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b-coloring of some bipartite graphs*

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Abstract

A b-coloring is a coloring of the vertices of a graph such that each color class contains a vertex that has a neighbor in all other color classes, and the b-chromatic number $b(G)$ of a graph G is the largest integer k such that G admits a b-coloring with k colors. Here we show that every graph G of order n that is not a complete graph satisfies the inequality $b(G) \leq \lfloor \frac{n+\omega(G)-1}{2} \rfloor$, where $\omega(G)$ is the maximum size of a clique in G, and we give a characterization of bipartite graphs that achieve equality in the above bound. Also we show that every graph G satisfies $b(G) - \chi(G) \leq \lceil \frac{n}{2} \rceil - 2$, and we characterize the graphs that achieve this bound. We also show that if G is a graph of order $n \geq 4$, then for any vertex v of G we have $b(G) - (\lfloor \frac{n}{2} \rfloor - 2) \leq b(G - v) \leq b(G) + \lfloor \frac{n}{2} \rfloor - 2$. We conjecture that this bound is achieved for every vertex of G if and only if $G = C_4, P_4$ or $2P_2$.

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1 Introduction

Let $G = (V, E)$ be a simple graph with vertex-set V and edge-set E. A coloring of the vertices of G is a mapping $c : V \to \{1, 2, \dots\}$. For every vertex $v \in V$ the integer $c(v)$ is called the color of v. A coloring is *proper* if any two adjacent vertices have different colors. The *chromatic number* $\chi(G)$ of a graph G is the smallest integer k such that G admits a proper coloring using k colors.

A *b-coloring* of a graph G by k colors is a proper coloring of the vertices of G such that in each color class there exists a vertex having neighbors in all the other $k-1$ colors classes. We call any such vertex a *b-vertex*. The b-chromatic number $b(G)$ of a graph G is the largest integer k such that G admits a b-coloring with k colors. The concept of a b-coloring was introduced by Irving and Manlove [12, 18]. They proved that determining $b(G)$ is NP-hard for general graphs, and this remains true when restricted to the class of bipartite graphs [16]. On the other hand, computing $b(G)$ can be done in polynomial time in the class of trees [12, 18]. The NP-completeness results have incited researchers to establish bounds on the b-chromatic number in general or to find its exact values for subclasses of graphs (see [2–18]).

We show that every graph that is not a clique satisfies the inequality $b(G) \leq \lfloor \frac{n+\omega(G)-1}{2} \rfloor$, where $\omega(G)$ is the maximum clique size in G, and we give a characterization of bipartite graphs that achieve equality in the above bound. Also we show that every graph G satisfies $b(G) - \chi(G) \leq \lceil \frac{n}{2} \rceil - 2$, and we characterize the graphs that achieve this bound.

For notation and graph theory terminology we generally follow [1]. Consider a graph $G = (V, E)$. For any vertex v of G, the *neighborhood* of v is the set $N_G(v) = \{u \in V(G) \mid (u, v) \in E\}$ (or $N(v)$ if there is no confusion). For any $A \subset V$, let $G[A]$ denote the subgraph of G induced by A. Let $\omega(G)$ denote the size of a maximum clique of G. If G and H are two vertex-disjoint graphs, the *union* of G and H is the graph $G+H$ whose vertex-set is $V(G) \cup V(H)$ and edge-set is $E(G) \cup E(H)$. For an integer $p \geq 2$, the union of p copies of a graph G is denoted pG. We let P_n denote the path with n vertices, and K_n denote the complete graph with n vertices. For $n \geq 3$, we let C_n denote the cycle with n vertices. For any graph G, we let $\overline{G}$ denote the complementary graph of G.

In a graph G, given a b-coloring c of G with k colors, we call any set $\{x_1, \dots, x_k\}$, such that x_i is a b-vertex of color i for each i, a *b-system*.

2 Upper bound on $b(G)$

In this section we give an upper bound for the b-chromatic number $b(G)$ in terms of n and $\omega(G)$. It is easy to see that $b(K_n) = \frac{n+\omega(K_n)}{2}$. The following theorem shows that the complete graph is the only graph that achieves this equality.

