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# On $k$-independence critical graphs

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**Abstract**

Let $k$ be a positive integer and $G = (V(G), E(G))$ a graph. A subset $S$ of $V(G)$ is a $k$-independent set of $G$ if the subgraph induced by the vertices of $S$ has maximum degree at most $k-1$. The maximum cardinality of a $k$-independent set of $G$ is the $k$-independence number $\beta_k(G)$. In this paper, we study the properties of graphs for which the $k$-independence number changes whenever an edge or vertex is removed or an edge is added.

## 1 Introduction and Terminology

We consider finite, undirected, and simple graphs $G$ with vertex set $V = V(G)$ and edge set $E = E(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N_G[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v$ of $G$, denoted by $d_G(v)$, is the size of its open neighborhood. Specifically, for a vertex $v$ in a rooted tree $T$, we denote by $C(v)$ and $D(v)$ the set of

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children and descendants, respectively, of $v$, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at $v$ is the subtree of $T$ induced by $D[v]$, and is denoted by $T_v$. For notation and graph theory terminology in general we follow [5].

In [2] and [3] Fink and Jacobson generalized the concept of independent sets. A subset $S$ of $V$ is *$k$-independent* if the maximum degree of the subgraph induced by the vertices of $S$ is less than or equal to $k-1$. A $k$-independent set $S$ of $G$ is maximal if for every vertex $v \in V \setminus S$, $S \cup \{v\}$ is not $k$-independent. The *$k$-independence number* $\beta_k(G)$ is the maximum cardinality of a $k$-independent set of $G$. Notice that for $k = 1$, the 1-independent sets are the classical independents sets. If $S$ is a $k$-independent set of $G$ of size $\beta_k(G)$, then we call $S$ a $\beta_k(G)$-set. A vertex $u$ of a $k$-independent set $S$ is said to be *full* if it has exactly $k-1$ neighbors in $S$. A vertex not in $S$ with at least $k$ neighbors in $S$ is said to be *$k$-dominated* by $S$. For a survey of results on $k$-independence in graphs see [1].

Much has been written about graphs where a parameter increases or decreases whenever an edge or vertex is removed or added. For references on the changing and unchanging concept in graph parameters see for example [5, 6].

Gunther et al. [4] studied graphs whose independence number is unaffected by single edge addition or deletion. In this paper, we study properties of graphs for which the $k$-independence number changes whenever an edge or vertex is removed or an edge is added.

## 2 Preliminary results

We consider three graph modifications, namely, adding an edge, and removing a vertex or an edge. Clearly, removing a vertex from $G$ or adding an edge to $G$ cannot increase the $k$-independence number. On the other hand, removing an edge from $G$ cannot decrease the $k$-independence number. Hence we have the following observations.

**Observation 2.1.** *Let $G$ be a graph and $k$ a positive integer. Then*

*a) For every vertex $v \in V(G)$, $\beta_k(G) \geq \beta_k(G - v)$.*

*b) For every edge $e \in E(G)$, $\beta_k(G - e) \geq \beta_k(G)$.*

**Observation 2.2.** *Let $G$ be a graph. If $uv \in E(G)$ and $\beta_k(G - uv) > \beta_k(G)$, then $u$ and $v$ are in every $\beta_k(G - uv)$-set.*

Our next result shows that under any of these three graph modifications, the $k$-independence number changes by at most 1.

**Proposition 2.3.** *Let $G$ be a graph and $k \geq 1$ an integer. Then* *a) For every vertex $v \in V(G)$, $\beta_k(G) - 1 \leq \beta_k(G - v) \leq \beta_k(G)$.*

*b) For every edge $e \in E(G)$, $\beta_k(G) \leq \beta_k(G - e) \leq \beta_k(G) + 1$.*

*c) For every edge $e \in E(\overline{G})$, $\beta_k(G) - 1 \leq \beta_k(G + e) \leq \beta_k(G)$.*

