AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **53** (2012), Pages 151–170

On the graceful conjecture for triangular cacti

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Abstract

A graceful labelling of a graph with n edges is a vertex labelling where the induced set of edge weights is $\{1,\ldots,n\}$. A near graceful labelling is almost the same, the difference being that the edge weights are $\{1,2,\ldots,n-1,n+1\}$. In both cases, the weight of an edge is the absolute difference between its two vertex labels. Rosa [8] in 1988 conjectured that all triangular cacti are either graceful or near graceful. He also suggested the use of Skolem sequences to label some types of triangular cacti. In this paper, we verify the conjecture for two families of triangular cacti, and extend the discussion for further research. Particular constructions of Skolem sequences are discussed, as well as a technique using Langford sequences to obtain Skolem and hooked Skolem sequences with specific sub-sequences. These special sequences are used to gracefully label the two families which are the focus of the paper.

1 Introduction

The vertices of a simple connected graph can be labelled using the integers from 0 to the number of edges, in such a way that each number is used at most once. Edge weights are then computed by taking the absolute value of the difference between its two incident vertices. Such a labelling is graceful if every possible edge weight from 1 to the number of edges occurs. A triangular cactus is a graph which is constructed by connecting copies of K_3, such that any two copies share at most one vertex. Throughout this paper, we assume the usual convention that the only cycles in a triangular cactus are the K_3 blocks. Graceful labellings were first introduced by Rosa [8], under the name β-labelling. Figure 1 features a gracefully labelled graph. It was conjectured by Rosa [8] that all triangular cacti are either graceful, or *near graceful*. A near graceful labelling is similar to a graceful labelling, except that the edge weights are $1,2,3,\ldots,n-1,n+1$, with n being the number of edges. He suggested the use of Skolem type sequences to label the graphs. To date, Skolem type

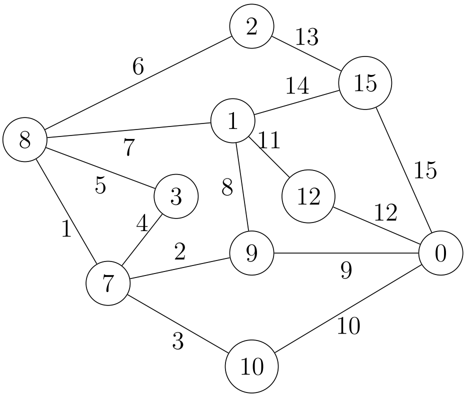

Figure 1: A gracefully labelled graph

sequences are the main technique used to gracefully or near gracefully label triangular cacti. Gallian stated in [4] that this conjecture seems “hopelessly difficult” to prove. However, some special cases of the conjecture have been proved. It is the goal of this paper to briefly discuss some of the known types of graceful/near graceful triangular cacti, and verify the conjecture for a new class of triangular cacti.

A triangular snake is a triangular cactus in which the blocks are arranged in a path. Moulton [6] proved that all triangular snakes are graceful or near graceful. Another type of triangular cactus is a Dutch windmill. These graphs have the property that all of the blocks share the same vertex. Dutch windmills were described by Bermond, and he proved in [1] that all Dutch windmills are either graceful or near graceful. Note that Bermond used the term Dutch windmill to describe what we will later define as a regular Dutch windmill. The technique used to prove this was modified slightly by Wicks [11] to prove that all Dutch windmills with one pendant triangle are either graceful, or near graceful. A Dutch windmill with one pendant triangle is a graph with an underlying regular Dutch windmill, and one extra K_3 block attached to one of the main blocks. The technique used by Wicks [11] is modified in this paper to prove that all Dutch windmills with two pendant triangles are either graceful, or near graceful.

2 Definitions and Known Results

Before getting into details about particular graceful graphs, we formalize the definition of a graceful labelling given in the introduction.

Definition 2.1. Let $G = (V, E)$ be a graph with $|E| = n$. Then G is graceful if there exists an injection $f : V \mapsto \{0, 1, 2, \ldots, n\}$ with the property that $\{|f(u) - f(v)|\}_{\{u,v\} \in E} = \{1, 2, 3, \ldots, n\}$. The function f is called a graceful labelling of G. Rosa originally called this a $\beta-$valuation.

A near graceful labelling is defined in the same way as a graceful labelling with the co-domain of f replaced by $\{0, 1, 2, \ldots, n+1\}$, and the "edge weight" set replaced by $\{1, 2, 3, \ldots, n-1, n+1\}$.

We now define triangular cacti, then proceed to define some special cases.

Definition 2.2. A triangular cactus is a connected graph which is made up of blocks which are all K_3, such that any two blocks share at most one vertex. Triangular cacti also have the property that if a cycle in the graph has only the start and end vertex repeated, then it is one of the K_3 blocks.

We include the definition of triangular snakes, as they are a known class of graceful/near graceful triangular cacti. As mentioned before, this result is due to Moulton [6]. We will not discuss triangular snakes in detail.

Definition 2.3. A triangular snake is a triangular cactus in which all but two of the blocks share vertices with exactly two other blocks.

Triangular snakes can be thought of as a "path" of K_3 blocks. Figure 2 is a gracefully labelled triangular snake with the edge weights included.

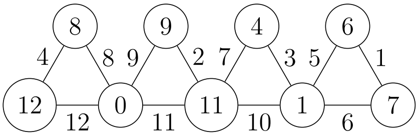

Figure 2: A gracefully labelled triangular snake

Now we define the regular Dutch windmill. All regular Dutch windmills are known to be graceful or near graceful, due to Bermond [1].

Definition 2.4. A regular Dutch windmill is a triangular cactus in which all blocks have a common vertex that we will call the *central vertex*. The blocks will be called *vanes*. A general Dutch windmill, or simply a Dutch windmill, is a graph that contains a regular Dutch windmill as a subgraph.

We now define pendant triangles, then discuss how they can be attached to a Dutch windmill.

Definition 2.5. A block is a pendant triangle of a Dutch windmill if the block is not a part of the largest regular Dutch windmill subgraph of a triangular cactus.

From here on, the order of a Dutch windmill, with or without pendant triangles, refers to the total number of blocks in the graph. If a Dutch windmill has one pendant triangle, there is, up to isomorphism, only one way to attach it. Figure 3 features a Dutch windmill of order 5, and a Dutch windmill of order 6 with one pendant triangle.