Theorem 2.1 *Every graph G of order n that is not a complete graph satisfies*

$$b(G) \leq \left\lfloor \frac{n + \omega(G) - 1}{2} \right\rfloor.$$

Proof. It is clear that if $\omega(G) = 1$ the inequality is valid. So assume that $\omega(G) \geq 2$. Let $b(G) = k$, let c be a b-coloring of G with k colors and let $S_1, S_2, \dots, S_k$ be the color classes of c. Let t be the number of color classes of size 1. If $t > 0$, let $S_1, \dots, S_t$ be these classes and $W = S_1 \cup \dots \cup S_t$. Note that W is a clique, because for any two integers $i, j \leq t$, the unique vertex in S_i is a b-vertex and must be adjacent to the unique vertex in S_j. If $t = 0$, then $n = \sum_{i=1}^{k} |S_i| \geq 2k$ and so $k \leq \frac{n}{2} \leq \lfloor \frac{n+\omega(G)-1}{2} \rfloor$. Now assume that $t \geq 1$. Note that $t < k$ since G is not a complete graph. Let x_ℓ be any b-vertex of color $\ell > t$. Then x_ℓ is adjacent to every vertex of W. It follows that $W \cup \{x_\ell\}$ induces a complete subgraph of G, so $t + 1 \leq \omega(G)$. We have $n = t + \sum_{i=t+1}^{k} |S_i| \geq t + 2(k - t) = 2k - t \geq 2k + 1 - \omega(G)$. This implies that $k \leq \frac{n+\omega(G)-1}{2}$. ■

Theorem 2.2 *For every integer $n \geq 4$, there exists a graph G_n of order n with*

$$b(G_n) = \left\lfloor \frac{n + \omega(G_n) - 1}{2} \right\rfloor.$$

Proof. Let $t = \lceil \frac{n}{3} \rceil$ and $r = n - 2t$. Note that $r \leq t$. We construct a graph G_n with vertex-set $A \cup B \cup C$, where A, B, C are pairwise disjoint sets, with $A = \{a_1, a_2, \dots, a_t\}$, $B = \{b_1, b_2, \dots, b_t\}$ and $C = \{c_1, c_2, \dots, c_r\}$, such that A and C are cliques, B is a stable set, there is no edge between A and C, vertex a_i is adjacent to vertex b_j if and only if $i \neq j$, every vertex of $C \setminus \{c_r\}$ is adjacent to every vertex of B, and if $n \equiv 0 \bmod 3$, then c_r has no neighbor in B, while if $n \not\equiv 0 \bmod 3$ then c_r is adjacent to every vertex of B.

It is easy to see that every maximal clique of G_n is either A, or $(A \setminus \{a_i\}) \cup \{b_i\}$, or $(C \setminus \{c_r\}) \cup \{b_i\}$, or $C \cup \{b_i\}$ for each i in $\{1, \dots, t\}$; and in any case (since $r \leq t$) we have $\omega(G_n) = t$. It follows that:

$$\left\lfloor \frac{n + \omega(G_n) - 1}{2} \right\rfloor = \left\lfloor \frac{n + t - 1}{2} \right\rfloor = \begin{cases} t + r - 1 & \text{if } n \equiv 0 \bmod 3 \\ t + r & \text{if } n \not\equiv 0 \bmod 3. \end{cases} \tag{1}$$

In order to show that G_n satisfies the statement, it suffices to exhibit a b-coloring of G with the number of colors indicated in (1). We do this as follows. For each integer $i \leq t$, assign color i to a_i and b_i. For each integer $i \leq r - 1$, assign color $t + i$ to c_i. If $n \equiv 0 \bmod 3$, assign color 1 to c_r. If $n \not\equiv 0 \bmod 3$, assign color $t + r$ to c_r. It is easy to check that this yields a b-coloring; indeed, a b-system is obtained by taking either $B \cup C \setminus \{c_r\}$ (when $n \equiv 0 \bmod 3$) or $B \cup C$ (when $n \not\equiv 0 \bmod 3$). ■

The following corollary is an immediate consequence of Theorem 2.1.

