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On locating-dominating codes for locating large numbers of vertices in the infinite king grid

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Abstract

Assume that $G = (V, E)$ is an undirected graph with vertex set V and edge set E. The ball $B_r(v)$ denotes the vertices within graphical distance r from v. A subset $C \subseteq V$ is called an $(r, \leq l)$-locating-dominating code of type B if the sets $I_r(F) = \bigcup_{v \in F}(B_r(v) \cap C)$ are distinct for all subsets $F \subseteq V \setminus C$ with at most l vertices. A subset $C \subseteq V$ is an $(r, \leq l)$-locating-dominating code of type A if sets $I_r(F_1)$ and $I_r(F_2)$ are distinct for all subsets $F_1, F_2 \subseteq V$ where $F_1 \neq F_2$, $F_1 \cap C = F_2 \cap C$ and $|F_1|, |F_2| \leq l$. We study $(r, \leq l)$-locating-dominating codes in the infinite king grid when $r \geq 1$ and $l \geq 3$. The infinite king grid is the graph with vertex set $\mathbb{Z}^2$ and edge set $\{\{(x_1, y_1), (x_2, y_2)\} \mid |x_1 - x_2| \leq 1, \ |y_1 - y_2| \leq 1\}$.

1 Introduction

Let $G = (V, E)$ be an undirected graph where V is a vertex set and E an edge set. Denote by $d(u, v)$ the distance between two vertices u and v, i.e. the number of edges on any shortest path between u and v. The ball with center v and radius r is

$$B_r(v) = \{u \in V \mid d(u, v) \leq r\}.$$

We call any set C with $C \subseteq V$ a *code.* The vertices of C are called *codewords.* In particular, C is an *$(r, \leq l)$-locating-dominating code of type B* if the sets

$$I_r(F) = \left(\bigcup_{v \in F} B_r(v) \right) \cap C = B_r(F) \cap C$$

are distinct for all subsets $F \subseteq V \setminus C$ with at most l non-codewords. The code is an *$(r, \leq l)$-locating-dominating code of type A* if $I_r(F_1) \neq I_r(F_2)$ for all subsets

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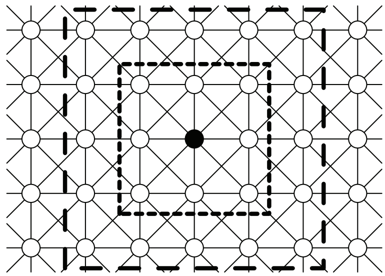


Figure 1: A part of the infinite king grid. The vertices within distance one and two from the black dot are surrounded by the dashed lines.

$F_1 \subseteq V$ and $F_2 \subseteq V$ where $|F_1| \leq l$, $|F_2| \leq l$, $F_1 \neq F_2$ and $F_1 \cap C = F_2 \cap C$. We call a locating-dominating code of type A (or type B, respectively) an LDA code (or an LDB code, respectively) for short. An LDA code is automatically an LDB code and if $l = 1$, then the definitions of LDA and LDB codes are equivalent. Also, an $(r, \leq l)$-LDA code (or -LDB code) is automatically an $(r, \leq k)$-LDA code (or -LDB code, respectively) if $k \leq l$.

Moreover we say that a codeword is a *special codeword* if exactly one vertex in its r-neighbourhood is a non-codeword. In particular in the case of LDB codes, a special codeword c is in an identifying set $I_r(F)$ if and only if the only non-codeword in the r-neighbourhood of c is in the set $F \subseteq V \setminus C$.

We study codes in the infinite king grid. The infinite king grid is the graph where $V = \mathbb{Z} \times \mathbb{Z}$ and vertices $u = (u_x, u_y)$ and $v = (v_x, v_y)$ are adjacent if $|u_x - v_x| \leq 1$ and $|u_y - v_y| \leq 1$. Thus vertices u and v are neighbours if the Euclidean distance between u and v is 1 or $\sqrt{2}$.

