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k-regular antichains on $[m]$ with $k \leq m-2$

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Abstract

Let $2^{[m]}$ be ordered by set inclusion and let $\mathcal{B} \subseteq 2^{[m]}$ be an antichain of size $n := |\mathcal{B}|$. An antichain $\mathcal{B}$ is k-regular for some non-negative integer k, if for each $i \in [m]$ there are exactly k sets $B_1, B_2, \ldots, B_k \in \mathcal{B}$ containing i. In this case we say that $\mathcal{B}$ is a (k,m,n)-antichain. Let $2 \leq k \leq m-2$ be positive integers. In this paper we show that a (k,m,n)-antichain exists if and only if $k+1 \leq n \leq \frac{km}{2}$.

1 Introduction

1.1 Notation

For nonnegative integers $k \leq m$, the sets $[k,m]$ and $[m]$ are defined by $[k,m] := \{k, k+1, \ldots, m-1, m\}$ and $[m] := [1,m]$. Let $\mathcal{B}$ be a subset of $2^{[m]}$, the power set of $[m]$. The size of $\mathcal{B}$ is $n := |\mathcal{B}|$. The collection $\mathcal{B}$ of sets is an *antichain* (AC) if there are no two sets in $\mathcal{B}$ which are comparable under set inclusion. An antichain $\mathcal{B}$ is *k-regular* for some non-negative integer k, if for each element $i \in [m]$ there are exactly k sets $B_1, B_2, \ldots, B_k \in \mathcal{B}$ containing i. In this case, $\mathcal{B}$ is a (k,m,n)-AC.

A *Separating System* $\mathcal{S}$ on $[n]$ is a collection of blocks of $[n]$ such that for each set $\{x,y\} \subseteq [n]$ of two distinct points, there is a block $S \in \mathcal{S}$ such that ($x \in S$ and $y \notin S$) or ($y \in S$ and $x \notin S$).

A *Completely Separating System (CSS)* $\mathcal{C}$ on $[n]$ is a collection of blocks of $[n]$ such that for any pair of distinct points $x, y \in [n]$, there exist blocks $A, B \in \mathcal{C}$ such that $x \in A - B$ and $y \in B - A$. A CSS on $[n]$ without restrictions on the size of the sets in the collection is said to be an (n)CSS. Let $k < n$. An (n,k) *Completely Separating System* (*(n,k)CSS*) is an (n)CSS in which each block is of size k.

The *volume* of a collection $\mathcal{B}$ of sets is $v(\mathcal{B}) := \sum_{A \in \mathcal{B}} |A|$. For a (k,m,n)-AC $\mathcal{B}$, $v(\mathcal{B}) = km$, and for an (n,k)CSS $\mathcal{C}$, $v(\mathcal{C}) = k|\mathcal{C}|$. Often we omit brackets and commas in our notation for sets. For example we write 1345 instead of $\{1,3,4,5\}$. A set is called a t-set if it contains t elements.

Let $\mathcal{B}$ be a collection of sets on $[m]$ and $i \in [m]$ an arbitrary fixed element then we define $\mathcal{B}_i := \{B \in \mathcal{B} : i \in B\}$. Obviously, for every element $i \in [m]$, if $\mathcal{B}$ is a (k, m, n)-AC then $|\mathcal{B}_i| = k$.

We remark that for every nonnegative integer m, we get by definition that there exists the $(0, m, 0)$-AC $\mathcal{B} := \emptyset$ and the $(0, m, 1)$-AC $\mathcal{B}' := \{\emptyset\}$. These regular antichains are the only ones for $k = 0$.

1.2 Motivation

Regular antichains have a strong connection to CSSs. In 1961, Rényi [8] found minimum Separating Systems in the context of solving certain problems in information theory, and in 1969, Dickson [5] introduced the notion of a Completely Separating System. Spencer [11] showed that antichains are the duals of *Completely Separating Systems.*

Definition 1 (dual)**.** *Let $\mathcal{M} = \{M_1, M_2, \ldots, M_m\}$ be a collection of subsets of $[n]$. We define the dual $\mathcal{M}^*$ of $\mathcal{M}$ to be the collection $\mathcal{M}^* := \{M_1^*, M_2^*, \ldots, M_n^*\}$ of subsets of $[m]$ given by $M_i^* := \{j \in [m] : i \in M_j\}$ $(i = 1, \ldots, n)$.*

Lemma 1. *If $\mathcal{M}$ is a CSS then its dual $\mathcal{M}^*$ is an antichain and vice versa.*