Corollary 2.3 *If G is a triangle-free graph of order n different from K_2, then $b(G) \leq \lceil \frac{n}{2} \rceil$. In particular, every bipartite graph different from K_2 satisfies this inequality.*

In the next section, we characterize the bipartite graphs that achieve equality in Corollary 2.3.

3 Bipartite graphs with $b(G) = \lceil \frac{n}{2} \rceil$

Lemma 3.1 *Let G be a bipartite graph with bipartition (X, Y). Let c be a b-coloring of G with k colors, and let S be any b-system of c. If $|S \cap X| \geq 2$, then all colors of c appear in Y. In particular, if $|Y| = k$, then all vertices of Y have distinct colors.*

Proof. Let $S = \{s_1, \ldots, s_k\}$, and assume without loss of generality that $s_1, s_2 \in X$. For each i in $\{1, 2\}$, vertex s_i must have a neighbor of color j for all $j \neq i$. This implies that Y contains a vertex of each color. In particular, if $|Y| = k$, then all vertices of Y have distinct colors. ■

Lemma 3.2 *Let G be a bipartite graph of order n with bipartition (X, Y) such that $|X| > |Y|$. If there exists a b-coloring of G with $\lceil \frac{n}{2} \rceil$ colors, then $|Y| = \lceil \frac{n}{2} \rceil - 1$ and all b-vertices of X have the same color.*

Proof. Let $k = \lceil \frac{n}{2} \rceil$, and let c be a b-coloring of G with k colors. We have $n = |X| + |Y| > 2|Y|$. Thus,

$$|Y| < \frac{n}{2} \leq \left\lceil \frac{n}{2} \right\rceil = k. \tag{2}$$

If X contains at least two b-vertices of c of distinct colors, then Lemma 3.1 implies that all colors appear in Y. Therefore $|Y| \geq \lceil \frac{n}{2} \rceil$, which contradicts the inequality in (2). Thus all b-vertices of X have the same color. Consequently, Y contains b-vertices of at least $k - 1$ colors, so $|Y| \geq k - 1$. Together with (2), this implies $|Y| = k - 1 = \lceil \frac{n}{2} \rceil - 1$. ■

A set M of edges in G is a *matching* if every vertex of G is incident with at most one edge in M. For integers p and q, we let $K_{p,q}$ denote the complete bipartite graph with parts of size p and q respectively, and we let $K^*_{p,p}$ denote the graph obtained from $K_{p,p}$ by removing a matching of p edges.

In order to characterize bipartite graphs whose b-chromatic number is equal to $\lceil \frac{n}{2} \rceil$, we define four families of bipartite graphs $\mathcal{F}_i$ $(i = 0, 1, 2, 3)$ as follows:

Class $\mathcal{F}_0$. This class consists of all bipartite graphs with at most four vertices different from K_2, $\overline{K_3}$ and $\overline{K_4}$.

Class $\mathcal{F}_1$. A bipartite graph $G = (X, Y; E)$ is in $\mathcal{F}_1$ if one can write $X = A \cup C$ and $Y = B \cup D$ where A, B, C and D are disjoint sets that satisfy the following conditions:

- $|A| = |D| = p$, $|B| = |C| = q$, with $q \geq p \geq 0$ and $p + q \geq 3$;
- $G[A \cup B] = K_{p,q}$ and $G[B \cup C] = K^*_{q,q}$;
- if $p > 1$, then $G[A \cup D] = K^*_{p,p}$, (and there may be edges between C and D);
- if $p = 1$, the vertex in D has a non-neighbor in X.

Class $\mathcal{F}_2$. A bipartite graph $G = (X, Y; E)$ is in $\mathcal{F}_2$ if one can write $X = A \cup \{x, u, v\}$ and $Y = B \cup \{y\}$ where A, B and $\{x, y, u, v\}$ are disjoint sets that satisfy the following conditions:

- $|A| = |B| = p$, with $p \geq 1$, and $G[A \cup B] = K^*_{p,p}$;
- x is adjacent to all of $B \cup \{y\}$, and y is adjacent to all of $A \cup \{x\}$;
- every vertex of B has a neighbor in $\{u, v\}$;
- y has at most one neighbor in $\{u, v\}$, and if y is adjacent to one of u, v, say to u, then v is adjacent to all of B.