The density of $C \subseteq \mathbb{Z}^2$ is

$$D(C) = \limsup_{n \to \infty} \frac{|C \cap B_n((0,0))|}{|B_n((0,0))|},$$

where $|C \cap B_n((0,0))|$ is the number of codewords in the ball $\{(x,y) \mid |x| \leq n, |y| \leq n\}$ and $|B_n((0,0))|$ is the number of all vertices in the ball. We also define $B_n((0,0)) = Q_n$. We search for the minimum density of locating-dominating codes of both types for given r and l in the infinite king grid.

Next, we define that a non-codeword is *isolated* if all the four vertices at Euclidean distance 1 from it are codewords. For example, the circled non-codewords in Figures 2(c) and 2(d) on page 132 are isolated non-codewords.

Moreover, we say that vertices $u_1, u_2, \ldots, u_k$ are *consecutive* in the infinite king grid if all of them have the same x-coordinate (or y-coordinate) and the y-coordinates (or x-coordinates, respectively) are consecutive. Finally, we define that $v_1, v_2, \ldots, v_k$ are k *successive non-codeword neighbours* of (x, y) if they are k successive non-

	$r=1$		$r=2$		$r\geq 3$	
$l=1$	$\frac{1}{5}$	[6]	$\frac{1}{10}\leq D\leq\frac{1}{8}$	[2], [10]	$\frac{1}{4r+2}\leq D\leq\frac{1}{4r}$ if $2\mid r$ $\frac{1}{4r+2}\leq D\leq\frac{1}{4r+\frac{2}{r+1}}$ if $2\nmid r$	[1], [10]
$l=2$	$\frac{1}{3}$	[11]	$\frac{1}{5}\leq D\leq\frac{1}{4}$	[11]	$\frac{1}{6}\leq D\leq\frac{r+1}{6r+3}$ if $r\equiv 0,2,5 \pmod 6$ $\frac{1}{6}\leq D\leq\frac{2r+3}{12r+6}$ if $r\equiv 1,3,4 \pmod 6$	[11]
$l\geq 3$	1	a	1	a	1	a

Table 1: The known lower and upper bounds for the density of LDA codes. Reference a refers to Theorem 1.

codewords of cycle

$$(x+1,y),(x+1,y+1),(x,y+1),(x-1,y+1),$$
$$(x-1,y),(x-1,y-1),(x,y-1),(x+1,y-1).$$

Locating-dominating codes (of types A and B) were introduced in the late of 1980s by Slater [13] and [14] when $l=1$ and in the 2000s by Honkala, Laihonen and Ranto [7] for general l. A motivation for such codes is a safeguard analysis of a facility [13].

We study $(r,\leq l)$-locating-dominating codes for large l. The emphasis of this paper is on LDB codes since a code is an LDA code for $l\geq 3$ only if there are no non-codewords (Theorem 1). For small l, LDA and LDB codes have been studied in the papers [6] and [10]–[12] and Tables 1 and 2 summarize what is known about the density of $(r,\leq l)$-locating-dominating codes of type A and B in the infinite king grid. Here, the upper bound means that there exists such a code with that density and the lower bound means that the density of every such code is at least the value given in the table.

Papers [1]–[4] and [9] study $(r,\leq l)$-identifying codes which is a closely related class of codes in the infinite king grid. More papers on such codes in the infinite king grid and many other graphs can be found in the web bibliography [15].

2 Lower bounds

Theorem 1. *Assume that $r\in\mathbb{N}$ and $l\geq 3$. Then C is an $(r,\leq l)$-LDA code in the infinite king grid if and only if C contains all vertices in the infinite king grid.*

Proof. Clearly, the code containing all vertices is an $(r,\leq l)$-LDA code for all r and l. On the other hand, since $B_r((x,y))\subseteq B_r(\{(x-1,y),(x+1,y)\})$, we see that $I_r(\{(x-1,y),(x+1,y)\})=I_r(\{(x-1,y),(x,y),(x+1,y)\})$. Thus, if C is an $(r,\leq l)$-LDA code, then $\{(x-1,y),(x+1,y)\}\cap C\neq\{(x-1,y),(x,y),(x+1,y)\}\cap C$ and so $(x,y)\in C$ for all $(x,y)\in\mathbb{Z}^2$. $\square$