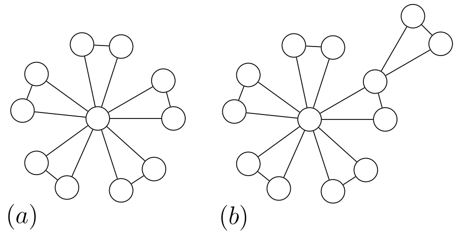

Figure 3: A regular Dutch windmill (a) and a Dutch windmill with one pendant triangle (b).

It was proved by Wicks in [11] that Dutch windmills with one pendant triangle of order $n \equiv 0$ or $1 \pmod 4$ are graceful. A summary of this proof will be given in the next section, along with a slight variation to show that Dutch windmills with order $n \equiv 2$ or $3 \pmod 4$ and one pendant triangle are near graceful. The focus of this paper is to prove that all Dutch windmills with 2 pendant triangles are graceful or near graceful. Unlike the case with one pendant triangle, a Dutch windmill with two pendant triangles is not unique. In fact, there are 4 cases. If the two pendant triangles are attached to different vanes, we will call them *independent*. If one of the triangles is attached to a vane, and the other is attached to the first pendant triangle, we will call them *stacked*. If the pendant triangles are attached to different vertices of the same vane, they will be called *split*. Finally, if they are attached to the same vertex on the same vane, they are called *double* pendant triangles.

Figures 4 $(a), (b), (c)$, and (d) are Dutch windmills with independent, stacked, split, and double pendant triangles, respectively.

Theorem 2.1 gives necessary conditions for gracefulness and near gracefulness of general triangular cacti. The result is well known and stated without proof.

Theorem 2.1. *Let G be a triangular cactus with n blocks. Then,*

1. *if G is graceful, then $n \equiv 0$ or $1 \pmod 4$, and,*
2. *if G is near graceful, then $n \equiv 2$ or $3 \pmod 4$.*

The goal of this paper is to gracefully or near gracefully label each of the 4 types of Dutch windmills of any order with 2 pendant triangles. To do this, we will use Skolem and hooked Skolem sequences with particular properties. It will be taken for granted that a Skolem sequence of order n exsits if and only if $n \equiv 0$ or $1 \pmod 4$, and that a hooked Skolem sequence of order n exists if and only if $n \equiv 2$ or $3 \pmod 4$. We will use the following notations for Skolem and hooked Skolem sequences, which can be found in [3].

Definition 2.6. A Skolem sequence of order n is a sequence $S = (s_1, s_2, \ldots, s_{2n})$ of $2n$ integers satisfying the conditions

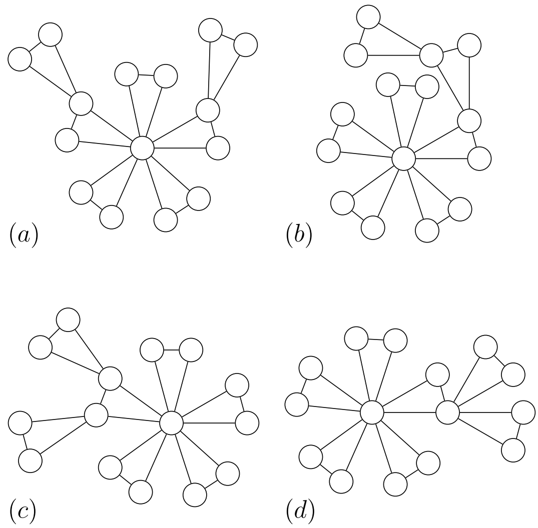

Figure 4: All possible Dutch windmills of order 7 with two pendant triangles.

1. for every $k \in \{1, 2, \ldots, n\}$ there exist exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$, and

2. if $s_i = s_j = k$ with $i < j$, then $j - i = k$.

Equivalently, Skolem sequences can also be written as collections of ordered pairs $\{(a_i, b_i) : 1 \leq i \leq n, b_i - a_i = i\}$ with $\bigcup_{i=1}^{n} \{a_i, b_i\} = \{1, 2, \ldots, 2n\}$.

Skolem sequences will sometimes be called perfect Skolem sequences to distinguish them from hooked Skolem sequences, which are defined next.

Definition 2.7. A hooked Skolem sequence, HS, is a sequence of $2n + 1$ integers satisfying conditions (1) and (2) of Definition 2.6, with the additional requirement that $s_{2n} = 0$. The element s_{2n} is called the hook of the sequence, and is sometimes represented by a $*$ rather than a zero.

Example 2.1. The sequence $(4, 1, 1, 5, 4, 2, 3, 2, 5, 3)$ is a Skolem sequence of order 5. It can be represented by the collection

$$\{(2, 3), (6, 8), (7, 10), (1, 5), (4, 9)\}.$$

Example 2.2. The sequence $(1, 1, 5, 2, 6, 2, 3, 5, 4, 3, 6, *, 4)$ is a hooked Skolem sequence of order 6, and it can be represented by

$$\{(1, 2), (4, 6), (7, 10), (9, 13), (3, 8), (5, 11)\}.$$

For the sequence notation of both Skolem and hooked Skolem sequences, we will usually omit the brackets. To avoid repetition, we will sometimes use the term "sequence" instead of "Skolem or hooked Skolem sequence." We now state two similar lemmas, one for each type of sequence. Lemmas 2.1 and 2.2 give solutions to Heffter's first and second difference problems, respectively [3].

Lemma 2.1. *Let* $S = \{(a_i, b_i)\}_{i=1}^n$ *be a Skolem sequence. Consider the set of triples* $T = \{(0, a_i + n, b_i + n)\}_{i=1}^n$. *Then the set of differences* $\bigcup_{i=1}^n \{|b_i + n - 0|, |a_i + n - 0|, |(b_i + n - (a_i + n)|\}$ *is precisely* $\{1, 2, 3, \ldots, 3n\}$.

These triples are the base blocks of a Steiner triple system.

Lemma 2.2. *Let* $S = \{(a_i, b_i)\}_{i=1}^n$ *be a hooked Skolem sequence. Consider the set of triples* $T = \{(0, a_i + n, b_i + n)\}_{i=1}^n$. *Then the set of differences* $\bigcup_{i=1}^n \{|b_i + n - 0|, |a_i + n - 0|, |(b_i + n - (a_i + n)|\}$ *is precisely* $\{1, 2, 3, \ldots, 3n - 1, 3n + 1\}$.