Lemma 2. *If $\mathcal{M}$ is a (n, k)CSS of size m then its dual $\mathcal{M}^*$ is a (k, m, n)-AC and vice versa.*

Completely Separating Systems have been studied in a lot of papers. In 1973, Katona [6] studied combinatorial search problems in general; Cai [4], Ramsay et al. [7] and Roberts et al. [10] studied Completely Separating Systems with restrictions. The relevant CSSs for the motivation of this paper are the (n, k)CSSs. Much of the recent CSS work is on the minimum size of these. We denote with $R := R(n, k)$ the minimum size of an (n, k)CSS, so $R := \min\{|\mathcal{C}| : \mathcal{C} \text{ is } (n, k) \text{ CSS}\}$. If we know that a (k, m, n)-AC exists then we get that $R \leq m$, and similarly, if we know that no (k, m, n)-AC exists then we get that $R > m$.

This connection between regular antichains and Completely Separating Systems is part of the motivation for having a closer look at regular antichains — especially at k-regular antichains on $[m]$ with $k \leq m - 2$ in this paper. The case $k = m$ was studied in [1] and the case $k = m - 1$ can be found in [2].

We remark that the case $k > m$ is much more difficult to analyse. Like in this paper, we tried to reach similar results for the case $k = m+s$ for some non-negative integer s. Especially, it is hard to prove *good* necessary conditions for the existence of (k, m, n)-ACs in general. As a special case, see for example the proof of $R(59, 14) = 12$ in [3], which includes the part that there does not exist a $(14, 11, 59)$-AC.

In [10], Roberts et al. developed a method to show the non-existence of an (n, k)CSS of size m for $k \geq m$, and so also the non-existence of a (k, m, n)-AC for given integers $k \geq m$. It also gives an overview of known values $R(n, k)$.
In this paper, we just remark that these values are known for $k \leq 14$ and $n \geq \binom{k-1}{2}$.

So the problem of determining the values $R(n,k)$ for given n and k is still open in general.

2 Necessary conditions of existence

In this part some easily derived but useful lower and upper bounds for the size of a k-regular antichain on m elements are given.

Lemma 3. *Let $k > 1$ be an integer and let $\mathcal{B}$ be an arbitrary (k,m,n)-AC. Then $1 < |B| < m$ for all $B \in \mathcal{B}$.*

Proof. Let $\mathcal{B}$ be a (k,m,n)-AC. For $i \in [m]$, if $\{i\} \in \mathcal{B}$ or $[m] \in \mathcal{B}$ then $|\mathcal{B}_i| = 1 < k$, and this is a contradiction. □

Lemma 4. *Let $\mathcal{B}$ be a (k,m,n)-AC with $k > 1$. Then*

$$n \le \frac{km}{2}.$$

Proof. According to Lemma 3 there does not exist a set with just one element. We count in two ways to obtain the following inequality:

$$2n = \sum_{B\in\mathcal{B}} 2 \le \sum_{B\in\mathcal{B}} |B| = v(\mathcal{B}) = \sum_{i\in[m]} |\mathcal{B}_i| = \sum_{i\in[m]} k = km.$$

□

Corollary 5. *For all $2 \le k < n$,*

$$R(n,k) \ge \left\lceil \frac{2n}{k} \right\rceil.$$

□

Remark. This result can also be found in [7].

Lemma 6. *Let $\mathcal{B}$ be an arbitrary (k,m,n)-AC with $m > k > 1$. Then*

$$n \ge k+1.$$

Proof. For all $B \in \mathcal{B}$, it is $|B| < m$ (Lemma 3). So we get

$$n(m-1) \ge \sum_{B\in\mathcal{B}} |B| = v(\mathcal{B}) = \sum_{i\in[m]} |\mathcal{B}_i| = km.$$

Hence, $n \ge k\frac{m}{m-1} > k$. □

Corollary 7. *Let $\mathcal{B}$ be an arbitrary (k,m,n)-AC with $m > k > 1$. Then*

$$\frac{km}{2} \ge n > k.$$

3 Constructions

This section contains some constructions which are mostly quite simple or known in their dual version.

3.1 Raising the ground set

Lemma 8. *Let $\mathcal{B}$ be an arbitrary (k, m, n)-AC. Then there exists a $(k, m+1, n)$-AC $\mathcal{B}'$.*

Proof. We fix any element $i \in [m]$ and define $\mathcal{B}_i' := \{B \ \cup \ \{m+1\} : B \in \mathcal{B}_i\}$.

$$\mathcal{B}' := (\mathcal{B} - \mathcal{B}_i) \cup \mathcal{B}_i'.$$

Obviously, $\mathcal{B}'$ is a $(k, m+1, n)$-AC.

