of G. A *cut vertex* (*bridge*, respectively) of a connected graph is a vertex (an edge, respectively), the removal of which would disconnect G.

In [2], Fink and Jacobson generalized the concept of dominating sets. Let k be a positive integer. A subset D of $V(G)$ is *k-dominating* if every vertex of $V(G) - D$ is adjacent to at least k vertices of D. Thus the 1-dominating set is a dominating set and so $\gamma_1(G) = \gamma(G)$. For more details on k-domination, see the recent survey of Chellali et al. [1].

A subset $D \subseteq V(G)$ is a *connected k-dominating set* of a connected graph G, if D is a k-dominating set of G and if the induced subgraph $G[D]$ is connected. The *connected k-dominating number* $\gamma_k^c(G)$ is the minimum cardinality among the connected k-dominating sets of G. A connected k-dominating set of minimum cardinality of a connected graph G is called a *$\gamma_k^c(G)$-set*.

In [3], Volkmann characterized connected graphs G with $\gamma_k^c(G) = n$ for every integer $k \geq 2$. He also characterized all connected graphs G with $\gamma_k^c(G) = n - 1$ when $\delta(G) \geq k \geq 2$. The problem of characterizing all connected graphs G with $\gamma_k^c(G) = n - 2$ when $\delta(G) \geq k \geq 2$ remained open.

In this paper, we first give some properties of connected graphs G with $\gamma_k^c(G) = n-2$. Then we provide a complete characterization of connected cubic graphs G such that $\gamma_2^c(G) = n - 2$ and connected 4-regular claw-free graphs with $\gamma_3^c(G) = n - 2$.

2 Properties of graphs G with $\gamma_k^c(G) = n - 2$

Lemma 1 *Let $k \geq 2$ be an integer and G a connected graph such that $\gamma_k^c(G) = n-2$. Then $\delta(G) \leq k + 1$.*

Proof. Let G be a connected graph such that $\gamma_k^c(G) = n-2$ for some integer $k \geq 2$, and assume that $\delta(G) \geq k + 2$. Let D be a $\gamma_k^c(G)$-set and x any vertex of D such that $G[D - \{x\}]$ is connected. Clearly such a vertex exists and since $d_G(x) \geq k + 2$, x has at least k neighbors in $D - \{x\}$. Also, every vertex of $V - D$ still has at least k neighbors in $D - \{x\}$. It follows that $D - \{x\}$ is a connected k-dominating set of G of cardinality $(n - 3)$, a contradiction. □

Recall that an *independent set* of a graph G is a set S of vertices such that no edge of G has its two endvertices in S. We mention that we often use in our proofs the fact that in every nontrivial connected graph G, there exist two vertices such that the removal of each one from G leaves the resulting graph connected.

Lemma 2 *Let $k \geq 2$ be an integer and G a connected graph with $\delta(G) \geq k$ such that $\gamma_k^c(G) = n - 2$. Then for every independent set S of cardinality at least three, $G - S$ is a disconnected graph.*

Proof. Let G be a connected graph with $\delta(G) \geq k \geq 2$ such that $\gamma_k^c(G) = n - 2$. Let S be an independent set with $|S| \geq 3$. Since every vertex of S has at least k

neighbors in $V-S$, the subgraph induced by $V-S$ is disconnected, for otherwise $V-S$ would be a connected k-dominating set of G of size less than $n-2$. □

Lemma 3 *Let $k \geq 2$ be a positive integer and G a connected graph of order n with $\delta(G)=k+1$. If $\gamma_k^c(G)=n-2$, then G contains no bridge.*

Proof. Let G be a connected graph with $\delta(G)=k+1$ such that $\gamma_k^c(G)=n-2$ for some integer $k \geq 2$. Assume that G contains a bridge uv. Let G_u and G_v denote the two components resulting from the removal of uv, where $u \in V(G_u)$ and $v \in V(G_v)$. We further assume that uv is chosen so that, say G_u, has no bridge. Clearly since $\delta(G)=k+1$ and $k \geq 2$, each of G_u and G_v has order at least three. Now let w and w' be any two adjacent vertices in G_u different from u and let $v' \neq v$ be a vertex of G_v such that $G_v-\{v'\}$ is connected. If $G_u-\{w,w'\}$ is connected, then $V(G)-\{w,w',v'\}$ is a connected k-dominating set of G of cardinality $(n-3)$, a contradiction. Thus let us assume that $G_u-\{w,w'\}$ is not connected. Let G_u^i denote the ith component of $G_u-\{w,w'\}$. Note that each G_u^i has order at least two, and since G_u is assumed to have no bridge, there are at least two edges between $\{w,w'\}$ and $V(G_u^i)$ for every i. We consider two cases.