Definition 2.8. Let $\{(a_i, b_i)\}_{i=1}^n$ be a Skolem or hooked Skolem sequence. Then i is said to be a *pivot* of a Skolem sequence if $b_i + i \leq 2n$. For i to be a pivot of a hooked Skolem sequence, $2n \neq b_i + i \leq 2n + 1$.

Definition 2.9. Let i and j be two pivots in a perfect or hooked Skolem sequence. We say i and j are compatible pivots if $b_i + i \neq a_j$ and $b_j + j \neq a_i$.

Example 2.3. In the Skolem sequence $\{(1, 2), (7, 9), (3, 6), (4, 8), (5, 10)\}$, which can also be represented by 1134532425, 1 and 3 are pivots. However, since $b_1 + 1 = 2 + 1 = 3 = a_3$, they are not compatible.

In the proofs of Theorems 3.2, 3.3, 3.4, and 3.5, we will use sequences with 2 compatible pivots. The following definition gives all possible ways two compatible pivots in the same sequence can interact. The names of the types of interaction correspond to the possible configuration of two pendant triangles. The definition is in terms of the ordered pair notation for sequences.

Definition 2.10. Let S be a sequence with two compatible pivots, j and k.

1. The pivots are *independent* if the following hold:
   (a) $b_j + j = l$,
   (b) $b_k + k = m$,
   (c) l, m, b_j, and b_k are pairwise different, and
   (d) there is no i such that $\{m, l\} = \{a_i, b_i\}$.
2. The pivots are *stacked* if either $b_k + k = b_j$ or $b_j + j = b_k$.
3. The pivots are *split* if there is some l, different from j and k, such that $b_j + j = a_l$ and $b_k + k = b_l$ or $b_j + j = b_l$ and $b_k + k = a_l$
4. The pivots are *double* if $b_j + j = b_k + k$.

Example 2.4. The Skolem sequence 23243114 has two independent pivots, 1 and 2. The sequence $34632425611*6$ has stacked pivots, 2 and 3. In the sequence $42524365311*6$, 2 and 4 are split pivots. Finally, 2 and 3 are double pivots of 3523245114.

Finally, we define Langford sequences. The existence of Langford sequences will be useful in finding Skolem and hooked Skolem sequences of arbitrary order that have particular subsequences.

Definition 2.11. A Langford sequence of order n and defect d is a set of ordered pairs $\{(a_i, b_i)\}_{i=d}^{n+d-1}$ where $\bigcup_{i=d}^{n}\{a_i, b_i\} = \{1, 2, 3, \ldots, 2n\}$ and $b_i - a_i = i$ for each i. It can also be interpreted as a sequence of $2n$ elements where there is an i in position a_i and b_i.

The definition of a hooked Langford sequence is the same as that for a Langford sequence except the $\{1, 2, 3, \ldots, 2n\}$ is replaced by $\{1, 2, 3, \ldots, 2n-1, 2n+1\}$. The following theorem states exactly when a Langford or hooked Langford sequence exists. The proof can be found in [9].

Theorem 2.2.

1. *A Langford sequence of order n and defect d exists if and only if $n \geq 2d-1$ and either*
   (a) $n \equiv 0$ *or* $1 \pmod 4$ *and d is odd, or*
   (b) $n \equiv 0$ *or* $3 \pmod 4$ *and d is even.*
2. *A hooked Langford sequence of order n and defect d exists if and only if $n(n-2d+1)+2 \geq 0$ and either*
   (a) $n \equiv 2$ *or* $3 \pmod 4$ *and d is odd, or*
   (b) $n \equiv 1$ *or* $2 \pmod 4$ *and d is even.*

Finally, we observe that a Langford sequence or a hooked Langford sequence can be used with a Skolem sequence to construct a Skolem sequence of larger order. This is illustrated by the following example.

Example 2.5. Let d be odd, and $n \equiv 0 \pmod 4$ be such that $n \geq 2d-1$. Further, suppose that there exists a Skolem sequence, S, of order $d-1$, which means $d-1 \equiv 0 \pmod 4$. By Theorem 2.2, there exists a Langford sequence, L, of order n and defect d. Let $S = \{(a_i, b_i)\}_{i=1}^{d-1}$ and $L = \{(a_i, b_i)\}_{i=d}^{n+d-1}$. Construct $L' = \{(a_i + 2(d-1), b_i + 2(d-1))\}_{i=d}^{n+d-1}$. Define $S \cup L' = \{(a_i', b_i')\}_{i=1}^{n+d-1}$ where (a_i', b_i') is the i^{th} pair from S if $1 \leq i \leq d-1$, and the i^{th} pair from L' if $d \leq i \leq n+d-1$. We claim that $S \cup L'$ is a Skolem sequence of order $n+d-1$. Clearly, $b_i' - a_i' = i$ for each $i = 1, \ldots, n+d-1$. Also,

$$\begin{aligned} & \bigcup_{i=1}^{n+d-1}\{a_i', b_i'\} = \bigcup_{i=1}^{d-1}\{a_i, b_i\} \cup \bigcup_{i=d}^{n+d-1}\{a_i + 2(d-1), b_i + 2(d-1)\} \\ = \; & \{1, 2, 3, \ldots, 2(d-1)\} \cup \{1 + 2(d-1), 2 + 2(d-1), \ldots, 2n + 2(d-1)\} \\ = \; & \{1, 2, 3, \ldots, 2(n+d-1)\}. \end{aligned}$$

So it is indeed a Skolem sequence of order $n+d-1$.

The next section is devoted to the proof of the main theorem.

3 New Results

In this section, we reduce the graceful and near graceful labelling of Dutch windmills to finding special Skolem and hooked Skolem sequences. It is shown in four similar lemmas. Before this is done, the proof by Wicks of the one pendant case is outlined. The following is the theorem as stated in [11].

Theorem 3.1. *A graceful labelling of a Dutch windmill of order n with one pendant triangle exists if and only if $n \equiv 0$ or $1 \pmod 4$.*

We summarize the proof , and discuss how the idea can be adapted to solve the near graceful case corresponding to $n \equiv 2$ or $3 \pmod 4$.