*Proof.* a) Let $v \in V(G)$. Observation 2.1-a gives the upper bound. Let $S$ be a $\beta_k(G)$-set. If $v \notin S$, then $S$ is a $k$-independent set for $G - v$, and so $\beta_k(G - v) \geq \beta_k(G) \geq \beta_k(G) - 1$. Thus we assume that $v \in S$. Then $S - \{v\}$ is a $k$-independent set for $G - v$, and so $\beta_k(G - v) \geq \beta_k(G) - 1$. b) Observation 2.1-b gives the lower bound. Suppose that $\beta_k(G - uv) \geq \beta_k(G) + 2$ for some edge $uv \in E(G)$. Let $S$ be a $\beta_k(G - uv)$-set. By Observation 2.2, both $u$ and $v$ are in $S$. But $S - \{u\}$ is a $k$-independent set of $G$, implying that $\beta_k(G) \geq |S| - 1 \geq \beta_k(G - uv) - 1$, a contradiction. c) Observation 2.1-b gives the upper bound. Suppose that $\beta_k(G + uv) \leq \beta_k(G) - 2$ for some edge $uv \in E(\overline{G})$. By Observation 2.2, both $u$ and $v$ are in every $\beta_k(G)$-set $S$. But $S - \{u\}$ is a $k$-independent set of $G + uv$ implying that $\beta_k(G + uv) \geq |S| - 1 \geq \beta_k(G) - 1$, a contradiction. □

Let $k \geq 1$ be an integer. Then from Observation 2.1, we define a graph $G$ to be :

- a *$\beta_k$-vertex critical graph* if for every vertex $v \in V(G)$, $\beta_k(G - v) < \beta_k(G)$. It follows from Proposition 2.3-a, that in a $\beta_k$-vertex critical graph $G$, $\beta_k(G - v) = \beta_k(G) - 1$ for every vertex $v \in V(G)$.
- a *$\beta_k^-$-edge critical graph* if for every edge $e \in E(G)$, $\beta_k(G - e) > \beta_k(G)$. It follows from Proposition 2.3-b, that in a $\beta_k^-$-edge critical graph $G$, $\beta_k(G - e) = \beta_k(G) + 1$ for every edge $e \in E(G)$. Also if $G$ is a $\beta_k^-$-edge critical graph of order $n$, then clearly $\beta_k(G) < n$.
- a *$\beta_k^+$-edge critical graph* if for every edge $e \in E(\overline{G})$, $\beta_k(G + e) < \beta_k(G)$. It follows from Proposition 2.3-c, that in a $\beta_k^+$-edge critical graph $G$, $\beta_k(G + e) = \beta_k(G) - 1$ for every edge $e \in E(\overline{G})$.

Now we characterize all $\beta_k$-vertex critical graphs. We recall that a graph $G$ has $\beta_k(G) = |V(G)|$ if and only if $d_G(v) \leq k - 1$ for every $v \in V(G)$.

**Theorem 2.4.** *A graph $G$ is $\beta_k$-vertex critical if and only if $\beta_k(G) = |V(G)|$.*

*Proof.* Let $G$ be a $\beta_k$-vertex critical graph. Assume that $\beta_k(G) < |V(G)|$ and let $S$ be a $\beta_k(G)$-set. Then for every vertex $v \notin S$, $S$ is a $k$-independent set for $G - v$. Hence for every $v \notin S$, $\beta_k(G - v) \geq \beta_k(G)$, a contradiction. Thus $\beta_k(G) = |V(G)|$. Conversely, assume that $\beta_k(G) = |V(G)|$. For every vertex $v \in V(G)$, $\beta_k(G - v) = |V(G - v)| = |V(G)| - 1 < |V(G)| = \beta_k(G)$. Therefore $G$ is $\beta_k$-vertex critical. □

## 3 $\beta_k^-$-edge critical graphs

Clearly if $G$ is a graph with $\beta_k(G) = |V(G)|$, then the deletion of any edge does not affect the $k$-independence number. Hence we may assume for the next that $\beta_k(G) < |V(G)|$ for every graph $G$. Obviously, such graphs have at least one vertex of degree at least $k$. We now give the following lemma.