Class $\mathcal{F}_3$. A bipartite graph $G = (X, Y; E)$ is in $\mathcal{F}_3$ if one can write $X = A \cup \{x\}$ where A, $\{x\}$ and Y are disjoint sets that satisfy the following conditions:

- $|A| = |Y| = p$, with $p \geq 2$, and $G[A \cup Y] = K^*_{p,p}$;
- x is adjacent to all vertices of Y.

Let $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.

Lemma 3.3 *If $G \in \mathcal{F}$, then $b(G) = \lceil \frac{n}{2} \rceil$.*

Proof. By the definition of $\mathcal{F}$, G is a bipartite graph and $G \neq K_2$. By Corollary 2.3, we have $b(G) \leq \lceil \frac{n}{2} \rceil$. Thus, to show equality it suffices to give a b-coloring c of G with $\lceil \frac{n}{2} \rceil$ colors. We do this as follows.

Suppose that $G \in \mathcal{F}_0$. If G is K_1 or $\overline{K_2}$, then we have $b(G) = 1 = \lceil \frac{n}{2} \rceil$. In the other cases, we have $b(G) = 2 = \lceil \frac{n}{2} \rceil$.

Suppose that $G \in \mathcal{F}_1$. Set $B = \{b_1, b_2, \ldots, b_q\}$ and $C = \{c_1, c_2, \ldots, c_q\}$, where the non-edges between B and C are $b_1c_1, \ldots, b_qc_q$. If $p \geq 1$, then set $A = \{a_1, a_2, \ldots, a_p\}$ and $D = \{d_1, d_2, \ldots, d_p\}$, where the non-edges between A and D are $a_1d_1, \ldots, a_pd_p$. Note that $\lceil \frac{n}{2} \rceil = p + q$. Assign color i to b_i and c_i $(1 \leq i \leq q)$, and, if $p \geq 1$, assign color $q+j$ to a_j and d_j $(1 \le j \le p)$. We obtain a b-coloring with $p+q$ colors, where $A \cup B$ is a b-system of this coloring.

Now suppose that $G \in \mathcal{F}_2$. Set $A = \{a_1, a_2, \ldots, a_p\}$ and $B = \{b_1, b_2, \ldots, b_p\}$, where the non-edges between A and B are $a_1b_1, \ldots, a_pb_p$. Note that $\lceil \frac{n}{2} \rceil = p+2$. Color a_i and b_i with i $(1 \le i \le p)$. Color x and y with $p+1$ and $p+2$ respectively. If y is adjacent to u, then color u with $p+1$ and v with $p+2$. If y is not adjacent to any of u and v, then color u and v with $p+2$. We obtain a b-coloring with $p+2$ colors, where $B \cup \{x, y\}$ is a b-system of this coloring.

Finally, suppose that $G \in \mathcal{F}_3$. Set $A = \{a_1, a_2, \ldots, a_p\}$ and $Y = \{y_1, y_2, \ldots, y_p\}$, where the non-edges between A and Y are $a_1y_1, \ldots, a_py_p$. Note that $\lceil \frac{n}{2} \rceil = p+1$. Color a_i and y_i with i $(1 \le i \le p)$ and x with $p+1$. We obtain a b-coloring with $p+1$ colors, where $B \cup \{x\}$ is a b-system of this coloring.

Thus we always obtain a b-coloring with $\lceil \frac{n}{2} \rceil$ colors. ■

Now we are in a position to prove our main result.

Theorem 3.4 *Let G be a bipartite graph of order n. Then, $b(G) = \lceil \frac{n}{2} \rceil$ if and only if $G \in \mathcal{F}$.*

Proof. Let (X, Y) be the bipartition of G. If $n \le 4$, then it is easy to see that $b(G) = \lceil \frac{n}{2} \rceil$ holds if and only if $G \in \mathcal{F}_0$. Now let $n \ge 5$. If $G \in \mathcal{F}$, then by Lemma 3.3, we have $b(G) = \lceil \frac{n}{2} \rceil$. To prove the converse, let c be a b-coloring with $\lceil \frac{n}{2} \rceil$ colors. Let $k = \lceil \frac{n}{2} \rceil$. Since $n \ge 5$, we have $k \ge 3$. Let S be a b-system of c and let $S_X = S \cap X$ and $S_Y = S \cap Y$. There are two cases to consider:

Case 1: n is even. Then $k = \frac{n}{2}$ and $n \ge 6$.