Proof. The necessary condition follows from Theorem 2.1. For the sufficient condition, we begin with Skolem's construction of a Skolem sequence which can be found in [10] and [11]. The two cases are in Tables 1 and 2.

	i	a_i	b_i	
1	$4m-2r+2$	$4m+r-1$	$8m-r+1$	$1 \le r \le 2m$
2	$4m-2r-1$	r	$4m-r-1$	$1 \le r \le m-2$
3	$2m-2r-1$	$m+r+1$	$3m-r$	$1 \le r \le m-2$
4	$2m+1$	$m-1$	$3m$	
5	1	m	$m+1$	
6	$2m-1$	$2m$	$4m-1$	
7	$4m-1$	$2m+1$	$6m$	

Table 1: The construction for $n = 4m$

	i	a_i	b_i	
1	$4m-2r+2$	$4m+r+1$	$8m-r+3$	$1 \le r \le 2m$
2	$4m-2r+1$	r	$4m-r+1$	$1 \le r \le m$
3	$2m-2r-1$	$m+r+2$	$3m-r+1$	$1 \le r \le m-2$
4	1	$m+1$	$m+2$	
5	$4m+1$	$2m+1$	$6m+2$	
6	$2m-1$	$2m+2$	$4m+1$	

Table 2: The construction for $n = 4m+1$

We will assume that $n > 5$. Triples of the form $(0, i, b_i + n)$ are constructed from either Table 1 or Table 2 depending on the congruence of n modulo 4. It is easily verified that these triples give a complete set of differences from 1 to $3n$, similar to Lemmas 2.1 and 2.2. We claim that 1 is a pivot of the Skolem sequence. In the $n \equiv 0$ case, $b_1 = m+1$, so $b_1 + 1 = m+2 < 8m = 2n$. In the other case, $b_1 = m+2$, so $b_1 + 1 = m + 3 < 8m + 2 = 2n$. Therefore, $b_1 + 1 \leq 2n$ in both cases and thus 1 is a pivot. The triple $(0, 1, b_1 + n)$ is replaced by $(1, 2, b_1 + n + 1)$. Table 3 contains the triples with this replacement outlined.

$n \equiv 0 \pmod 4$	
$(0, 4m - 2r + 2, 12m - r + 1)$	$1 \leq r \leq 2m$
$(0, 4m - 2r - 2, 8m - r - 1)$	$1 \leq r \leq m - 2$
$(0, 2m - 2r - 1, 7m - r)$	$1 \leq r \leq m - 2$
$(0, 2m + 1, 7m)$	
$(1, 2, 5m + 2)$	
$(0, 2m - 1, 8m - 1)$	
$(0, 4m - 1, 10m)$	
$n \equiv 1 \pmod 4$	
$(0, 4m - 2r + 2, 12m - r + 4)$	$1 \leq r \leq 2m$
$(0, 4m - 2r + 1, 8m - r + 2)$	$1 \leq r \leq m$
$(0, 2m - 2r - 1, 7m - r + 2)$	$1 \leq r \leq m - 2$
$(1, 2, 5m + 4)$	
$(0, 4m + 1, 10m + 3)$	
$(0, 2m - 1, 8m + 2)$	

Table 3: Triples used to label Dutch windmills with one pendant triangle.

Next, using the construction of the sequence, we need to verify that $b_1 + n + 1$ is not in any other triple. For $n \equiv 0 \pmod 4$, we have that $b_1 + n + 1 = m + 1 + 4m + 1 = 5m + 2$. This element is easily verified from the construction to be a_{2m-4}. The only way this can be in another triple is if it is $b_i + n$ for some i. However, $5m + 2 - n = 5m + 2 - 4m = m + 2$, and $m + 2$ is not b_i for any i. Similar reasoning works for the $n \equiv 1 \pmod 4$ case. Therefore, once the replacement is made, the only elements that occur in more than one triple are 0 and 2. Note that 1 still only occurs once because the only original triple containing it was removed. The windmill is then labelled using the triples. Each of the vanes which are not attached to the pendant triangle are labelled by one of the triples containing 3 through n. This is done so that the non-zero entries label the vertices which are unique to the vane, and each triple is used exactly once. The central vertex is labelled by 0. The vane connected to the pendant triangle is labelled by the triple containing 2, such that the 2 labels the vertex shared with the pendant triangle. Finally, the pendant triangle is labelled using 1 and $b_1 + n + 1$. They can be placed in either of the two possible orders on the remaining unlabelled vertices of the pendant triangle. When $n = 4$ and $n = 5$, the graphs can easily be labelled by hand. □

A proof of the analogous result for near gracefulness when $n \equiv 2$ or $3 \pmod 4$ is not given in [11]. However, it can be done using a similar argument with hooked Skolem sequences. This will be the approach taken to prove the result for the two pendant triangle case.

3.1 Preliminary Results

Lemma 3.1. *Let G be a Dutch windmill of order n with two independent pendant triangles. If there exists a Skolem or hooked Skolem sequence of order n with two independent pivots, then G is graceful or near graceful. In particular, G is graceful for $n \equiv 0$ or $1 \pmod 4$, and near graceful for $n \equiv 2$ or $3 \pmod 4$.*

Proof. Let G be a Dutch windmill of order n with two independent pendant triangles. Let S be a sequence of order n with two independent pivots, say j and k. Construct the set of triples as in Lemmas 2.1 and 2.2, $T = \{(0, a_i + n, b_i + n)\}_{i=1}^{n}$. Since the a_i and b_i are all unique, the only repeated element throughout these triples is 0. We now create a new set of triples, T', by making the following replacements: $(0, a_j + n, b_j + n) \mapsto (j, a_j + n + j, b_j + n + j)$ and $(0, a_k + n, b_k + n) \mapsto (k, a_k + n + k, b_k + n + k)$. Observe the following:

1. Since $0 < j, k \leq n$, j and k do not appear in any triples of T. This means they occur only once each throughout T'.

2. The numbers, $a_j + n + j$ and $a_k + n + k$ equal $b_j + n$ and $b_k + n$ respectively, but the triples in T containing these numbers are not in T', so they are still only used once each.

3. Since j and k are independent pivots, the numbers $b_j + n + j$ and $b_k + n + k$ each occur in exactly two triples in T', and neither of them occurs in the j^{th} or k^{th} triples.

Putting all of this together, we see that the only repeated elements are $0, b_j + n + j$, and $b_k + n + k$. Furthermore, since adding a constant to each element of a triple preserves the differences (edge weights) between the elements, by Lemma 2.1 or Lemma 2.2, we still get all of the differences for a graceful or near graceful labelling. Therefore, if we can assign the triples to the vertices of blocks in G, so that repeated labels are used only once, we will have gracefully or near gracefully labelled G. We do this as follows:

1. Label the central vertex with 0.

2. Label the pendant triangles with the new triples in T' (corresponding to the pivots) such that the $b_j + n + j$ and $b_k + n + k$ label the vertices shared with vanes.