□

In the dual version it is the same as copying a block and including it for a second time in the CSS. See for example [9].

3.2 Construction with the help of 2-sets

Theorem 9. *Let $\mathcal{B}^1$ be an (m_1+t, m_1, n_1)-AC and let $\mathcal{B}^2$ be an (m_2+t, m_2, n_2)-AC with $t \geq -\min\{m_1, m_2\}$ and with the property that each set in $\mathcal{B}^1 \cup \mathcal{B}^2$ has cardinality at least two. Then there exists an $(m_1+m_2+t, m_1+m_2, n_1+n_2+m_1m_2)$-AC $\mathcal{C}$.*

Remark. Since $t \geq -\min\{m_1, m_2\}+2$, it follows that every set in $\mathcal{B}^1 \cup \mathcal{B}^2$ contains at least two elements, by Lemma 3.

Proof. We choose the ground set $[m_1]$ for the antichain $\mathcal{B}^1$ and $[m_1+1, m_1+m_2]$ for the antichain $\mathcal{B}^2$. We define:

$$\mathcal{C} := \mathcal{B}^1 \cup \mathcal{B}^2 \cup \underbrace{\{\{i,j\} : i \in [m_1], j \in [m_1+1, m_1+m_2]\}}_{=:\mathcal{D}}.$$

Obviously, $\mathcal{C}$ is a collection of $n_1+n_2+m_1m_2$ sets on an (m_1+m_2)-ground set. With the help of a short case analysis it is easy to prove that $\mathcal{C}$ is an antichain. Now we want to check the conditions of regularity. We choose an element $i \in [m_1]$. There are exactly m_1+t sets in $\mathcal{B}^1$ and m_2 sets in $\mathcal{D}$ which contain the element i. Any set in $\mathcal{B}^2$ does not contain i. Similarly, we can check the regularity for every element $j \in [m_1+1, m_1+m_2]$.

□

It is useful to explicitly state a corollary for the particular case when $m_2+t = n_2 = 0$:

Corollary 10. *Let $\mathcal{B}$ be an $(m-s, m, n)$-AC with the property that $|B| \geq 2$ for all $B \in \mathcal{B}$. Then there exists an $(m, m+s, n+ms)$-AC $\mathcal{C}$.*

Proof. Using Theorem 9 with $\mathcal{B}^1 := \mathcal{B}$ and $\mathcal{B}^2$ defined as the $(0, s, 0)$-AC $\emptyset$ we know that an $(m, m+s, n+ms)$-AC exists. □

3.3 Merging two antichains

Lemma 11. *Let $\mathcal{B}^1$ be a (k, m_1, n_1)-AC and $\mathcal{B}^2$ be a (k, m_2, n_2)-AC. Then there exists a $(k, m_1 + m_2, i)$-AC $\mathcal{B}$ for all i with* $\max\{n_1, n_2\} \leq i \leq n_1 + n_2$.

Proof. Let $\mathcal{B}^1$ be a collection of sets on $[m_1]$ and $\mathcal{B}^2$ be a collection of sets on $[m_1 + 1, m_1 + m_2]$. The sets of $\mathcal{B}^j$ $(j = 1, 2)$ are called B_1^j, B_2^j, ..., $B_{n_j}^j$. Let i be an arbitrary fixed positive integer with $\max\{n_1, n_2\} \leq i \leq n_1 + n_2$. We define $t := n_1 + n_2 - i$ and

$$\mathcal{B} := \{B_j^1 \cup B_j^2 : j \leq t\} \cup \{B_j^1 : t < j \leq n_1\} \cup \{B_j^2 : t < j \leq n_2\}.$$

Then $n_{\mathcal{B}} = t + (n_1 - t) + (n_2 - t) = n_1 + n_2 - t = i$, and so $\mathcal{B}$ is a $(k, m_1 + m_2, i)$-AC. □

Remark. With the help of this construction we reach the dual result: $R(i, k) \leq R(n_1, k) + R(n_2, k)$ with $\max\{n_1, n_2\} \leq i \leq n_1 + n_2$.