Case 1. $i \geq 3$. Without loss of generality, let G_u^1 and G_u^2 be two components that do not contain u. Now let x be any vertex of G_u^1 such that $G_u^1-\{x\}$ is connected. Likewise let y be a vertex of G_u^2 defined as x. Observe that $S=\{x,y,v'\}$ is an independent set and so by Lemma 2, $G-S$ is disconnected. Hence either $G_u-\{x\}$ or $G_u-\{y\}$ is disconnected, say $G_u-\{x\}$. Since there are at least two edges between $\{w,w'\}$ and $V(G_u^1)$, we conclude that w and w' are both adjacent to x and have no other neighbor in G_u^1 different from x. Now since $\delta(G)=k+1 \geq 3$, G_u^1 has order at least three and so let $x' \neq x$ be a vertex of G_u^1 such that $G_u^1-\{x'\}$ is connected. Likewise, let y' be a vertex of G_u^2 with $y' \neq y$ if y is the unique neighbor of w and w' in G_u^2 and $y'=y$ otherwise. Clearly now $\{x',y',v'\}$ is an independent set and $G-\{x',y',v'\}$ is connected, which contradicts Lemma 2.

Case 2. $i=2$. Thus u belongs to either G_u^1 or G_u^2, say G_u^1. Clearly one of w and w' must have at least one neighbor in each component. Assume that w is a such vertex. Let y be any vertex of G_u^2 such that $G_u^2-\{y\}$ is connected. If w has another neighbor in G_u^2 different from y, then $V(G)-\{w',y,v'\}$ is a connected k-dominating set of G of cardinality $(n-3)$, a contradiction. Hence y is the unique neighbor of w in G_u^2. Since $\delta(G)=k+1 \geq 3$, G_u^2 has order at least three and so there is a vertex $y' \neq y$ in such that $G_u^2-\{y'\}$ is connected. Therefore $V(G)-\{w',y',v'\}$ is a connected k-dominating set of G of cardinality $(n-3)$, a contradiction too. The proof of Lemma 3 is complete. □

Lemma 4 *Let $k \geq 2$ be an integer and G a connected graph with $\delta(G)=k+1$. If $\gamma_k^c(G)=n-2$, then for every pair of adjacent vertices x,y, $V(G)-\{x,y\}$ is a minimum connected k-dominating set of G.*

Proof. Let G be a connected graph with $\delta(G)=k+1$ such that $\gamma_k^c(G)=n-2$ for some integer $k \geq 2$. Let x,y be two adjacent vertices of G. Clearly since

$\delta(G) = k + 1$, $V(G) - \{x, y\}$ k-dominates G. Hence to show that $V(G) - \{x, y\}$ is a $\gamma_k^c(G)$-set, it suffices to show that $G' = G - \{x, y\}$ is connected. Thus assume to the contrary that G' is not connected and let C_i be the ith component of G'. Note that each C_i is nontrivial and there are at least two edges between $V(C_i)$ and $\{x, y\}$, otherwise we have a bridge which contradicts Lemma 3. Now let x_i be a vertex of C_i such that $C_i - \{x_i\}$ is connected. If the subgraph induced by the vertices of $(V(C_i) - \{x_i\}) \cup \{x, y\}$ is not connected, then x and y are both adjacent to x_i and have no other neighbor in C_i besides x_i. But then there exists another vertex x_i' in C_i such that the subgraph G_i' induced by $(V(C_i) - \{x_i'\}) \cup \{x, y\}$ is connected. Thus, without loss of generality, we may assume that such a vertex x_i' exists in each component C_i. Now if G' has three components or more, then vertices x_i' form an independent set whose removal does not disconnect G, contradicting Lemma 2. Therefore G' has exactly two components. Moreover, since xy is not a bridge, one of x and y, say x, has neighbors in both C_1 and C_2. If C_1 has order two, then obviously $k = 2$, $\delta(G) = 3$ and so $\{x, y\}$ 2-dominates $V(C_1)$, implying that $V(G) - (V(C_1) \cup \{x_2'\})$ is a connected 2-dominating set of G of size $n - 3$, a contradiction. Hence we can assume that $|V(C_1)| \geq 3$.