**Lemma 3.1.** *If $uv$ is an edge in a $\beta_k^-$-edge critical graph $G$, then either $d_G(u) \geq k$ or $d_G(v) \geq k$.*

*Proof.* Let $G$ be a $\beta_k^-$-edge critical graph. Let $uv \in E(G)$ and $S$ be a $\beta_k(G-uv)$-set. By Observation 2.2, $u, v \in S$. Now if $\max\{d_G(u), d_G(v)\} < k$, then $S$ is a $k$-independent set for $G$, implying that $\beta_k(G) \geq \beta_k(G-uv)$, a contradiction. □

As an immediate consequence of Lemma 3.1, we obtain the following.

**Corollary 3.2.** *If $G$ is a $\beta_k^-$-edge critical graph, then the set of vertices of degree less than $k$ is independent.*

Next we give a necessary and sufficient condition for $\beta_k^-$-edge critical graphs.

**Theorem 3.3.** *A graph $G$ is $\beta_k^-$-edge critical if and only if for every edge $uv \in E(G)$ at least one of the following is true:*

*i) there exists a $\beta_k(G)$-set $S$, where $u \in S$ and $v \notin S$ and $v$ has exactly $k$ neighbors in $S$ and no vertex in $N(v) \cap (S - \{u\})$ is full.*

*ii) there exists a $\beta_k(G)$-set $S$, where $u, v \in S$, $u$ is a full vertex, $deg(u) \geq k$, and there is a vertex $x \in N(u) \cap (V - S)$ such that $x$ is not $k$-dominated by $S$ and $u$ is the unique full vertex of $S$ adjacent to $x$.*

*Proof.* Let $G$ be a $\beta_k^-$-edge critical graph and $u, v$ any two adjacent vertices. Assume to the contrary that neither (i) nor (ii) is verified. By Observation 2.2, $u$ and $v$ belong to every $\beta_k(G-uv)$-set. Let $S$ be any $\beta_k(G-uv)$-set. Since $\beta_k(G-uv) = \beta_k(G)+1$, there exists a vertex $w \in S$ such that $S' = S - \{w\}$ is a $\beta_k(G)$-set. Clearly $w = u$ or $v$ or one of their neighbors in $S$. Note that if $w \in \{u, v\}$, then no neighbor of $w$ in $S'$ is full for otherwise $S' \cup \{w\} = S$ is not a $k$-independent set for $G - uv$, a contradiction. Also it clear that at least one of $u$ and $v$, say $v$, is full in $S$ for otherwise $S$ would be a $k$-independent set for $G$, a contradiction too. Thus $v$ has exactly $k$ neighbors in $S'$ and no vertex in $N(v) \cap (S' - \{u\})$ is full. Hence condition (i) holds, a contradiction. Thus we assume that $w \notin \{u, v\}$, and without loss of generality, $w$ is a neighbor of $v$ in $S$. It follows that $v$ is a full vertex in $S$ and hence in $S'$, implying that $v$ has degree at least $k$, $w$ has less than $k$ neighbors in $S'$ and no neighbor of $w$ in $S'$ besides $v$ is full but then condition (ii) holds, a contradiction too.

Conversely, suppose that for every edge $uv \in E(G)$ at least one of the conditions (i) and (ii) holds. Clearly if the condition (i) holds, then $S \cup \{v\}$ is a $k$-independent set of $G - uv$, and so $\beta_k(G - uv) \geq \beta_k(G) + 1$. The equality follows from Proposition 2.3-b, implying that $G$ is $\beta_k^-$-edge critical graph. Thus assume that the condition (ii) holds. Let $x \in N(u) \cap (V - S)$ be a vertex such that $x$ is not $k$-dominated by $S$. Then $S \cup \{x\}$ is a $k$-independent set of $G - uv$, and so $\beta_k(G - uv) \geq \beta_k(G) + 1$. The equality follows from Proposition 2.3-b, implying that $G$ is $\beta_k^-$-edge critical graph. □

In the following we characterize all $\beta_k^-$-edge critical trees.