3.4 Direct construction with a starter

Lemma 12. *Let $k < m$. Then there exists a (k, m, m)-AC $\mathcal{B}$.*

Proof.

$$\mathcal{B} := \{\{1, 2, \ldots, k\}, \{2, 3, \ldots, k + 1\}, \ldots, \{m, 1, 2, \ldots, k - 1\}\}.$$

Obviously, $\mathcal{B}$ is a (k, m, m)-AC. □

4 (k, m, n)-Antichains with fixed k

4.1 $(1, m, n)$-antichains

Lemma 13. *Let m and n be positive integers with $m \geq n$. Then there exists a $(1, m, n)$-AC $\mathcal{B}$.*

Proof. We define a $(1, m, n)$-AC $\mathcal{B}$ by

$$\mathcal{B} := \{\{1\}, \{2\}, \{3\}, \ldots, \{n - 1\}, \{n, n + 1, n + 2, \ldots, m\}\}.$$

Obviously, $\mathcal{B}$ satisfies all conditions. □

The sufficient condition $m \geq n$ is also necessary:

Lemma 14. *Let m and n be nonnegative integers with $m < n$. Then there does not exist any $(1, m, n)$-AC $\mathcal{B}$.*

Proof. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ be an arbitrary $(1, m, n)$-AC. Because of $k = 1$ it is obvious that $B \cap C = \emptyset$ for all $B, C \in \mathcal{B}$. Otherwise, there would be an element $i \in [n]$, which is in more than one set. Hence,

$$n = \sum_{i=1}^{n} 1 \leq \sum_{i=1}^{n} |B_i| = \left| \bigcup_{i \in [n]} B_i \right| = m$$

□

If we especially ask for a $(1, m, n)$-AC $\mathcal{B}$ with the property (a) that $|B| > 1$ for all $B \in \mathcal{B}$, then we get the following equivalence.

Lemma 15. *Let m and n be positive integers. Then there exists a $(1, m, n)$-AC $\mathcal{B}$ with property (a) if and only if $m \geq 2n$.*

4.2 $(2, m, n)$-AC

Lemma 16. *Let m and n be positive integers with $3 \leq n \leq m$. Then there exists a $(2, m, n)$-AC.*

Proof. We prove Lemma 16 with the help of an induction on m. Let $m = 3$. Then there exists a $(2, 3, 3)$-AC (Lemma 12). Now, let $m \geq 4$ and $3 \leq n \leq m$. If $n = m$ then there exists a $(2, m, n)$-AC (Lemma 12). If $3 \leq n \leq m - 1$, we know that a $(2, m - 1, n)$-AC exists. Using Lemma 8 we know there is also a $(2, m, n)$-AC. □

Using Corollary 7 we also get that, if $\mathcal{B}$ is a $(2, m, n)$-AC then $3 \leq n \leq m$. So in the case $k = 2$ we have the following equivalence.

Corollary 17. *Let $m \geq 3$ be a positive integer. Then there exists a $(2, m, n)$-AC if and only if $3 \leq n \leq m$.*

4.3 $(3, m, n)$-AC

It is relatively easy to check the following results for $m \leq 5$. There is no 3-regular antichain on $[m]$ with $m = 1$, 2, 3. If $m = 4$ then there exists a 3-regular antichain on $[m]$ if and only if $n = \binom{4}{2}$ or $n = \binom{4}{3}$ (see also [2]). If $m = 5$ then there is a 3-regular antichain if and only if $n = 4, 5, 6, 7$ (we will give examples in Table 1).

Lemma 18. *Let $m \geq 6$. If m is even then there exists a $(3, m, \frac{3m}{2})$-AC $\mathcal{E} := \mathcal{E}(m)$ and a $(3, m, \frac{3m}{2} - 1)$-AC $\mathcal{E}' := \mathcal{E}'(m)$. If m is odd then there exists a $(3, m, \frac{3m-1}{2})$-AC $\mathcal{F} := \mathcal{F}(m)$.*

Proof. For $m \geq 6$ define a collection $\mathcal{C}(m)$ of sets as follows

$$\mathcal{C}(m) := \{12, 23, \ldots, (m-1)m, 1m\}.$$

For $m=8$ we define $\mathcal{E}'$ as $\mathcal{C}(8)\cup\{135,468,27\}$ and $\mathcal{E}$ as $\mathcal{C}(8)\cup\{15,26,37,48\}$, and for $m=7$ we define $\mathcal{F}$ as $\mathcal{F}:=\mathcal{C}(7)\cup\{147,25,36\}$. Let $m_1\in\{6,10,12,\ldots\}$ and let $m_2>7$ be odd. Define