Case 1.1: $|X| = |Y| = k$. Let $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$.

First suppose that $|S_X| = 0$. Then all vertices of Y are b-vertices with distinct colors. Lemma 3.1 implies that all vertices of X have distinct colors. So we may assume that $c(x_i) = c(y_i) = i$ for each i in $\{1, \ldots, k\}$. For each i in $\{1, \ldots, k\}$, since y_i is a b-vertex it must be adjacent to all vertices of $X \setminus \{x_i\}$. Thus G is equal to $K^*_{k,k}$, which is a member of $\mathcal{F}_1$ (with $q = k$ and $p = 0$).

Now suppose that $|S_X| = 1$. Then $|S_Y| = k - 1 \ge 2$. By Lemma 3.1, all vertices of X have distinct colors. Without loss of generality, we may assume that $S_X = \{x_1\}$, $S_Y = \{y_2, y_3, \ldots, y_k\}$, $c(x_1) = 1$, $c(x_i) = c(y_i) = i$ for each i in $\{2, \ldots, k\}$, and $c(y_1) = \ell$ for some ℓ in $\{1, \ldots, k\}$. For each i in $\{2, \ldots, k\}$, since y_i is a b-vertex it must be adjacent to all vertices of $X \setminus \{x_i\}$. Hence, x_1 is adjacent to all vertices of S_Y. Vertex y_1 is not adjacent to x_ℓ and may be adjacent to any vertex of $X \setminus \{x_\ell\}$. Thus G is a member of $\mathcal{F}_1$ (with $q = k-1$, $p = 1$, $A = \{x_1\}$, $B = S_Y$, $C = X \setminus \{x_1\}$ and $D = \{y_1\}$).

Now suppose that $|S_X| > 1$ and $|S_Y| > 1$. Lemma 3.1 implies that all vertices of X (respectively, of Y) have distinct colors. Without loss of generality, we may suppose that $c(x_i) = c(y_i) = i$ for each i in $\{1, \ldots, k\}$ and $S_X = \{x_1, x_2, \ldots, x_p\}$

(with $p \geq 2$). Therefore $S_Y = \{y_{p+1}, y_{p+2}, \ldots, y_k\}$ (with $k - p \geq 2$). Let $A = S_X$, $B = S_Y$, $C = X \setminus S_X$ and $D = Y \setminus S_Y$. Then $|A| = |D|$ and $|B| = |C|$. Each b-vertex x_i in S_X is adjacent to all vertices of $Y \setminus \{y_i\}$, and each b-vertex y_i in S_Y is adjacent to all vertices of $X \setminus \{x_i\}$. Hence we have $G[A \cup B] = K_{p,k-p}$, $G[A \cup D] = K^*_{p,p}$, $G[B \cup C] = K^*_{k-p,k-p}$, and there may be arbitrary edges from C and D. Thus G is a member of $\mathcal{F}_1$.

Case 1.2: $|X| \neq |Y|$. Without loss of generality, we may assume that $|X| > |Y|$. Lemma 3.2 implies that $|Y| = k - 1$. Consequently, $|X| = n - |Y| = k + 1$. So let $X = \{x_1, x_2, \ldots, x_{k+1}\}$ and $Y = \{y_1, y_2, \ldots, y_{k-1}\}$. Lemma 3.2 implies that S_X contains only one vertex, say x_1. This implies that all vertices of Y are b-vertices with distinct colors. So $S_Y = Y$. Also x_1 is adjacent to all vertices of Y. We may assume that $c(x_1) = 1$ and $c(y_i) = i + 1$ for each i in $\{1, \ldots, k-1\}$. Since $k - 1 \geq 2$, Lemma 3.1 implies that all colors of c appear in X. Hence we may assume that for each i in $\{2, \ldots, k\}$ we have $c(x_i) = i$, and $c(x_{k+1}) = h$ for some h in $\{1, \ldots, k\}$. Without loss of generality, let $h \in \{1, k\}$.