	$r=1$		$r=2$		$r \geq 3$	
$l=1$	$\frac{1}{5}$	[6]	$\frac{1}{10} \leq D \leq \frac{1}{8}$	[2], [10]	$\frac{1}{4r+2} \leq D \leq \frac{1}{4r}$ if $2 \mid r$ $\frac{1}{4r+2} \leq D \leq \frac{1}{4r+\frac{2}{r+1}}$ if $2 \nmid r$	[1], [10]
$l=2$	$\frac{1}{3}$	[11]	$\frac{1}{6} \leq D \leq \frac{1}{4}$	[11]	$\frac{1}{6} \leq D \leq \frac{r+1}{6r+3}$ if $r \equiv 0,2,5 \ (6)$ $\frac{1}{6} \leq D \leq \frac{2r+3}{12r+6}$ if $r \equiv 1,3,4 \ (6)$	[11]
$l=3$	$\frac{3}{5}$	[12]	$\frac{2}{3}$	[12]	$\frac{r}{r+1}$	[12]
$4 \leq l \leq 4r$	$\frac{2}{3}$	b, d	$\frac{2}{3} \leq D \leq \frac{4}{5}$	[12], d	$\frac{r}{r+1} \leq D \leq \frac{2r}{2r+1}$	[12], d
$l > 4r$	$\frac{2}{3}$	b, d	$\frac{4}{5}$	c, d	$\frac{2r}{2r+1}$	c, d

Table 2: The known lower and upper bounds for the density of LDB codes. References b, c and d refer to Theorems 5, 9 and 10, respectively.

Now we have shown when the code is an $(r, \leq l)$-LDA code for any $r \in \mathbb{Z}_+$ and $l \geq 3$. Then, we shall consider only LDB codes in the future.

2.1 LDB codes when $r = 1$

Lemma 2. *[12] If C is a $(1, \leq 4)$-LDB code in the infinite king grid, then at most two of four consecutive vertices can be non-codewords, which also means that at least two of four consecutive vertices must be codewords.*

Proof. If there were three non-codewords among four consecutive vertices, then the identifying set of the set of all these three non-codewords would be the same as the identifying set of the set of the outermost non-codewords. □

Corollary 3. *If C is a $(1, \leq 4)$-LDB code in the infinite king grid, then at least one of three consecutive vertices must be a codeword.*

Corollary 4. *If C is a $(1, \leq 4)$-LDB code in the infinite king grid, then there can exist at most three successive non-codeword neighbours.*

In the next proof, we use an averaging method, which is often called voting or discharging. The idea in the method is the following: Initially, each codeword has some fixed number (t_1) of votes. After the initial state, we transfer votes from vertices to others, i.e., we add a certain number of votes to some vertex and subtract the same number votes from another vertex at the same time. Thus, the total number of votes does not change when votes are transferred. Finally, we show that every vertex has at least t_2 votes. This then implies that the density of the code is $\frac{t_2}{t_1}$.

In what follows, we say that a non-codeword v or the neighbourhood of v *covers* the vertex u if $u \in B_r(v)$.

Theorem 5. *The density of a $(1, \leq 4)$-LDB code is at least $\frac{2}{3}$ in the infinite king grid.*

Proof. Let C be a $(1, \leq 4)$-LDB code.

First, every codeword gives one vote to itself and all neighbours. Then, we transfer two more votes from each special codeword to the unique non-codeword in its neighbourhood and half a vote from each codeword with exactly six neighbours in the code to the two non-codewords in its neighbourhood. This is called **Rule 1**.

Now, every codeword with at least five neighbours in the code has at least six votes and we also show in Step 1 that every non-codeword has at least six votes. However, codewords can still have fewer than six votes, but in that case we can transfer extra votes from non-codewords to the codewords by Rule 2 (which will be defined later).