$$\begin{aligned}
\mathcal{E} &:= \mathcal{C}(m_1)\cup\{\{1,\frac{m_1}{2}+1\},\{2,\frac{m_1}{2}+2\},\ldots,\{\frac{m_1}{2},m_1\}\}\\
\mathcal{E}' &:= \mathcal{C}(m_1)\cup\{\{1,3,5\},\{2,4,6\},\{7,\tfrac{m_1+6}{2}+1\},\{8,\tfrac{m_1+6}{2}+2\},\ldots,\{\tfrac{m_1+6}{2},m_1\}\}\\
\mathcal{F} &:= \mathcal{C}(m_2)\cup\{\{1,3,5\},\{2,4\},\{6,\tfrac{m_2+5}{2}+1\},\{7,\tfrac{m_2+5}{2}+2\},\ldots,\{\tfrac{m_2+5}{2},m_2\}\}.
\end{aligned}$$

□

Example. For $m=14$ we get $\mathcal{E}:=\{12,\ 23,\ 34,\ 45,\ 56,\ 67,\ 78,\ 89,\ 9A,\ AB,\ BC,\ CD,\ DE,\ 1E,\ 18,\ 29,\ 3A,\ 4B,\ 5C,\ 6D,\ 7E\}$ and $\mathcal{E}':=\{12,\ 23,\ 34,\ 45,\ 56,\ 67,\ 78,\ 89,\ 9A,\ AB,\ BC,\ CD,\ DE,\ 1E,\ 135,\ 246,\ 7B,\ 8C,\ 9D,\ AE\}$.

Theorem 19. *Let $m\geq 5$ be an integer. Then there exists a $(3,m,n)$-AC if and only if $4\leq n\leq\lfloor\frac{3m}{2}\rfloor$.*

Proof. The proof of the sufficiency of the condition is by induction on m. Let $m=5$. Then it is true (all necessary examples are given in subsection 4.5). Now, let $m\geq 6$. If m is even there is a $(3,m-1,n')$-AC if $n'\in[4,\lfloor\frac{3(m-1)}{2}\rfloor]=[4,\frac{3m}{2}-2]$. Using Lemmas 8 and 18 there is also a $(3,m,n)$-AC with $n\in[4,\frac{3m}{2}]$. If m is odd there is a $(3,m-1,n')$-AC if $n'\in[4,\lfloor\frac{3(m-1)}{2}\rfloor]=[4,\lfloor\frac{3m}{2}\rfloor-1]$. Using Lemmas 8 and 18 again, there is also a $(3,m,n)$-AC with $n\in[4,\lfloor\frac{3m}{2}\rfloor]$.
The necessity of the conditions follows from Corollary 7. □

4.4 $(m-1,m,n)$-AC

The author analysed $(m-1)$-regular antichains on $[m]$ in earlier work [2] and obtained the following result:

Theorem 20. *Let m be a positive integer. An $(m-1,m,n)$-AC exists if and only if $n\in[m+2,\binom{m}{2}-2]\cup\{m,\binom{m}{2}\}$.*

4.5 $(m-2,m,n)$-AC

With the help of Theorem 20 and Lemma 8, we can show that for every positive integer $m\geq 4$ an $(m-2,m,n)$-AC exists for $n\in[m+1,\binom{m-1}{2}-2]\cup\{m-1,\binom{m-1}{2}\}$. So the first natural question is what will happen within the gap? Does an $(m-2,m,m)$-AC and an $(m-2,m,\binom{m-1}{2}-1)$-AC exist? By Lemma 12 we know that an $(m-2,m,m)$-AC exists. The other case is answered in the following lemma.

Lemma 21. *Let $m\geq 5$ be an integer. Then there exists an $(m-2,m,\binom{m-1}{2}-1)$-AC $\mathcal{B}$.*

Proof. Let $m \geq 5$ be an integer. We construct an $(m-2, m, \binom{m-1}{2}-1)$-AC $\mathcal{B} \subseteq \binom{[m]}{2} \cup \binom{[m]}{3}$. Let a be the number of 2-sets and let b be the number of 3-sets in $\mathcal{B}$. To obtain a and b, we solve the following system of equations