**Theorem 3.4.** *Let $k$ be a positive integer. Then a tree $T$ is $\beta_k^-$-edge critical if and only if $T = K_{1,k}$.*

*Proof.* Clearly a star $K_{1,k}$ is a $\beta_k^-$-edge critical tree. Assume that $T$ is a $\beta_k^-$-edge critical tree. If $\text{diam}(T) = 1$, then $T = P_2$ and obviously $k = 1$ since we assumed that $\beta_k(G) < |V(G)|$. Thus we assume that $\text{diam}(T) \geq 2$. Let $x, y$ be two leaves such that $d(x, y) = \text{diam}(T)$, and let $x$-$x_1$-$x_2$...-$x_t$ be a diametrical path with $x_t = y$. By Lemma 3.1, $d_T(x_1) \geq k$. Assume that $d_T(x_1) \geq k + 1$. Then there are at least $k$ leaves adjacent to $x_1$. Let $S_1$ be a $\beta_k(T - x_1x_2)$-set. Since $\beta_k(T - e) > \beta_k(T)$, for every edge $e$ of $T$, by Observation 2.2, $x_1, x_2 \in S_1$. It follows that some leaf $w$ adjacent to $x_1$ does not belong to $S_1$ but then $(S_1 - \{x_1\}) \cup \{w\}$ is a $\beta_k(T - x_1x_2)$-set that does not contain $x_1$, contradicting Observation 2.2. Therefore $d_T(x_1) = k$. We next show that $t = 2$. Assume that $t \geq 3$ and let $S_2$ be a $\beta_k(G - x_2x_3)$-set. By Observation 2.2, $x_2, x_3 \in S_2$. If $x_1 \in S_2$, then there is a leaf $w \in N(x_1)$ such that $w \notin S_2$. Clearly, then $(S_2 - \{x_2\}) \cup \{w\}$ is a $\beta_k(G - x_2x_3)$-set that does not contain $x_2$, a contradiction with Observation 2.2. Thus we assume that $x_1 \notin S_2$ but then $(S_2 - \{x_2\}) \cup \{x_1\}$ is a $\beta_k(G - x_2x_3)$-set, a contradiction too. Hence $t = 2$. Consequently, $T = K_{1,k}$. □

We call a graph $G$, $t$-$\beta_k^-$-edge critical if $G$ is $\beta_k^-$-edge critical and $\beta_k(G) = t$.

**Proposition 3.5.** *A graph $G$ of order $n$ is $(n-1)$-$\beta_k^-$-edge critical if and only if $G$ has a vertex $x$ such that $d_G(x) = k$ and $\Delta(G - x) < k$.*

*Proof.* Assume that $G$ is a $(n-1)$-$\beta_k^-$-edge critical graph of order $n$. Let $S$ be a $\beta_k(G)$-set. Since $|S| = n - 1$, let $v$ be the unique vertex not in $S$. If $d_G(v) > k$, then clearly removing any edge incident with $v$ does not increase the $k$-independence number, a contradiction. Hence $d_G(v) \leq k$. Now if $d_G(v) = k$, then we are done. Thus assume that $d_G(v) \leq k - 1$. Since $S \cup \{v\}$ is not $k$-independent, $v$ is adjacent to a full vertex in $S$, say $x$. Thus $d_G(x) = k$ and since $\beta_k(G - vx) = n > \beta_k(G) = n - 1$, it follows that $\Delta(G - x) < k$. The converse is obvious. □

Note that $\beta_1(C_n) = \lfloor \frac{n}{2} \rfloor$ and $\beta_1(P_n) = \lceil \frac{n}{2} \rceil$.

**Lemma 3.6.** *Let $n \equiv r$ (mod 3), where $r \in \{0, 1, 2\}$. Then:*
*(i) $\beta_2(P_n) = 2\lfloor \frac{n}{3} \rfloor + r$.*
*(ii) $\beta_2(C_n) = 2\lfloor \frac{n}{3} \rfloor + 1$ if $r = 2$, and $\beta_2(C_n) = 2\lfloor \frac{n}{3} \rfloor$ if $r \neq 2$.*