Let $A = \{x_2, \ldots, x_{k-1}\}$ and $B = \{y_1, \ldots, y_{k-2}\}$. Since all vertices of Y are b-vertices, it follows that each vertex y_i of B is adjacent to all of $A \setminus \{x_{i+1}\}$. So $G[A \cup B] = K^*_{k-2,k-2}$. Moreover, y_{k-1} is adjacent to all of A. If $h = 1$, then every vertex of B is adjacent to x_k, and vertex x_{k+1} may be adjacent to any vertex of Y. On the other hand, if $h = k$, then x_k and x_{k+1} are not adjacent to y_{k-1}, and each vertex y_i of B has a neighbor in $\{x_k, x_{k+1}\}$. Thus, whatever the value of h may be, G is a member of $\mathcal{F}_2$ (with $x = x_1$, $\{u, v\} = \{x_k, x_{k+1}\}$ and $y = y_{k-1}$).

Case 2: n is odd. Then $k = \frac{n+1}{2}$ and $n \geq 5$. Without loss of generality, we may assume that $|X| > |Y|$. Lemma 3.2 implies that $|Y| = k - 1$, hence $|X| = k$. So let $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_{k-1}\}$. Lemma 3.2 implies that S_X contains only one vertex, say x_1. Therefore, all of Y are b-vertices with distinct colors ($S_Y = Y$). Since x_1 is a b-vertex, it is adjacent to all vertices of Y. So we may assume that $c(x_1) = 1$ and $c(y_i) = i + 1$ for each i in $\{1, \ldots, k-1\}$. Since $|S_Y| = k - 1 \geq 2$, all colors of c appear in X. Without loss of generality, we may suppose, for each i in $\{2, \ldots, k\}$, $c(x_i) = i$. It follows that for each i in $\{1, \ldots, k-1\}$, each b-vertex y_i of Y is adjacent to all vertices of $X \setminus \{x_{i+1}\}$. Thus G is a member of $\mathcal{F}_3$. ■

Let $\alpha(G)$ denote the *stability* number of graph G, i.e., the maximum size of a set of pairwise non-adjacent vertices in G. Every bipartite graph G of order n satisfies $\alpha(G) \geq \lceil \frac{n}{2} \rceil$, so Corollary 2.3 implies that every bipartite graph G of order $n \geq 5$ satisfies $b(G) \leq \alpha(G)$. We can characterize the bipartite graphs that achieve equality $b(G) \leq \alpha(G)$, as follows. Let $\mathcal{F}'_0 = \{K_1, K_2 + K_1, P_3, P_4, C_4, 2K_2\}$.

Theorem 3.5 *Let G be a bipartite graph. Then $b(G) = \alpha(G)$ if and only if $G \in \mathcal{F}'_0 \cup \mathcal{F}_1 \cup \mathcal{F}_3$.*

Proof. If $G \in \mathcal{F}'_0$, then it is easy to check that either $b(G) = 1 = \alpha(G)$ or $b(G) = 2 = \alpha(G)$. If $G \in \mathcal{F}_1$, then it is easy to check that $\alpha(G) = p + q$, and we observed in the proof of Lemma 3.3 that $b(G) = p + q$. So $b(G) = \alpha(G)$. If $G \in \mathcal{F}_3$, then it is easy to check that $\alpha(G) = p + 1$, and we observed in the proof of Lemma 3.3 that $b(G) = p + 1$. So $b(G) = \alpha(G)$.