$$(m-2)m \;=\; v(\mathcal{B}) \;=\; 2a+3b;$$

$$\binom{m-1}{2} - 1 \;=\; n \;=\; a+b.$$

We find that $a = \frac{m(m-5)}{2}$ and $b = m$. Define $\mathcal{C} := \{123, 234, \ldots, m12\} \subseteq \binom{[m]}{3}$ and $\mathcal{D} := \mathcal{D}(\mathcal{C}) := \{\{i,j\} \subset [m] \mid \forall C \in \mathcal{C} : \{i,j\} \subsetneq C\} \subseteq \binom{[m]}{2}$. It is easy to see that $\mathcal{C}$ is a collection of $b = m$ 3-sets with $|\mathcal{C}_i| = 3$ and $|\bigcup_{C \in \mathcal{C}_i} C| = 5$ for every element $i \in [m]$. Thus for every $i \in [m]$, we have $|\{\{i,j\} \in \mathcal{D} : j \in [m]\}| = m-5$. Hence $\mathcal{D}$ is a collection of $a = \frac{m(m-5)}{2}$ 2-sets. Define $\mathcal{B} := \mathcal{C} \cup \mathcal{D}$ which is obviously an $(m-2, m, \binom{m-1}{2}-1)$-AC. □

Remark. If $m \leq 4$ then there is no $(m-2, m, \binom{m-1}{2}-1)$-AC. This follows immediately from the previous subsections.

Corollary 22. *For every positive integer $m \geq 5$ there exists an $(m-2, m, n)$-AC if $n \in [m-1, \binom{m-1}{2}]$*

Table 1 provides a brief overview of all possible (m,n) such that an $(m-2, m, n)$-AC with $m \leq 6$ exists.

Theorem 23. *Let $m \geq 5$ be a positive integer. An $(m-2, m, n)$-AC exists if and only if $n \in [m-1, \lfloor\frac{(m-2)m}{2}\rfloor]$.*

Proof. According to Corollary 7 there is no $(m-2, m, n)$-AC with $n > \lfloor\frac{(m-2)m}{2}\rfloor$ or $n < m-1$.

The proof of the sufficiency of the condition is by induction on m. We start with $m = 5, 6$ (examples in Table 1), and the induction is from $m-2$ to m.

Let $m \geq 7$ be a positive integer. Using Corollary 22 we know there is an $(m-2, m, n)$-AC for every $n \in [m-1, \binom{m-1}{2}]$. By the induction hypothesis there is also an $(m-4, m-2, n')$-AC for every $n' \in [m-3, \lfloor\frac{(m-2)(m-4)}{2}\rfloor]$. Using the $(0,2,0)$-AC and Corollary 10 there is also an $(m-2, m, n)$-AC if $n \in [m-3+2(m-2), \lfloor\frac{(m-2)(m-4)}{2}\rfloor + 2(m-2)] = [3m-7, \lfloor\frac{(m-2)m}{2}\rfloor]$. As $3m-7 \leq \binom{m-1}{2}$ for every $m \geq 7$, it follows that an $(m-2, m, n)$-AC exists for every $n \in [m-1, \lfloor\frac{(m-2)m}{2}\rfloor]$. □

5 General case

In this section the main theorem of this paper is stated and proved, which is a necessary and sufficient condition for the existence of (k, m, n)-ACs with $k \leq m-2$.

m	n	example for an $(m-2, m, n)$-AC
2	0	$\emptyset$
3	1	123
	2	12, 3
	3	1, 2, 3
4	3	123, 124, 34
	4	13, 14, 23, 24
5	4	1234, 1235, 1245, 345
	5	123, 125, 145, 234, 345
	6	123, 125, 145, 24, 34, 35
	7	123, 14, 15, 24, 25, 34, 35
6	5	12345, 12346, 12356, 12456, 3456
	6	1234, 1236, 1256, 1456, 2345, 3456
	7	126, 1346, 1356, 1456, 234, 235, 245
	8	126, 136, 146, 156, 234, 235, 245, 345
	9	123, 126, 156, 234, 345, 456, 14, 25, 36
	10	126, 136, 146, 156, 23, 24, 25, 34, 35, 45
	11	123, 124, 15, 16, 25, 26, 34, 35, 36, 45, 46
	12	13, 14, 15, 16, 23, 24, 25, 26, 35, 36, 45, 46

Table 1: Examples of $(m-2, m, n)$-ACs with $m \le 6$.