*Proof.* We prove (i). Let $V(P_n) = \{v_1, v_2, ..., v_n\}$, where $v_i$ is adjacent to $v_{i+1}$ for $i = 1, 2, ..., n-1$. For $n \geq 2$, let $S$ be a $\beta_2(P_{n+3})$-set. Then $|S \cap \{v_{n+1}, v_{n+2}, v_{n+3}\}| \leq 2$ and $S \cap \{v_1, v_2, ..., v_n\}$ is a 2-independent set for $P_n$. This implies that $\beta_2(P_n) \geq |S| - 2 = \beta_2(P_{n+3}) - 2$. Also if $D$ is a $\beta_2(P_n)$-set, then $D \cup \{v_{n+2}, v_{n+3}\}$ is a 2-independent set for $P_{n+3}$, implying that $\beta_2(P_{n+3}) \geq \beta_2(P_n) + 2$. We deduce that $\beta_2(P_{n+3}) = 2 + \beta_2(P_n)$. Now using the values $\beta_2(P_2) = \beta_2(P_3) = 2$ and $\beta_2(P_4) = 3$, we obtain the result. (ii) is similarly verified. □

**Proposition 3.7.** *A cycle $C_n$ is $\beta_k^-$-edge critical if and only either $k = 1$ and $n$ is odd, or $k = 2$ and $3 \nmid n$.*

*Proof.* Clearly if $k \geq 3$, then clearly $\beta_k(C_n) = n$ and $C_n$ is not $\beta_k^-$-edge critical. Thus $k \in \{1, 2\}$. If $k = 1$, then $\beta_1(P_n) = \lceil \frac{n}{2} \rceil > \beta_1(C_n) = \lfloor \frac{n}{2} \rfloor$ if and only if $n$ is odd. If $k = 2$, then Lemma 3.6 implies the result. □

## 4 $\beta_k^+$-edge critical graphs

We begin by mentioning that graphs $G$ with $\Delta(G) \leq k - 2$ are not $\beta_k^+$-edge critical. Hence it is obvious to only consider graphs with maximum degree at least $k - 1$. We also note that complete graphs are not $\beta_k^+$-edge critical. Next we give a necessary and sufficient condition for $\beta_k^+$-edge critical graphs.

**Theorem 4.1.** *A graph $G$ is $\beta_k^+$-edge critical if and only if any pair of nonadjacent vertices $u$ and $v$ are in every $\beta_k(G)$-set $S$ and at least one of $u$ and $v$ is a full vertex in $S$.*

*Proof.* Let $G$ be a $\beta_k^+$-edge critical graph and $u, v$ any two nonadjacent vertices. If there exist a $\beta_k(G)$-set that does not contain both $u$ and $v$, then such a set is a $k$-independent set of $G + uv$, implying that $\beta_k(G + uv) \geq \beta_k(G)$, a contradiction. Thus $u, v$ are contained in every every $\beta_k(G)$-set. Now assume that neither $u$ nor $v$ is full in a $\beta_k(G)$-set $S$, then $S$ remains a $k$-independent set of $G + uv$, and so $\beta_k(G + uv) \geq |S| = \beta_k(G)$, a contradiction too. Conversely, let $e = uv$ be any edge of $E(\overline{G})$ and $S$ a $\beta_k(G+e)$-set. Assume that $\beta_k(G+e) \geq \beta_k(G)$. Then by Proposition 2.3-c, $\beta_k(G + e) = \beta_k(G)$. Hence $S$ is a $\beta_k(G)$-set for which neither $u$ nor $v$ is full in $S$, a contradiction. This achieves the proof. □

The following corollary is an immediate consequence to Theorem 4.1.

**Corollary 4.2.** *If $G$ is a $\beta_k^+$-edge critical graph, then for every $\beta_k(G)$-set $S$, the subgraph induced by $V - S$ is complete and every vertex of $V - S$ is adjacent to all vertices of $S$.*

We call a graph $G$, $t$-$\beta_k^+$-edge critical if $G$ is $\beta_k^+$-edge critical and $\beta_k(G) = t$. Let $\mathcal{G}_k$ be the class of all graphs $G$ such that $G$ is not a complete graph, $\Delta(G) = k - 1$ and the subgraph induced by the vertices of degree less than $k - 1$ is complete.