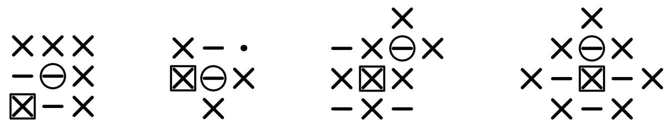
Step 1. Every non-codeword has at least six votes after Rule 1.

Let v be a non-codeword. First, assume that v has a special codeword c in its neighbourhood. Now v has at least four codewords in its neighbourhood. Indeed, the intersection of the neighbourhood of v and the neighbourhood of c contains at least four vertices and apart from v these (at least three) vertices have to be in the code. Furthermore, there are three consecutive vertices that are in the neighbourhood of v, but are not in the neighbourhood of c, and at least one of these consecutive vertices must be in the code by Corollary 3. Thus, there are at least four codewords in the neighbourhood of v and at least one of them is a special codeword. Then, v has at least six votes.

Second, we assume that v has no special codeword in its neighbourhood. Assume further that v is not isolated, i.e., there is another non-codeword u_1 at Euclidean distance one from v. Let v' be the unique vertex with Euclidean distance one from v and two from u_1. Now, v' has to be a codeword or else there would be three consecutive non-codewords, which contradicts Corollary 3. Then v' has another non-codeword neighbour $u_2 \neq v$ since we assume that v does not have a special codeword in its neighbourhood. Now, u_1 and u_2 cover the neighbourhood of v except for one vertex v'' (or zero vertices, but then $B_1(\{v, u_1, u_2\}) = B_1(\{u_1, u_2\})$). Again, since v has no special codeword in its neighbourhood, then v'' has another non-codeword $u_3 \neq v$ in the neighbourhood of v''. If v'' is a non-codeword, then we choose $v'' = u_3$. Thus $B_1(v) \subseteq B_1(\{u_1, u_2, u_3\})$ and so C is not a $(1, \leq 4)$-LDB code. In particular, **non-codewords without special codeword neighbours are isolated.**

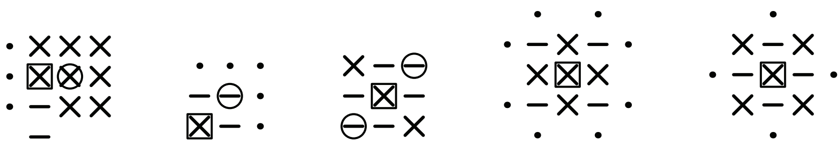
Finally, we assume that v is isolated and it has no special codewords in its neighbourhood. If v has at most five codewords in its neighbourhood, then at least three corners in its neighbourhood have to be non-codewords. Let u_1 and u_2 be two of the non-codewords that are in the opposite corners and the third non-codeword be u_3. Let u_4 ($\neq v$) be a non-codeword that covers the fourth corner of the neighbourhood of v. (Possibly, u_4 is the fourth corner.) Then, $I_1(v) \subseteq B_1(\{u_1, u_2, u_4\})$ and again C would not be a $(1, \leq 4)$-LDB code. Hence, v has at least six neighbours in the code and so also at least six votes.

Next, we transfer votes from non-codewords to codewords as in Figure 2 (symmetries such as reflections allowed). The amount of transferred votes is given in Figure



(a) 2 votes (b) 1 vote (c) $\frac{1}{4}$ of a vote (d) $\frac{1}{4}$ of a vote

Figure 2: The circled non-codeword gives away 2, 1 or $\frac{1}{4}$ votes to squared codeword. The crosses are codewords and the lines are non-codewords. The black dot may be a codeword or a non-codeword.



(a) (b) (c) (d) (e)

Figure 3: Some cases of the proof of Theorem 5

2. This is called **Rule 2**.

Step 2. Every codeword now has at least six votes after Rule 2.