Now let z be a vertex of G_1' different from x and y. Recall that x_1' does not belong to G_1'. Now if $G_1' - \{z\}$ is connected, then $V(G) - \{z, x_1', x_2'\}$ is a connected k-dominating set of G, a contradiction. Hence $G_1' - \{z\}$ is not connected, and so z is a cut vertex of G_1'. Clearly each component of $G_1' - \{z\}$ is nontrivial, and one of them contains both x and y. In this case, let z' be any vertex in the component, say C^*, that does not contain x and y, for which $C^* - \{z'\}$ is connected. Note that if z' is the unique neighbor of z in C^*, then we can find another vertex z'' such that $C^* - \{z''\}$ is connected. So we may assume that z' is not the unique neighbor of z in C^*. Now clearly we have $V(G) - \{z', x_1', x_2'\}$ is a connected k-dominating set of G, a contradiction too. This achieves the proof of Lemma 4. □

Lemma 5 *Let $k \geq 2$ be an integer and G a connected graph of order n and minimum degree $\delta(G) = k + 1$. If D is a $\gamma_k^c(G)$-set of size $n - 2$ such that the subgraph induced by $N(V - D)$ is connected, then for every vertex $x \in D$, $N(x) \cap (V - D) \neq \emptyset$.*

Proof: Let G be a a connected graph of order n and minimum degree $\delta(G) = k + 1$ for some integer $k \geq 2$. Let D be a $\gamma_k^c(G)$-set with $|D| = n - 2$, $A = N(V - D)$ and $B = D - A$. Assume that $G[A]$ is connected and $B \neq \emptyset$. If there is a vertex $x \in B$ such that $G[D - \{x\}]$ is connected, then $D - \{x\}$ is a connected k-dominating set of G smaller than D, a contradiction. Hence every vertex of B is a cut vertex in $G[D]$. It follows that some vertex of B, say y has no neighbor in A, for otherwise $G[D - \{y\}]$ is connected, contradicting the fact that y is a cut vertex in $G[D]$. Thus $N(y) \cap A = \emptyset$. Now it is clear that some component of $G[D - \{y\}]$ contains all of A. Let C be a component of $G[D - \{y\}]$ that does not contain A. Thus every vertex of C belongs to B, that is, has no neighbor in $V - D$. Note that C is nontrivial. Let y' be a vertex of C such that $C - \{y'\}$ is connected. In the case that y' is the unique neighbor of y in C, then C has another vertex y'' such that $C - \{y''\}$ is connected. In this case we consider y'' instead of y'. Hence we may assume that y has an neighbor in C besides y'. It follows that $D-\{y'\}$ is a connected k-dominating set of G smaller than D, a contradiction. Therefore $B=\emptyset$ and we obtain the desired result. □

3 Cubic graphs with $\gamma_2^c(G)=n-2$

In this section, we give a complete characterization of cubic graphs with $\gamma_k^c(G)=n-2$, when $k=2$. It well known that a cubic graph contains a bridge if and only if it contains a cut vertex.

Theorem 6 *Let G be a connected cubic graph of order n. Then $\gamma_2^c(G)=n-2$ if and only if $G=K_4, K_{3,3}$ or G is the complement graph of C_6.*

Proof. It is easy to check that if $G=K_4, K_{3,3}$ or G is the complement graph of C_6, then $\gamma_2^c(G)=n-2$.