3. Finish labelling the vanes attached to the pendant triangles with the remaining number from the corresponding triples.

4. Label the remaining vanes so that the two labels for each vane come from the same triple, and each remaining triple is used.

□

The proofs of the next three lemmas are very similar to the proof of Lemma 3.1. They are not given in detail, but the main differences are outlined.

Lemma 3.2. *Let G be a Dutch windmill of order n with two stacked pendant triangles. If there exists a sequence of order n with two stacked pivots, then G is graceful or near graceful.*

Proof. We construct the set of triples, T', as in the proof of Lemma 3.1, and label the graph in a very similar fashion. The triples corresponding to the pivots, however, interact differently. The new triples are $(j, a_j + n + j, b_j + n + j)$ and $(k, a_k + n + k, b_k + n + k)$ where $a_j + n + j = a_k + n + k$, and $b_k + n + k$ is in some triple other than the two associated with the pivots. The triple associated with k labels the pendant triangle attached to a vane, with $b_k + n + k$ the label for the vertex it shares with the vane. The triple associated with j labels the pendant triangle attached to the pendant triangle with $b_j + n + j$ the label for the shared vertex. The other vanes of the windmill are labelled in the same way as Lemma 3.1. □

Lemma 3.3. *Let G be a Dutch windmill of order n with two split pendant triangles. If there exists a sequence of order n with two split pivots, then G is graceful or near graceful.*

Proof. In this case, the triples associated with the pivots have the property that $b_j + n + j$ and $b_k + n + k$ are in the same triple in T'. This triple is used to label the vane from which the two pendant triangles stem. Then the pendant triangles are labelled appropriately, since the points of attachment are labelled already. □

Lemma 3.4. *Let G be a Dutch windmill of order n with double pendant triangles. If there exists a sequence of order n with double pivots, then G is graceful or near graceful.*

Proof. In this case, $b_j + n + j = b_k + n + k$, so this number is used to label the vertex where the pendant triangles are both attached. □

We now have reduced the problem of gracefully labelling Dutch windmills with two pendant triangles to finding sequences with particular pivots. Using Lemmas 3.1-3.4, and Langford sequences, we can easily prove that for all but finitely many n, a Dutch windmill of order n with two pendant triangles is graceful or near graceful.

3.2 Independent Pendants

Theorem 3.2. *Every Dutch windmill of order at least 4 with two independent pendant triangles is graceful or near graceful.*

Proof. We start by noticing that the Skolem sequence of order 4, $S = \{(6,7), (1,3), (2,5), (4,8)\}$, has two independent pivots, 1, and 2. By Theorem 2.2, and since 5 is odd, a Langford sequence of order n and defect 5 exists for $n \equiv 0$ or 1 (mod 4) as long as $n \geq 2 \times 5 - 1 = 9$. The proof of this can be found in [9]. We can append such a sequence to S, to obtain a Skolem sequence of order $n+4$. This means that for every $n \geq 9$ and $n \equiv 0$ or $1 \pmod 4$, we get a Skolem sequence of order $n+4$ with two independent pivots. Note that $n+4 \equiv 0$ or $1 \pmod 4$. So by Lemma 3.1, if $n \geq 9+4 = 13$ and $n \equiv 0$ or $1 \pmod 4$, a Dutch windmill of order n with two independent pendant triangles is graceful. Similarly, if $n(n-2\times 5+1)+2 \geq 0$ then $n(n-9) \geq -2$, and this is true for positive n only if $n \geq 9$. So a hooked Langford sequence of order $n \geq 9$ and defect 5 exists whenever $n \equiv 2$ or $3 \pmod 4$. We can append this to S to get a hooked Skolem sequence of order $n+4$ whenever $n+4 \geq 14$ and $n+4 \equiv 2+4$ or $3+4 \equiv 2$ or $3 \pmod 4$. By Lemma 3.1, if $n \geq 14$ and $n \equiv 2$ or $3 \pmod 4$, then a Dutch windmill of order n with two independent pendant triangles is near graceful.

Table 4 contains Skolem and hooked Skolem sequences with two independent pivots of order $5 \leq n \leq 12$. By applying Lemma 3.1, we get the cases for the above theorem with $5 \leq n \leq 12$. For $n \leq 4$, the theorem degenerates. For orders $1, 2$, and 3, there are not enough triangles, and in the case $n = 4$, the graph is a triangular snake of order 4, which is known to be graceful. In this table, and all tables to follow, we will represent 10 by A, 11 by B, 12 by C, and so on.

n	Sequence	Pivots
5	2325341154	1 and 2
6	23243564115∗6	1 and 2
7	2527115463743∗6	2 and 5
8	4857411568723263	1 and 2
9	292411748596375386	2 and 4
10	2529115784A69473863∗A	2 and 5
11	2A2B1135938A57B469847∗6	2 and 3
12	2529115B86AC947684B3A73C	2 and 5

Table 4: Sequences with two independent pivots.

□

3.3 Stacked Pendants

Theorem 3.3. *A Dutch windmill of order at least* 3 *with two stacked pendant triangles is graceful or near graceful.*

Proof. Using the same technique as in the previous section, we begin with the Skolem sequence $S = \{(9,10),(5,7),(1,4),(2,6),(3,8)\}$. The numbers 2 and 3 are stacked pivots of this sequence. We now construct Langford and hooked Langford sequences with defect 6. We can construct Langford sequences of order $n \geq 2\times 6-1 = 11$, and hooked Langford sequences whenever $n \geq n(n-2\times 6+1)+2 \geq 0$, or $n \geq 11$. Since the defect is 6, we get Langford sequences of order congruent to 0 and $3 \pmod 4$. This gives Skolem sequences of order $0+5 \equiv 1$ and $3+5 \equiv 0 \pmod 4$. Similarly, we get hooked Skolem sequences of order congruent to 2 and $3 \pmod 4$ as long as $n \geq 5+11 = 16$. Therefore, as long as $n \geq 16$, a Dutch windmill of order n with two stacked pendant triangles is either graceful or near graceful.