Theorem 24. *Let k and m be positive integers with $2 \le k \le m-2$. A (k, m, n)-AC exists if and only if $k+1 \le n \le \frac{km}{2}$.*

***Proof*.** The theorem is proved by induction on k. For $k = 2, 3$ the theorem is true (see Corollary 17 and Theorem 19). Let $k > 3$ be an arbitrary fixed integer. The first step is to show that for given m there is a k-regular antichain on $[m]$ with many elements, so we show that for fixed $k \ge 4$ and given (m, n) with $\lfloor \frac{k(m-1)}{2} \rfloor < n \le \lfloor \frac{km}{2} \rfloor$, there is a (k, m, n)-AC.
Let $t := t(k, m, n) := \lfloor \frac{km}{2} \rfloor - n$, i.e. $0 \le t \le \lfloor \frac{k}{2} \rfloor$.
Let $s := s(k, m)$ with:

$$s \equiv m \bmod (k+1)$$
$$k+2 \le s \le 2k+2$$

and let $r := r(k, m)$ with:

$$r := \frac{m-s}{k+1}.$$

We partition the ground set $[m]$ into $r+1$ classes with the condition that each of the first r classes $P^1, P^2, \ldots, P^r$ contain exactly $k+1$ elements $p_1^j, p_2^j, \ldots, p_{k+1}^j$ $(j \in [r])$ and the last class P^{r+1} contains s elements. Without loss of generality, $P^{r+1} := [s]$. On every class $P^j (j \in [r])$ we construct a $\left(k, k+1, \binom{k+1}{2}\right)$-AC and on P^{r+1} we construct a $(k, s, \lfloor \frac{ks}{2} \rfloor - t)$-AC:

- $s = k + 2$
  We know that there is a $(k, k+2, \lfloor \frac{k(k+2)}{2} \rfloor - t)$-AC on $[s]$ (Theorem 23).

- $s = k + i$ $(i \in [3, k-2])$
  Choose a $(k-i, k, \lfloor \frac{k(k-i)}{2} \rfloor - t)$-AC on $[k]$ (induction hypothesis) and a $(0, i, 0)$-AC on $[k+1, s]$. Using Lemma 10 a $(k, k+i, \lfloor \frac{k(k-i)}{2} \rfloor - t + ki) = (k, k+i, \lfloor \frac{k(k+i)}{2} \rfloor - t) = (k, s, \lfloor \frac{ks}{2} \rfloor - t)$-AC on $[s]$ is obtained.

- $s = 2k - 1$
  Choose a $(1, k, \lfloor \frac{k}{2} \rfloor - t)$-AC on $[k]$ with the property that every set contains at least two elements (see also Lemma 15) and a $(0, k-1, 0)$-AC on $[k+1, s]$. Using Lemma 10 again, a $(k, k+(k-1), \lfloor \frac{k}{2} \rfloor - t + k(k-1)) = (k, 2k-1, \lfloor \frac{k(2k-1)}{2} \rfloor - t)$-AC is obtained.

- $s = 2k$
  In the case that $t \neq 1$, choose a $(k-1, k, \frac{k(k-1)}{2} - t)$-AC on $[k]$, a $(k-1, k, \frac{k(k-1)}{2})$-AC on $[k+1, s]$ and append the following sets $\{i, k+i\}$, $(i \in [k])$. That is a $(k, 2k, 2\frac{k(k-1)}{2} - t + k) = (k, 2k, k^2 - t) = (k, 2k, \lfloor \frac{ks}{2} \rfloor - t)$-AC.
  If $t = 1$ and k is even, choose a $(k-2, k, \frac{k(k-2)}{2} - 1)$-AC on $[k]$, a $(k-2, k, \frac{k(k-2)}{2})$-AC on $[k+1, s]$ and append the following sets $\{i, k+i\}$ for $i \in [k]$, $\{j, k+j+1\}$ for $j \in [k-1]$, $\{k, k+1\}$. And if k is odd, choose a $(k-2, k, \frac{k(k-2)}{2} - \frac{1}{2})$-AC on $[k]$, a $(k-2, k, \frac{k(k-2)}{2} - \frac{1}{2})$-AC on $[k+1, s]$ and append the same 2-sets as before. In both cases a $(k, 2k, \lfloor \frac{ks}{2} \rfloor - 1)$-AC is obtained.