Conversely, let G be a connected cubic graph such that $\gamma_2^c(G)=n-2$. We first assume that G has a vertex x whose neighborhood, say $\{x_1,x_2,x_3\}$, is an independent set. By Lemma 4, $V(G)-\{x,x_1\}=V'$ is a $\gamma_2^c(G)$-set. Let $A=\{x_2,x_3\}$ and $B=N(x_1)-\{x\}=\{y_1,y_2\}$. Clearly $A\cap B=\emptyset$ since $N(x)$ is independent. Let $V''=V'-(A\cup B)$. We shall show that $V''=\emptyset$. Suppose to the contrary that $V''\neq\emptyset$ and let z be any vertex of V''. Then z is a cut vertex in $G[V']$, for otherwise $V(G)-\{x,x_1,z\}$ is a connected 2-dominating set of G of cardinality $(n-3)$, a contradiction. Note that each component of $G[V'-\{z\}]$ contains at least one vertex of $A\cup B$, for otherwise z is a cut vertex in G, implying that G has a bridge, a contradiction with Lemma 3. Consider the following two cases.

Case 1. $G[V'-\{z\}]$ contains three connected components C_1, C_2 and C_3. Clearly each C_i is nontrivial and z has exactly one neighbor in each C_i. Also, since each component contains at least one vertex of $A\cup B$, we may assume, without loss of generality, that $y_1\in V(C_1)$, $x_2\in V(C_2)$ and $x_3,y_2\in V(C_3)$. Now, let $a\in V(C_1)$ such that $a=y_1$ if $y_1z\notin E(G)$ and $a\in N(y_1)-\{z,x_1\}$ otherwise. Observe that $C_1-\{a\}$ is connected, for otherwise a is a cut vertex in C_1 and so in G, a contradiction. Likewise let $b\in V(C_2)$ such that $b=x_2$ if $x_2z\notin E(G)$ and $b=N(x_2)-\{z,x\}$ otherwise. Then $C_2-\{b\}$ is also connected. In addition, let $c\in V(C_3)$ such that $cz\notin E(G)$ and $C_3-\{c\}$ is connected. Now it is evident that $\{a,b,c\}$ is an independent set whose removal does not disconnect G, contradicting Lemma 2.

Case 2. $G[V'-\{z\}]$ contains two connected components C_1 and C_2. Clearly each C_i is nontrivial. Also, let us assume, without loss of generality, that $|N(z)\cap C_1|=1$. Let w be a vertex of C_1 such that $C_1-\{w\}$ is connected. Note that C_1 contains at least two such vertices w; one of them is not adjacent to z. We now examine the following situations depending on whether C_1 contains one, two or three vertices of $A\cup B$.

a) $|(A\cup B)\cap V(C_1)|=1$. Without loss of generality, let $\{x_2\}\subset V(C_1)$. Here we only suppose for w to be different from x_2. Observe that the subgraph induced by $V(G)-\{z,w\}$ is connected. Now since x has a neighbor in C_2, we obtain that $V(G)-\{z,w,x_1\}$ is a connected 2-dominating set of G of cardinality $(n-3)$.

b) $|V(C_1) \cap A| = 1$ and $|V(C_1) \cap B| = 1$. Without loss of generality, let $\{x_2, y_2\} \subset V(C_1)$. Suppose that $wz \notin E(G)$ and let $w' \in V(C_2)$ such that $C_2 - \{w'\}$ is connected. Then $V(G) - \{w, w', z\}$ is a connected 2-dominating set of G of cardinality $(n-3)$.

c) $A \subset V(C_1)$ and $B \subset V(C_2)$. Suppose that $wz \notin E(G)$ and let $w' \in V(C_2)$ such that $C_2 - \{w'\}$ is connected. Clearly $V(G) - \{w, w', x\}$ is a connected 2-dominating set of G of cardinality $(n-3)$.

d) $|(A \cup B) \cap V(C_1)| = 3$. Without loss of generality, suppose that $\{x_3, y_1, y_2\} \subset V(C_1)$ and let $x_2' \in N(x_2) \cap V(C_2)$. Then according to the Lemma 4, $V(G) - \{x_2, x_2'\}$ is a $\gamma_c^2(G)$-set. Also observe that z is adjacent to at most one of x_2 and x_2', for otherwise x_2' would be a cut vertex in C_2 and so in G, which is excluded. Now it is evident that $V(G) - \{x_2, x_2', x_1\}$ is a connected 2-dominating set of G of cardinality $(n-3)$.

Clearly, each of the previous situations leads to a contradiction. Therefore we conclude that $V'' = \emptyset$, implying that G is isomorphic to $K_{3,3}$.