Table 5 contains particular Skolem sequences with stacked pivots for each order from 5 to 15. For $n = 2$, there are not enough triangles, and for $n = 3$ and $n = 4$, the problem is reduced to a triangular snake, which is known to be either graceful or near graceful.

n	Sequence	Pivots
5	3453242511	2 and 3
6	34632425611∗5	2 and 3
7	3473242567115∗6	2 and 3
8	3853262578461147	2 and 3
9	385329257861149764	2 and 3
10	39532A256497846A117∗8	2 and 3
11	3B732A211786B95A46854∗9	2 and 3
12	3B63242C6489BA71158C975A	2 and 3
13	39B32D258A9C5B6784DA647C11	2 and 3
14	3B932A211DE9B6CA485647D5E8C∗7	2 and 3
15	3CF32729$ABDE$7C869FA5B68D5E411∗4	2 and 3

Table 5: Sequences with two stacked pivots.

□

3.4 Split Pendants

Theorem 3.4. *A Dutch windmill with two split pendant triangles is graceful or near graceful as long as its order is at least* 3*.*

Proof. The following Skolem sequence has two split pivots:

$$\{(12, 13), (2, 4), (6, 9), (1, 5), (11, 16), (8, 14), (3, 10), (7, 15)\}.$$

The pivots are 2 and 4. It is of order 8, and through similar Langford sequence constructions, we get Skolem or hooked Skolem sequences with two split pivots of order greater than or equal to 24. For $n = 3$, the graph is a triangular snake, which is known to be near graceful. Table 6 shows particular Skolem and hooked Skolem sequences of orders 6 through 23 that have split pivots. For $n = 4$, we use the triples

$$\{(0, 2, 8), (0, 1, 12), (1, 4, 11), (3, 7, 12)\}$$

where the first two triples label the vanes. For $n = 5$, we use

$$\{(0, 3, 9), (0, 2, 12), (0, 1, 15), (1, 6, 14), (4, 8, 15)\},$$

where the first three triples label the vanes. These constructions were both found in [8].

□

n	Sequence	Pivots
6	$42524365311{*}6$	2 and 4
7	$3113456742526{*}7$	3 and 4
8	4272438637511685	2 and 4
9	427243893756118596	2 and 4
10	$42A243783569A758611{*}9$	2 and 4
11	$425243B53789A6117B869{*}A$	2 and 4
12	$42C2436A395B68C57A9118B7$	2 and 4
13	$42B243CD369A5B8675C9DA8711$	2 and 4
14	$42C2437B3AE5D7C659BA8611ED9{*}8$	2 and 4
15	$42C243EA3D11F8CB7A69E8D765BF9{*}5$	2 and 4
16	$42D243EF3A679CGD6B7AE9F85C11B5G8$	2 and 4
17	$42E243DF3B8G9ACHE78DB9FA76CG5116H5$	2 and 4
18	$428243GA3F87IEC6DA7HB6G5F9CE5DIB119{*}H$	2 and 4
19	$42C243BD3HFJ6ACGIB6ED75A8FH579JG8EI$ $11{*}9$	2 and 4
20	$42H243J736BK8E76CFIH8BGADJ9EC11KFA59$ $IDG5$	2 and 4
21	$4282437L368IF756JDE5KABGH9CFLIDAEB9J1$ $1CGKH$	2 and 4
22	$42J243CF3LM8BIEK9AC8DJFBH9GAE7LIMD$ $6K71156HG{*}5$	2 and 4
23	$42B2436L35116B5IEFMJ9KHNACG8L9EDFIA8$ $7CJHMKG7D{*}N$	2 and 4

Table 6: Sequences with two split pivots.

3.5 Double Pendants

Theorem 3.5. *A Dutch windmill with two double pendant triangles is graceful or near graceful as long as its order is at least* 3.

Proof. The Skolem sequence $\{(7,8),(3,5),(1,4),(2,6)\}$ has double pivots, 2 and 3. The same argument as the independent pendant section applies and we get that Dutch windmills of order at least 13 with double pendant triangles. Table 7 has Skolem and hooked Skolem sequences with double pivots of order 5 through 12. For $n = 2$, there are not enough triangles, for $n = 3$, the case is the same as a Dutch windmill of order 3 with no pendant triangles, and for $n = 4$, the case is the same as a Dutch windmill of order 3 with one pendant triangle.

□

n	Sequence	Pivots
5	3523245114	2 and 3
6	42624511635∗3	1 and 4
7	3623275611457∗4	2 and 3
8	3723258476541186	2 and 3
9	372329687115649854	2 and 3
10	36232A768119574A854∗9	2 and 3
11	35232B549A841167B98A6∗7	2 and 3
12	3A232C78119AB7685C49654B	2 and 3

Table 7: Sequences with two double pivots.

4 Further Results and Observations

While the results in this paper only cover two small instances of Rosa's conjecture, the technique used may be a pathway to many other cases. In particular, by looking for Skolem or hooked Skolem sequences with many pivots, it may be possible to label Dutch windmills with many pendant triangles. Working case by case quickly becomes overwhelming. In general, there are 11 nonisomorphic ways to place three pendant triangles on a Dutch windmill. This number grows very fast. It seems that we must turn to more general results, similar to those following in this section.

Theorem 4.1. *Let S be a Skolem sequence of order $n \equiv 0 \pmod 4$. Let $i \leq \frac{n}{2}$. Then either $a_i - i \geq 1$ or $b_i + i \leq 2n$.*

Proof. Suppose $a_i - i < 1$. Then $a_i - i + 3i < 1 + 3i$, which is the same as $a_i + i + i < 3i + 1$. Substituting, we get $b_i + i < 3\frac{n}{2} + 1 \leq 2n$. Therefore, $b_i + i \leq 2n$. This completes the proof. ☐

A similar result holds for $n \equiv 1 \pmod 4$. This result means that $\frac{n}{2}$ numbers in the sequence are pivots in either S, or the reverse of S. By the pigeon hole principle, we get that for any Skolem sequence, either S, or its reverse has $\lfloor \frac{n}{4} \rfloor$ pivots. It should be noted that it is not guaranteed that the pivots are pairwise compatible.

The next result shows that there exist constructions for any n where the sequence has about $\frac{n}{4}$ pivots that behave independently.

Theorem 4.2. *For any $n \equiv 0$ or $1 \pmod 4$, there exists a Skolem sequence of order n that has $\lfloor \frac{n}{4} \rfloor - 1$ independent pivots.*

Proof. We consider 4 cases:

1. $n = 4m$ with $m \equiv 0$ or $1 \pmod 3$
2. $n = 4m$ with $m \equiv 2 \pmod 3$
3. $n = 4m + 1$ with $m \equiv 0$ or $2 \pmod 3$
4. $n = 4m + 1$ with $m \equiv 1 \pmod 3$.