**Proposition 4.3.** *A graph $G$ of order $n$ is $n$-$\beta_k^+$-edge critical if and only if $G \in \mathcal{G}_k$.*

*Proof.* Assume that $G$ is an $n$-$\beta_k^+$-edge critical graph of order $n$. It is obvious that $\Delta(G) \leq k - 1$, since $\beta_k(G) = n$. Also since $G \neq K_n$, let $x, y$ be two non-adjacent vertices of $G$. If $\Delta(G) < k - 1$, then $V(G)$ is a $\beta_k(G + xy)$-set, a contradiction to the fact that $G$ is $\beta_k^+$-edge critical. Thus $\Delta(G) = k - 1$. Likewise if $G$ has two non-adjacent vertices, each of degree less than $k - 1$, then adding an edge between these two vertices does not decrease the $k$-independence number of $G$. Hence the subgraph induced by the vertices of degree less than $k - 1$ is complete. Conversely, assume that $G \in \mathcal{G}_k$. Clearly $\beta_k(G) = n$ and any added edge has at least an incident vertex of maximum degree. Hence $\Delta(G + xy) = k$, and so $\beta_k(G + e) = n - 1$ for every $e \notin E(G)$. Therefore $G$ is $n$-$\beta_k^+$-edge critical. □

**Theorem 4.4.** *A graph $G$ of order $n$ is $\beta_k^+$-edge critical if and only if one of the following holds:*
*(1) $G \in \mathcal{G}_k$, or*
*(2) $G$ is obtained from a graph $H \in \mathcal{G}_k$ by joining every vertex of $K_{n-\beta_k(G)}$ to all vertices of $H$.*

*Proof.* Assume that $G$ is $\beta_k^+$-edge critical. If $\beta_k(G) = n$, then by Proposition 4.3, $G$ belongs to $\mathcal{G}_k$. Hence we may assume that $\beta_k(G) < n$. It follows that $k < n$. Now let $S$ be a $\beta_k(G)$-set. Note that since any subset of $k$ vertices of $G$ is a $k$-independent set, we deduce that $|S| \geq k$. Also by Corollary 4.2, $G[V(G) - S]$ is complete (isomorphic to $K_{n-\beta_k(G)}$), and every vertex of $V(G) - S$ is adjacent to all vertices of $S$. Hence $d_G(x) \geq |S| \geq k$ for each vertex $x \in V(G) - S$. Since $G$ is not complete, there are two non-adjacent vertices in $G[S]$. Obviously, $\beta_k(G[S]) = |S|$. Furthermore, since $G$ is $\beta_k^+$-edge critical, $\beta_k(G[S] + xy) < |S|$ for every pair of non-adjacent vertices $x, y$ in $S$. Consequently $G[S]$ belongs to $\mathcal{G}_k$, and so the result follows. Conversely, if $G \in \mathcal{G}_k$, then by Proposition 4.3, $G$ is $\beta_k^+$-edge critical. Thus assume that $G$ is obtained from a graph $H \in \mathcal{G}_k$ by joining any vertex of $K_{n-\beta_k(G)}$ to all vertices of $H$. It is clear that $V(H)$ is a $\beta_k(G)$-set. Note that every vertex not in $V(H)$ has degree at least $k$. Now let $x, y$ be two non-adjacent vertices in $G$. Then $x, y \in V(H)$ and since $H \in \mathcal{G}_k$, $\beta_k(H + xy) < |V(H)| = \beta_k(G)$. This implies that at least one of $x$ and $y$ has exactly $k - 1$ neighbors in $H$. Now if $\beta_k(G + xy) = \beta_k(G) = |V(H)|$, then there is some $\beta_k(G + xy)$-set $D$ such that $x \notin D$ and $D$ contains a vertex, say $z \notin V(H)$. Recall that $z$ is adjacent to all vertices of $G$. So $|D| = k$. It follows that $V(H) = \{x\} \cup D - \{z\}$. Since $H$ belongs to $\mathcal{G}_k$ and $H \neq K_k$, each of $x$ and $y$ has degree less than $k - 1$, but this contradicts the fact that the set of vertices of degree less than $k - 1$ induces a complete graph. It follows that $\beta_k(G + xy) < \beta_k(G)$ and hence $G$ is $\beta_k^+$-edge critical. □

We next give for some particular classes of graphs a complete characterization of $\beta_k^+$-edge critical graphs. Let $\mathcal{G}_k^*$ be a subclass of $\mathcal{G}_k$ consisting of graphs with girth at least four such that every vertex has degree $k-1$ except possibly one or two adjacent vertices.