- $s = 2k + 1$
  In the case that $t \neq 1$, choose a $(k-1, k, \frac{k(k-1)}{2} - t)$-AC on $[k]$, a $(k-1, k, \frac{k(k-1)}{2})$-AC on $[k+1, 2k]$ and a $(0, 1, 0)$-AC on $\{s\}$. If k is even we append the following sets $\{i, s\}$, $\{k+i, s\}$ for $i \in [\frac{k}{2}]$ and $\{j, k+j\}$ for $\frac{k}{2} < j \leq k$. That is a $(k, 2k+1, 2 \cdot \frac{k(k-1)}{2} - t + 2 \cdot \frac{k}{2} + \frac{k}{2}) = (k, 2k+1, k^2 + \frac{k}{2} - t) = (k, 2k+1, \frac{2k^2+k}{2} - t) = (k, 2k+1, \frac{sk}{2} - t)$-AC. If k is odd we append $\{i, s\}$, $\{k+i, s\}$ for $i \in [\frac{k-1}{2}]$, $\{\frac{k+1}{2}, k + \frac{k+1}{2}, s\}$ and $\{j, k+j\}$ for $\frac{k+1}{2} < j \leq k$. That is a $(k, 2k+1, 2\frac{k(k-1)}{2} - t + 2 \cdot \frac{k-1}{2} + 1 + \frac{k-1}{2}) = (k, 2k+1, k^2 - k + k - 1 + 1 + \frac{k-1}{2} - t) = (k, 2k+1, \frac{2k^2+k-1}{2} - t) = (k, 2k+1, \frac{sk-1}{2} - t) = (k, 2k+1, \lfloor \frac{sk}{2} \rfloor - t)$-AC.

  If $t = 1$ and k is even, choose a $(k-2, k, \frac{k(k-2)}{2} - 1)$-AC on $[k]$, a $(k-2, k, \frac{k(k-2)}{2})$-AC on $[k+1, s-1]$, a $(0, 1, 0)$-AC on $\{s\}$ and append the same sets like in the case k even, $t \neq 1$ as well as $\{j, k+j+1\}$ for $j \in [k-1]$ and $\{k, k+1\}$. The case when k is odd is done in a similar way to the case when k is even. In both cases a $(k, 2k, \lfloor \frac{ks}{2} \rfloor - 1)$-AC is obtained.

- $s = 2k + 2$
  Choose a $(k, k+1, \binom{k+1}{2})$-AC on $[k+1]$ and a $(k, k+1, \binom{k+1}{2})$-AC on $[k+2, s]$. By Lemma 11, this gives a (k, s, i)-AC with $i \in [\binom{k+1}{2}, k(k+1)] = [\binom{k+1}{2}, \frac{ks}{2}]$. As $\frac{ks}{2} - \binom{k+1}{2} = \binom{k+1}{2} > \frac{k}{2}$, a $(k, s, \frac{ks}{2} - t)$-AC on $[s]$ is obtained for every possible value t.

Using Lemma 11, a k-regular AC on $[m]$ of size n is obtained with

$$\begin{aligned} n =& r\overbrace{\binom{k+1}{2}}^{even} + \left\lfloor \frac{ks}{2} \right\rfloor - t \\ =& \frac{k(m-s)}{2} + \left\lfloor \frac{ks}{2} \right\rfloor - t \\ =& \left\lfloor \frac{km}{2} \right\rfloor - t. \end{aligned}$$

This finishes the first step.

Now use the same style of argument as used in the proofs of Theorem 16 or 19 with fixed k and induction on m. If $m = k+2$ Theorem 24 is correct (using Theorem 23). So let $m > k+2$ be fixed. For $k+1 \leq n \leq \lfloor \frac{k(m-1)}{2} \rfloor$ we know that a $(k, m-1, n)$-AC exists. Using Lemma 8, a (k, m, n)-AC is obtained for these values of n. Together with the first step, this proves that a (k, m, n)-AC with $2 \leq k \leq m-2$ exists if $k+1 \leq n \leq \frac{km}{2}$.

The necessity of the conditions follows from Corollary 7. □

6 Outlook

In connection with [2], this paper answers completely the question for which parameters k, m and n a (k, m, n)-AC with $k < m$ exists. But as we mentioned in the introduction, in general it is an open problem, if a (k, m, n)-AC exists for given values $k \geq m$ and n. Solving this problem is equivalent to determining all values $R(n, k)$ for given integers n and k.

Perhaps, this problem can be solved in two independent steps. We conjecture that for every (k, m, n)-AC $\mathcal{B}$ there also exists a flat (k, m, n)-AC $\mathcal{B}'$, where flat means that $\mathcal{B}' \subseteq \binom{[m]}{l} \cup \binom{[m]}{l+1}$ for some nonnegative integer l. If this conjecture is correct, then it is sufficient to consider flat antichains and thus, it is much easier to prove if a (k, m, n)-AC exists or not.

Another possibility, how to attack this problem, is also mentioned in [3], in the fifth section "*A related problem.*"

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