Here, we prove only case 1. The proofs for the other cases are similar. The construction in Table 8 is due to Köhler, and can be found in [2] and [5]:

	i	a_i	b_i	
1	$4m-2r$	r	$4m-r$	$1 \le r \le m-1$
2	$2m-2r$	$m-1+r$	$3m-1-r$	$1 \le r \le m-1$
3	$4m$	$2m-1$	$6m-1$	
4	1	$3m-1$	$3m$	
5	$4m+1-2r$	$4m-1+r$	$8m-r$	$1 \le r \le 2m-1$
6	$2m$	$6m$	$8m$	

Table 8: A Skolem sequence of order $n \equiv 0 \pmod 4$ due to Köhler.

We now show that as long as $m \equiv 0$ or $1 \pmod 3$, each i in the first row of the construction is a pivot, and they are all independent. That is, $\{i = 4m-2r : 1 \le r \le m-1\}$ are independent pivots. We first show that they are pivots. Since $b_i = 4m-r$ (from the construction), the largest b_i occurs when r is smallest, or $r = 1$. Therefore, the largest b_i is $4m-1$. Conveniently, $r = 1$ also maximizes $i = 4m-2r$ for $1 \le r \le m-1$. So for any of the i in question, we have that

$$b_i + i \le b_{4m-2} + 4m - 2 = 4m - 1 + 4m - 2 = 8m - 3 = 2n - 3 < 2n,$$

so i is a pivot. Now consider $(b_{4m-2r} + (4m-2r)) - (b_{4m-2(r+1)} + 4m - 2(r+1))$ for $1 \le r \le m-2$. By substitution, we get

$$\begin{aligned} & (b_{4m-2r} + (4m-2r)) - (b_{4m-2(r+1)} + 4m - 2(r+1)) \\ = \; & (4m-r) + (4m-2r) - (4m-(r+1)) - (4m-2(r+1)) \\ = \; & 8m - 3r - 8m + 3r + 1 + 2 \\ = \; & 3 \end{aligned}$$

This shows that if i and j are "consecutive" pivots (say for r_0 and r_0+1), then $(b_j + j) - (b_i + i) = 3$. So we have that

$$X = \{b_{4m-2r} + 4m - 2r : 1 \le r \le m-1\} = \{5m+3, 5m+6, \ldots, 8m-6, 8m-3\}.$$

This is an arithmetic progression with common difference equal to 3, $m-1$ terms, and final term $8m-3$. Observe that the smallest value of $b_i + i$ for these pivots is $5m+3$, which is greater than $m-1$, which is the largest of the a_i among the pivots. This guarantees that the pivots are pairwise compatible. Next, we will show that the pivots are independent. Since the values in X are all less than or equal to $2n$, they each must be a_i or b_i for some $1 \le i \le n$. With this in mind, we can construct a well defined function $f : X \to \{1, 2, \ldots n\}$ by $f(x) = i$ if $x = a_i$ or $x = b_i$. To show that the pivots are independent, we need only show that f is injective. First, suppose $m \equiv 0 \pmod 3$. In line 5 of the construction, we take $r = 3, 6, 9, \ldots, 2m-3$. These r correspond to $i = 4m-5, 4m-11, 4m-17, \ldots, 7$, which are all different and congruent to $1 \pmod 3$. Furthermore, these i have b_i

values of $8m-3, 8m-6, 8m-9, \ldots, 6m+3$, which are all in X. So far, we have shown that $f(6m+3), \ldots, f(8m-3)$ are all different. We also have that $a_{2m} = 6m$, so $f(6m) = 2m$, which is $0 \pmod 3$, so $f(6m) \neq f(6m+3), \ldots, f(8m-3)$. We now take, again in the fifth line of the construction, $r = m+4, m+7, m+10, \ldots, 2m-2$. These correspond to a_i values of

$$\begin{aligned} 4m-1+m+4 &= 5m+3 \\ 4m-1+m+7 &= 5m+6 \\ \vdots &= \vdots \\ 4m-1+2m-2 &= 6m-3, \end{aligned}$$

which is the remainder of the elements of X. Also, the i corresponding to these a_i are

$$\begin{aligned} 4m+1-2(m+4) &= 2m-7 \\ 4m+1-2(m+7) &= 2m-13 \\ \vdots &= \vdots \\ 4m+1-2(2m-2) &= 5, \end{aligned}$$

which are all different and congruent $\pmod 3$. Hence they are congruent to $5 \equiv 2 \pmod 3$. Therefore, $f(5m+3), \ldots, f(6m-3)$ are all different, and congruent to $2 \pmod 3$. So f must be injective, and the proof is complete for $m \equiv 0 \pmod 3$. The argument works similarly for $m \equiv 1 \pmod 3$. For the case when $m \equiv 2 \pmod 3$, the i corresponding to the a_i and b_i are congruent modulo 3, so there are repetitions, and the pivots are not independent. For case 2, the construction due to Anderson is used, for case 3, the construction due to Köhler [5] is used, and for case 4, the construction due to Hanani is used. All of the constructions can be found in [2]. ☐

It is also possible to construct sequences of most orders, n, that have about $\frac{n}{3}$ pivots, but they may not be pairwise compatible. To do this, begin with a Skolem or hooked Skolem sequence of order $\frac{n}{3}$. Then there is a hooked or perfect Langford sequence of order $\frac{2}{3}n+1$. Two sequences can be joined in this way, except in the case where a hooked Skolem sequence, and a perfect Langford sequence are used. This case doesn't work because the zero will appear in the interior of the sequence, rather than the $(2n)^{\text{th}}$ position. A way to append a hooked Langford sequence to a hooked Skolem sequence will be described in the discussion following Corollary 4.1.

We now state a corollary, which is essentially a generalization of the technique used to gracefully or near gracefully label all Dutch windmills with two pendant triangles.