From now on we can assume that the neighborhood of every vertex of G contains at least two adjacent vertices, and so G is a claw free graph. Let x be a vertex of G with $N_G(x) = \{x_1, x_2, x_3\}$ and $x_1x_2 \in E(G)$. By Lemma 4, $V(G) - \{x, x_3\} = V'$ is a $\gamma_2^c(G)$-set. Let $V'' = V' - N_G(\{x, x_3\})$. We shall show that $V'' = \emptyset$. Assume to the contrary that $V'' \neq \emptyset$ and let z be any vertex of V''. Then z is a cut vertex in $G[V']$, for otherwise $V(G) - \{x, x_3, z\}$ is a connected 2-dominating set of G of cardinality $(n-3)$, a contradiction. Note that since $V'' \neq \emptyset$, we have $N_G(\{x, x_3\}) \neq \{x_1, x_2\}$. Also, all neighbors of x_1 and x_3 in V' do not belong to a same component of $G[V' - \{z\}]$, for otherwise z would be a cut vertex of G. On the other hand, since G is claw free, $G[V' - \{z\}]$ contains exactly two connected components C_1 and C_2. Without loss of generality, let $|N(z) \cap C_1| = 1$. We consider the following two cases:

f) $N_G(x_3) \cap \{x_1, x_2\} = \emptyset$. Let $N_G(x_3) = \{x, y_1, y_2\}$. Then $y_1y_2 \in E(G)$ since G is a claw free. Suppose that $\{x_1, x_2\} \subset V(C_1)$, and let w be a vertex of C_1 such that $C_1 - \{w\}$ is connected and $wz \notin E(G)$. Also let $w' \in V(C_2)$ such that $C_2 - \{w'\}$ is connected. Then $V(G) - \{w, w', x\}$ is a connected 2-dominating set of G of cardinality $(n-3)$, a contradiction.

g) $N_G(x_3) \cap \{x_1, x_2\} \neq \emptyset$. Since $V'' \neq \emptyset$, we have that $|N(x_3) \cap \{x_1, x_2\}| = 1$. Let $x_3x_2 \in E(G)$ and $N_G(x_3) = \{x, x_2, y\}$. Assume that y belongs to the component C_i, where $i = 1$ or 2. It follows that $\{x_1, x_2\} \subset V(C_{3-i})$. Note that if $y \in V(C_1)$, then $yz \notin E(G)$, for otherwise y would be a cut vertex in C_1 and so in G. Now let w' be a vertex in the component that contains y such that $w'z \in E(G)$ and $w' \neq y$. By Lemma 4, $V(G) - \{w', z\}$ is a $\gamma_2^c(G)$-set and hence $V(G) - \{w', z, x\}$ is a connected 2-dominating set of G of cardinality $(n-3)$, a contradiction.

According to the previous situations, we conclude that $V'' = \emptyset$. Since G has an even order $n = 4$ or $n = 6$, we deduce that G is isomorphic to K_4 or the complement of C_6, respectively. □

4 4-regular claw-free graphs with $\gamma_3^c(G) = n - 2$

In this section we consider connected 4-regular claw-free graphs, where we give a characterization of such graphs G with $\gamma_3^c(G) = n-2$. Note that by Lemma 1, there are no connected 4-regular graphs such that $\gamma_2^c(G) = n - 2$.

Theorem 7 *Let G be a connected 4-regular claw-free graph of order n. Then $\gamma_3^c(G) = n - 2$ if and only if G is isomorphic to K_5 or $K_{2,2,2}$.*