**Theorem 4.5.** *A connected graph $G$ of order $n \geq 3$ and girth $g(G) \geq 4$ is $\beta_k^+$-edge critical if and only $G \in \mathcal{G}_k^*$ or $k = 1$ and $G$ is a star.*

*Proof.* By Theorem 4.4, every graph of $\mathcal{G}_k^*$ is $\beta_k^+$-edge critical. Also it is straightforward to see that a star is $\beta_1^+$-edge critical. Now let $G$ be a $\beta_k^+$-edge critical graph of order $n \geq 3$ and girth $g(G) \geq 4$. First assume that $G$ contains a cycle $C$. By Corollary 4.2 all vertices of this cycle belong to every $\beta_k(G)$-set $S$. If $V - S$ is non empty, then again by Corollary 4.2 any vertex of $V - S$ is adjacent to all $C$ and so $G$ would contain a triangle which contradicts the fact that $g(G) \geq 4$. Hence $V - S = \emptyset$, and so $\beta_k(G) = n$. By Proposition 4.3, $G \in \mathcal{G}_k$. Now let $A$ be the set of vertices of $G$ of degree less than $k-1$. By the definition of $\mathcal{G}_k$, $A$ induces a complete graph. But since $g(G) \geq 4$ it follows that $|A| \leq 2$, implying that $G \in \mathcal{G}_k^*$. Now assume that $G$ contains no cycle. Thus $G$ is a tree. Let $u$ and $v$ be leaves of $G$. By Theorem 4.1, both $u$ and $v$ are in every $\beta_k(G)$-set and at least one of $u$ and $v$ is full in every $\beta_k(G)$-set $S$ implying that $k \leq 2$. If $k = 2$, then since $G$ has at least two leaves, there is at least one edge in the subgraph induced by $S$. Moreover, the maximum degree of the subgraph induced by $S$ is one. But Corollary 4.2 implies that every vertex in $V - S$ must be adjacent to every vertex in $S$. Thus $V - S$ is empty or $k = 1$ and $S$ is an independent set (otherwise a cycle would be formed). By the connectivity of $G$, $V - S$ is not empty, so $k = 1$ and $S$ is an independent set. Furthermore, by Corollary 4.2, $V - S$ induces a complete graph and since $G$ is a tree, $|V - S| \leq 2$. But since every vertex in $V - S$ is adjacent to every vertex in $S$ and $G$ is a tree, $|V - S| = 1$. Hence $G$ is a star. $\square$

**Corollary 4.6.** *A tree $T$ of order $n \geq 3$ is $\beta_k^+$-edge critical if and only $k = 1$ and $T$ is a star.*

We remark that by Theorem 4.5 and since a cycle on three vertices is not $\beta_k^+$-edge critical it follows that cycles $C_n$ are $\beta_3^+$-edge critical. Let $H$ be a graph obtained from a star $K_{1,3}$ by adding one edge between two leaves of the star. In the following we characterize $\beta_k^+$-edge critical unicyclic graphs.

**Theorem 4.7.** *A connected unicyclic graph $G$ of order $n \geq 3$ is $\beta_k^+$-edge critical if and only if $k = 2$ and $G = H$ or $k = 3$ and $G = C_n$ with $n \geq 4$.*

*Proof.* It is easy to verify that $H$ is a $\beta_2^+$-edge critical graph and cycles $C_n$, with $n \geq 4$, are $\beta_3^+$-edge critical. Let $G$ be a $\beta_k^+$-edge critical unicycle graph. Denote the unique cycle of $G$ by $C$. First assume that $|V(C)| \geq 4$. Then by Theorem 4.5, $G$ belongs to $\mathcal{G}_k^*$. Now since $G$ has at most one vertex or two adjacent vertices of degree less than $k-1$, it follows that $G$ has no leaves and every vertex on $C$ has degree