Corollary 4.1. *Let S be a Skolem or hooked Skolem sequence of order n with m pivots. Suppose the pivots can be used to label some configuration of m pendant triangles. Then every Dutch windmill of order $3n+1$ or higher with that particular configuration of m pendant triangles is graceful or near graceful.*

This corollary is really just a restatement of the result used in the main proofs of the paper. We note that if a perfect and hooked sequence are to be joined, the hooked sequence goes on the right. A hooked Langford sequence can be attached to a hooked Skolem sequence by reversing the Langford sequence, and "linking" the sequences together by the hooks to construct a perfect Skolem sequence. For example, the Skolem sequence $1, 1, 2, *, 2$ and the hooked Langford sequence $8, 4, 7, 3, 6, 4, 3, 5, 8, 7, 6, *, 5$ can be joined by reversing the hooked Langford sequence and linking the sequences to create the Skolem sequence $1, 1, 2, 5, 2, 6, 7, 8, 5, 3, 4, 6, 3, 7, 4, 8$.

Corollary 4.1 gives a way to add vanes to a gracefully or near gracefully labelled Dutch windmill in such a way that the new windmill is still graceful or near graceful. This new windmill will have exactly the same number of pendant triangles with exactly the same structure as the original windmill. As described in the proof, this will work for all but finitely many numbers of vanes that can be added to the windmill. Corollary 4.1 does not mention possible pivots in the appended Langford or hooked Langford sequence. If these pivots were used, one could gracefully or near gracefully label Dutch windmills with many other configurations of pendant triangles. Corollary 4.2 formalizes this idea for Skolem and Langford sequences. Similar results hold for hooked Skolem and langford sequences as well.

Corollary 4.2. *Let S be a Skolem sequence of order n with k pairwise compatible pivots, and let L be a Langford sequence of order t, defect $n+1$, and with l pairwise compatible pivots. Then there is a graceful Dutch windmill of order $t+n$, with $k+l$ pendant triangles, where k of them are configured in the same way as those on the Dutch windmill that could be labelled with S.*

Proof. Construct a Skolem sequence of order $t+n$ by appending L to S. This sequence will have $k+l$ pairwise compatible pivots, which can be used to gracefully label a Dutch windmill of order $t+n$ with $k+l$ pendant triangles, configured in some way. Observe that the way the pivots in S interact with each other does not change with the addition of L. This means that the pendant triangles associated with those pivots will be configured in the same way as they would be in the windmill labelled just with S. ☐

The following construction, due to Simpson, can be found in [9]. It gives the construction of a Langford sequence of order $n \equiv 0 \pmod 4$, with defect $d \equiv 0 \pmod 4$. We set $n = 4m$ and $d = 4s$. We also require that $s \geq 1$, and $m \geq 4m$.

It should be noted that when $s = 1$, rows $7, 8$, and 9 should be omitted, and when $m = 2s$, rows 1 and 3 should be omitted. There are similar constructions given in [9] for all other admissible cases of n and d. The constructions all have several nested parts. For example, in the given construction, when $m = 7$ and $s = 2$, we get that the second row tells us that the sequence starts with $26, 24, 22, 20$, and so somewhere in the sequence we get $20, 22, 24, 26$ as a subsequence. In this case, $20, 22, 24$, and 26 are independent pivots. If this construction were used to extend a fixed Skolem sequence to arbitrarily large lengths, the extended part itself would have approximately $m - 2s + 1$ pairwise compatible pivots. If they are added to

	i	a_i	b_i	
1	$4s+1+2r$	$2m-3s+1-r$	$2m+s+2+r$	$0 \le r \le m-2s-1$
2	$2m+3s+2r$	$m-2s+1-r$	$3m+s+1+r$	$0 \le r \le m-2s$
3	$4s+2r$	$6m-s+1-r$	$6m+3s+1+r$	$0 \le r \le m-2s-1$
4	$2m+3s-1+2r$	$5m-s+1-r$	$7m+2s+r$	$0 \le r \le m-2s$
5	$2m$	$3m-s+2$	$5m-s+2$	
6	$2m+1+2r$	$2m-r$	$4m+1+r$	$0 \le r \le s-1$
7	$2m+2s+r$	$4m-s+4+r$	$6m+s+2+2r$	$0 \le r \le s-2$
8	$2m+2+2r$	$3m+1-r$	$5m+3+r$	$0 \le r \le s-2$
9	$4m+1+2r$	$3m+s-r$	$7m+s+1+r$	$0 \le r \le s-2$
10	$4m+2+2r$	$m-s+1-r$	$5m-s+3+r$	$0 \le r \le s-1$
11	$4m+2s+1+r$	$2m-3s+2+r$	$6m-s+3+2r$	$0 \le r \le 2m-2$
12	$4m-s+1+r$	$2m+1+r$	$6m-s+2+2r$	$0 \le r \le s-1$
13	$4m+2s-1$	$2m+s+1$	$6m+3s$	

Table 9: A Langford sequence with $n \equiv d \equiv 0 \pmod 4$

Skolem sequences of fixed length, the defect is constant, therefore s is constant, and so $m-2s+1$ approaches $\frac{n}{4}$ as $n \to \infty$.

Up to this point, we have started with Dutch windmills, and found appropriate sequences that would allow us to gracefully or near gracefully label them. It is also possible to start with a sequence, and create a graph, or several graphs that can be labelled using the given sequence. Rosa [8] was able to label half of the triangular cacti on 4 blocks with a single Skolem sequences, by using the pivots in different ways. There are many known constructions of Skolem and related sequences. Skolem's and Köhler's are mentioned in this paper. Any one of these sequences could possibly label a large class of triangular cacti, similar to the method of [8].

5 Conclusion and Open Questions

We have seen a technique that gracefully or near gracefully labels a large class of triangular cacti, as well as other techniques that could lead to the (near)graceful labelling of other large classes. The problem remains difficult, but perhaps it is not as hopeless as Gallian thought. To finish off, we state some open questions, the solutions to which would be helpful steps in proving or disproving Rosa's conjecture.

1. Define $S(n)$ to be the number of triangular cacti of order n which can be labelled using Skolem sequences and their pivots. Define $T(n)$ to be the number of triangular cacti of order n. What is $\lim_{n\to\infty} \frac{S(n)}{T(n)}$?

2. Let X be some collection of sequences of order n, each with some large number of pivots, say $\frac{n}{4}$. Is it possible to find such a collection that can label all possible Dutch windmills of order n with $\frac{n}{4}$ pivots?

3. How many nonisomorphic configurations of m pendant triangles are there on sufficiently many vanes? Is there a way to concisely categorize them?

While the conjecture remains open and difficult, there still seems to be some hope for further research.

Acknowledgments

The authors would like to thank the anonymous referees, and also Professor Clayton Halfyard for carefully reading the final version of the paper.

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(Received 3 Aug 2011; revised 9 Apr 2012)