Conversely, suppose that $b(G) = \alpha(G)$. Then $b(G) \geq \lceil \frac{n}{2} \rceil$, so, by Theorem 3.4, G belongs to $\mathcal{F}$. If G is in $\mathcal{F}_0 \setminus \mathcal{F}_0'$, then it is easy to see that either $b(G) = 1$ and $\alpha(G) = 2$ or $b(G) = 2$ and $\alpha(G) = 3$. If G is in $\mathcal{F}_2$, then it is easy to check that $\alpha(G) = p + 3$, whereas we observed in the proof of Lemma 3.3 that $b(G) = p + 2$. So $b(G) \neq \alpha(G)$. Thus $G \in \mathcal{F}_0' \cup \mathcal{F}_1 \cup \mathcal{F}_3$. ■

4 $b(G) - \chi(G)$ arbitrarily large

Using Theorem 2.1, we can deduce the following result:

Theorem 4.1 *Every graph G satisfies*

$$b(G) - \chi(G) \leq \left\lceil \frac{n}{2} \right\rceil - 2,$$

and equality holds if and only if $G \in \mathcal{F} \cup \{K_3, K_4\}$.

Proof. If either $\chi(G) = 1$ or G is a complete graph, then $b(G) - \chi(G) = 0$. So we may assume that $\chi(G) \geq 2$ and G is not a complete graph. Then Theorem 2.1 implies that $b(G) - \chi(G) \leq \frac{1}{2}(n + \omega(G) - 2\chi(G) - 1)$. Since $\omega(G) \leq \chi(G)$, it follows that $b(G) - \chi(G) \leq \frac{1}{2}(n - \chi(G) - 1) \leq \frac{1}{2}(n - 3)$. Thus, $b(G) - \chi(G) \leq \lfloor \frac{n-3}{2} \rfloor = \lceil \frac{n-4}{2} \rceil = \lceil \frac{n}{2} \rceil - 2$.

It is clear that the bound in Theorem 4.1 is satisfied with equality for the complete graphs K_3 and K_4. Now suppose that G is not a complete graph. If $b(G) - \chi(G) = \lceil \frac{n}{2} \rceil - 2$, then $\frac{1}{2}(n - \chi(G) - 1) = \frac{1}{2}(n - 3)$. Therefore $\chi(G) = 2$. Hence, G is a bipartite graph and $b(G) = \lceil \frac{n}{2} \rceil$. Thus $G \in \mathcal{F}$. ■

Theorem 4.2 *For each vertex v in a graph G of order $n \geq 4$,*

$$b(G - v) \leq b(G) + \left\lfloor \frac{n}{2} \right\rfloor - 2. \qquad (3)$$

Proof. Since $|G - v| \geq 3$, Theorem 4.1 implies that $b(G - v) \leq \chi(G - v) + \lceil \frac{n-1}{2} \rceil - 2$. Since $\chi(G - v) \leq \chi(G) \leq b(G)$ and $\lceil \frac{n-1}{2} \rceil = \lfloor \frac{n}{2} \rfloor$, it follows that $b(G - v) \leq b(G) + \lfloor \frac{n}{2} \rfloor - 2$. ■

Note that Francis Raj and Balakrishnan [8, 9] have recently found the same upper bound for connected graphs of order at least 5.

We have seen that if the bound (3) is achieved, then G belongs to the class of graphs satisfying $\chi(G - v) = \chi(G)$, for every vertex v of G.

Lemma 4.3 *Let v be any vertex of a graph G of order $n \geq 4$. If $b(G-v) = b(G) + \lfloor \frac{n}{2} \rfloor - 2$, then $\chi(G-v) = \chi(G)$.*

Proof. Theorem 4.1 implies that

$$b(G-v) \leq \chi(G-v) + \left\lfloor \frac{n}{2} \right\rfloor - 2 \leq \chi(G) + \left\lfloor \frac{n}{2} \right\rfloor - 2 \leq b(G) + \left\lfloor \frac{n}{2} \right\rfloor - 2.$$

If $b(G-v) = b(G) + \lfloor \frac{n}{2} \rfloor - 2$, then $\chi(G-v) = \chi(G)$. ∎

Based upon this lemma, we state the following conjecture:

Conjecture 4.4 *Let G be a graph of order $n \geq 4$. Then equality $b(G-v) = b(G) + \lfloor \frac{n}{2} \rfloor - 2$ holds for every vertex v of G if and only if G is either C_4, P_4 or $2K_2$.*

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