$k-1$, that is $G = C_n$ and $k = 3$. Assume now that $|V(C)| = 3$. By the above remark $G \neq C_3$, that is $G$ has at least one leaf. Let $w$ be any leaf of $G$. Also $\beta_k(G) < |V|$, for otherwise $G \in \mathcal{G}_k$ (by Proposition 4.3) and so the subgraph induced by the vertices of degree less than $k-1$ is complete, which is impossible. On the other hand, since $\beta_k(G) < |V|$, Corollary 4.2 implies that no $\beta_k(T)$-set $S$ contains all vertices of $V(C)$. It follows that all support vertices of $G$ belong to $V(C)$. We suppose that $x \in V(C)$ is the support vertex adjacent to $w$. Clearly by Theorem 4.1, every leaf belongs to every $\beta_k(T)$-set. Also if a $\beta_k(T)$-set $S$ does not contain all non-neighbors of some leaf $y$, then by Corollary 4.2, every vertex of $V - S$ is adjacent to $y$, which is impossible. Hence every $\beta_k(T)$-set contains the non-neighborhood of any leaf. This implies that $G$ has a unique support. Let $x'$ and $x''$ be the two neighbors of $x$ on $C$ and let $L_x$ the set of leaves adjacent to $x$. Clearly $L_x \cup \{x'\}$ is a $\beta_1(G)$-set and so adding the edge $wx''$ does not decrease $\beta_1(G)$, a contradiction. It follows that $k \geq 2$. If $w' \neq w$ is a leaf adjacent to $x$, then let $G + ww'$. By Theorem 4.1, both $w$ and $w'$ are in every $\beta_k(T)$-set $S$ and at least one of them is a full vertex in $S$. But since $x \notin S$, we deduce that $k = 1$, a contradiction. Therefore $L_x = \{w\}$ implying that $G = H$ and so $k = 2$. □

Let $R$ be the graph formed by at least two triangles sharing the same vertex. Recall that a cactus graph is a graph in which every edge belongs to at most one cycle. Finally we characterize $\beta_k^+$-edge critical cactus graphs.

**Theorem 4.8.** *A cactus graph $G$ of order $n \geq 3$ is $\beta_k^+$-edge critical if and only if*

*(1) $k = 1$ and $T$ is a star, or*

*(2) $k = 2$ and $G = H$ or $k = 3$ and $G = C_n$ with $n \geq 4$, or*

*(3) $k = 2$ and $G = R$.*

*Proof.* Let $G$ be a $\beta_k^+$-edge critical cactus graph. If $G$ has no cycle, then $G$ is a tree and item (1) follows by Corollary 4.6. If $G$ has only one cycle, then item (2) follows by Theorem 4.7. Now let us assume that $G$ has at least two cycles. Note that $G$ has at least once vertex of degree at least three and two non-adjacent vertices $u$ and $v$, each of degree at most two. We first suppose that $G$ has two cycles $C'$ and $C''$ with no common vertices. Then by Theorem 4.1, all vertices of $C'$ and $C''$ are in any $\beta_k(G)$-set. If $V(G) - S \neq \emptyset$, then by Corollary 4.2, every vertex of $V - S$ is adjacent to all vertices of $C'$ and clearly each edge of $C'$ will belong to two cycles, contradicting the fact that $G$ is a cactus graph. Thus $V(G) - S = \emptyset$, that is $\beta_k(G) = n$, and so by Proposition 4.3, $G \in \mathcal{G}_k$. By definition of the family $\mathcal{G}_k$ and the above remark we deduce that $k \leq 2$. But then the vertex of degree at least three cannot have all its neighbors in every $\beta_k(G)$-set, a contradiction. Therefore we may assume that all cycles of $G$ share the same vertex, say $x$. Thus $x$ has degree at least four. In $G + uv$, both $u$ and $v$ are in every $\beta_k(T)$-set $S$ and at least one of them is a full vertex in $S$. It follows that $k \leq 3$ and so $\beta_k(G) < |V|$. By Corollary 4.2, we deduce that $x$ does not belong to any $\beta_k(T)$-set and since $x$ must be adjacent to all vertices in every $\beta_k(T)$-set we deduce that $G$ has no leaf and all cycles of $G$ are $C_3$. Hence $G = R$ and $k = 2$. Conversely, if $G$ is a graph as described in items (1) and (2), then the result follows by Corollary 4.6 and Theorem 4.7, respectively. Now if $k = 2$ and $G = R$, then it is easy to check that $G$ is $\beta_2^+$-edge critical. □

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