Proof. It is a simple matter to check that if $G \in \{K_5, K_{2,2,2}\}$, then $\gamma_3^c(G) = n-2$. To prove the necessity, let G be a connected 4-regular claw-free graph such that $\gamma_3^c(G) = n-2$. We first assume that G contains a vertex whose open neighborhood induces a disconnected subgraph. Let x be such a vertex with $N_G(x) = \{x_1, x_2, x_3, x_4\}$. Then by Lemma 4, $V' = V(G) - \{x, x_4\}$ is a $\gamma_3^c(G)$-set. Let $V'' = V' - N_G(\{x, x_4\})$. We shall show that $V'' = \emptyset$. Suppose to the contrary that $V'' \neq \emptyset$ and let z be any vertex of V''. Clearly z is a cut vertex in $G[V']$, for otherwise $V(G) - \{x, x_4, z\}$ is a connected 3-dominating set of G of cardinality $(n-3)$, a contradiction. Also, since G is a claw-free graph, the removal of z in $G[V']$ provides two nontrivial connected components, say C_1 and C_2. Observe that the subgraph induced by $N_G(\{x, x_4\})$ is not connected, for otherwise by Lemma 5, z would be adjacent to x or x_4, which is impossible. If all vertices of $N_G(\{x, x_4\})$ belong to a same component in $G[V' - \{z\}]$, say C_1, then let t be a vertex of C_2 such that $C_2 - \{t\}$ is connected. Note that if z has a unique neighbor in C_2, then t is chosen so that $tz \notin E(G)$. Now it is clear that $V(G) - \{x, x_4, t\}$ is a connected 3-dominating set of G of cardinality $(n-3)$, a contradiction. Hence each of C_1 and C_2 contains at least one vertex of $N_G(\{x, x_4\})$. Since is a claw-free graph, we have two cases depending on whether $G[N(x)] = K_3 \cup K_1$ or $K_2 \cup K_2$.

Case 1. $G[N(x)] = K_3 \cup K_1$. There are two situations depending on whether x_4 belongs to K_3 or not. So let us assume that $\{x_1, x_2, x_3\}$ induces a K_3. In this case, let $N_G(x_4) = \{x, y_1, y_2, y_3\}$. Then $\{y_1, y_2, y_3\}$ induces a K_3, since G is claw-free. Without loss of generality, we can assume that $\{x_1, x_2, x_3\} \subset V(C_1)$ and hence $\{y_1, y_2, y_3\} \subset V(C_2)$. Now let $w \in V(C_1)$ and $w' \in V(C_2)$ such that $C_1 - \{w\}$ and $C_2 - \{w'\}$ are connected. Note that if z has a unique neighbor in C_1, then w is chosen so that $wz \notin E(G)$. Likewise for w' in C_2. Then $V(G) - \{w, w', x\}$ is a connected 3-dominating set of G of size $(n-3)$, a contradiction.

Now suppose that $K_3 = \{x_2, x_3, x_4\}$ and let y_1 be the fourth neighbor of x_4. Since $x_2x_3 \in E(G)$, we may assume that $\{x_2, x_3\} \subset V(C_1)$ and so C_2 contains at least one of x_1 and y_1.

a) $\{x_1, y_1\} \subset V(C_2)$. Suppose that z has at least two neighbors in C_1. Then $zx_1 \notin E(G)$ for otherwise the closed neighborhood of x_1 induces a claw. Likewise $zy_1 \notin E(G)$. In this case let w' be any neighbor of z in C_2. By Lemma 4, $V(G) - \{z, w'\}$ is a $\gamma_3^c(G)$-set and so $V(G) - \{z, w', x\}$ is a connected 3-dominating set of G of size less than $n - 2$, a contradiction. Now suppose that z has exactly one neighbor in C_1. Clearly the neighborhood of z in C_2 induces a clique K_3. Let w' be any vertex of C_2 adjacent to z and let w be a vertex of C_1 such that $wz \notin E(G)$ and $C_1 - \{w\}$ is connected. Using Lemma 4, it is clear that $V(G) - \{z, w', w\}$ is a connected 3-dominating set of G of size less than $n-2$, a contradiction too.

b) Now suppose $y_1 \in V(C_2)$ and $x_1 \in V(C_1)$. Note that if z has a unique neighbor in C_2, then such a vertex is different from y_1, for otherwise the closed neighborhood of y_1 induces a claw. So we can assume that z has a neighbor in C_2, say w, such that $w \neq y_1$. By Lemma 4, $V(G) - \{z, w\}$ is a $\gamma_3^c(G)$-set and hence $V(G) - \{z, w, x\}$ is a connected 3-dominating set of G of size less than $n-2$, a contradiction. The remaining case $y_1 \in V(C_1)$ and $x_1 \in V(C_2)$ can be seen by using a similar argument to that used for the previous situation.

Case 2. $G[N(x)] = K_2 \cup K_2$. Without loss of generality, let $x_1x_2 \in E(G)$ and $x_3x_4 \in E(G)$. We also let y_1, y_2 be the third and fourth neighbors of x_4. It follows that $y_1y_2 \in E(G)$, for otherwise $\{x, x_4, y_1, y_2\}$ induces a claw. Also, without loss of generality, we can assume that $\{x_1, x_2\} \subset V(C_1)$. We consider the following situations.

c) $\{y_1, y_2\} \subset V(C_2)$. Clearly x_3 belongs to either C_1 or C_2 and so let us choose the component that contains exactly two vertices of $N_G(\{x, x_4\})$. Note that such a component contains at least four vertices. Let now w be a neighbor of z in the selected component. By Lemma 4, $V(G) - \{z, w\} = S$ is a $\gamma_3^c(G)$-set. Now if C_1 is the selected component, then x_3 is in C_2 and so $S - \{x_4\}$ is a connected 3-dominating set of G of size $(n-3)$. If C_2 is the selected component, then x_3 is in C_1 and so $S - \{x\}$ is a connected 3-dominating set of G of size $(n-3)$. In each case we have a contradiction.

d) $\{y_1, y_2\} \subset V(C_1)$. It follows that x_3 belongs to C_2. Note that if z has a unique neighbor in C_2, then such a vertex is different from x_3 for otherwise the closed neighborhood of x_3 induces a claw. So we can assume that z has a neighbor in C_2, say w, such that $w \neq x_3$. By Lemma 4, $V(G) - \{z, w\}$ is a $\gamma_3^c(G)$-set and therefore $V(G) - \{z, w, x_4\}$ is a connected 3-dominating set of G of size $(n-3)$, a contradiction.

According to the previous cases we conclude that $V'' = \emptyset$. Using the fact that $G[N(x)]$ is not connected and up to isomorphism, one can see that the only 4-regular claw-free graph is the graph with 8 vertices $x, x_1, x_2, x_3, x_4, y_1, y_2, y_3$ such that each of $\{x_2, x_3, x_4\}$ and $\{y_1, y_2, y_3\}$ induces a clique K_3, x_1y_1, x_2y_2, x_3y_3 and $x_4y_i \in E(G)$ for every i. But then $V(G) - \{y_1, y_2, x\}$ is a connected 3-dominating set of G of size $(n-3)$, a contradiction.

From now on, we can assume that the subgraph induced by the neighborhood of every vertex is connected. Let x be any vertex of G with $N_G(x) = \{x_1, x_2, x_3, x_4\}$. Recall that by Lemma 4, $V' = V(G) - \{x, x_4\}$ is a $\gamma_3^c(G)$-set. Clearly the set $V'' = V' - N_G(\{x, x_4\})$ is empty, for otherwise every vertex of V'' will be a cut vertex in $G[V']$, contradicting the fact that the open neighborhood of every vertex induces a connected subgraph. Also $G[N(x)]$ contains a path P_4 not necessarily induced, say x_1-x_2-x_3-x_4. Let $\{y_1, y_2\} = N(x_4) - (\{x, x_3\})$. Suppose that $y_1, y_2 \notin N(x)$. Since G is claw-free, $y_1y_2 \in E(G)$. On the other hand, the fact $G[N(x_4)]$ is connected implies that one of y_1 and y_2 is adjacent to x_3, say $y_1x_3 \in E(G)$. It follows that x_1y_1, x_1y_2 and $x_2y_2 \in E(G)$. But then $\{x_1, x_2, x_3, x_4\}$ is a connected 3-dominating set of G of size $(n-3)$, a contradiction. Thus at least one of y_1 and y_2 belongs to $N(x)$, say y_2. If $y_2 = x_2$, then it is easy to see that G is not 4-regular. Thus $y_2 = x_1$, and so $N(y_1) = N(x)$. Therefore $G = K_{2,2,2}$. Finally if $y_1, y_2 \in N(x)$, then $G = K_5$. □

References

[1] M. Chellali, O. Favaron, A. Hansberg and L. Volkmann, k-Domination and k-independence in graphs: a survey, *Graphs Combin.* 28 (2012), 1–55.

[2] J. F. Fink and M. S. Jacobson, n-domination in graphs, *Graph Theory with Applications to Algorithms and Computer Science*, Wiley, New York (1985), 283–300.

[3] L. Volkmann, Connected p-domination in graphs, *Util. Math.* 79 (2009), 81–90.

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