First, we observe that if a codeword c has two consecutive non-codewords u and v with Euclidean distance one and two, respectively, from c, then c has six votes after Rule 1. Indeed, if c is a special codeword, the claim is clear; so assume that it is not. Because u is not isolated, it must have a special codeword in its neighbourhood (cf. Step 1), and such a special codeword can only be either of two vertices at Euclidean distance one from c and $\sqrt{2}$ from u. Now, we can without loss of generality consider the constellation in Figure 3(a). By Corollary 3, at least one of the black dots has to be a codeword, and c has at least six votes.

Next, we assume that a codeword c has at least three successive non-codeword neighbours and by the Corollary 4, there cannot be more than three successive codewords. See Figure 3(b). If c has at least six votes after Rule 1, then the claim is clear. Therefore, we also assume that c has fewer than six votes after Rule 1. Then, the black dots must be codewords in Figure 3(b). Indeed, the leftmost and lowest black dots are codewords by the observation made in the previous paragraph. Moreover, the black dot in the top right corner has to be a special codeword, because it is the only vertex in the neighbourhood of the circled non-codeword that can be a special codeword (and since the circled codeword is not isolated, there has to be one). Thus, the squared codeword gets two votes by Figure 2(a) in Rule 2. Now, if c has at least four votes after Rule 1, then it has enough votes after Rule 2. Assume then that c has fewer than four votes after Rule 1, then the neighbourhood of c has to be as the constellation of Figure 3(c) or its rotation. Then c gets two votes from both circled codewords and it has seven votes after Rule 2.

Next, we assume that there are two (but not three) successive non-codeword neighbours of the codeword c. Moreover, we assume that c has fewer than six votes after Rule 1. Then, the neighbourhood of the codeword is as in Figure 2(b) or its reflection or rotation where c is the squared codeword. Indeed, the cross on the right has to be a codeword by the observation made at the beginning of Step 2. Thus, the squared codeword gets one vote from the circled non-codeword in Rule 2. Now, if c has at least five votes after Rule 1, then we are done. If c has fewer votes, then there are two pairs of consecutive non-codewords in the neighbourhood of c, and then c gets two votes – one vote from two non-codewords each – by Rule 2. Anyway, c has at least four votes after Rule 1 (or else c has three successive non-codeword neighbours), and six votes after Rule 2.

Assume finally that the codeword c has no two successive non-codeword neighbours. Now there are only two possible constellations given in Figures 3(d) and 3(e) where c has fewer than six votes after Rule 1. All the black dots have to be codewords since at most two of four consecutive vertices can be non-codewords. Now, each of the four non-codewords gives $\frac{1}{4}$ of a vote to c by Figures 2(c) or 2(d) of Rule 2 and so c has enough votes.

Step 3. Every non-codeword still has at least six votes after Rule 2.

First, the circled non-codeword in Figure 2(a) has to have at least eight votes after Rule 1, since it gets one vote from all six codewords and two more votes from a special codeword. The non-codeword must have a special codeword neighbour because it is not isolated. Thus, the circled non-codeword has at least six votes after Rule 1 since it does not give votes to other neighbours than the squared codeword.

The circled non-codeword in Figure 2(b) can only give a vote to the codewords to its left and right. If it gives one vote to the codeword on the right, then the black dot has to also be a codeword. In any case, one of the three lowest vertices in the neighbourhood of the circled non-codeword is a special codeword for the same reason as in the previous case. Then the non-codeword has at least eight votes after Rule 1, if it gives a vote to two codewords, or else at least seven votes after Rule 1. Anyway, it has at least six votes after Rule 2.

Next, we show that the circled non-codeword in Figures 2(c) or 2(d) has at least six votes after the voting. The circled non-codeword can give $\frac{1}{4}$ of a vote to at most four codewords i.e. in total one vote. Then, if the circled non-codeword has at least seven votes after Rule 1, the claim is true. We still assume that the squared codeword is in the origin, so we can use coordinates in the proof.

We show first that non-codeword in $(1,1)$ in Figure 2(c) has at least six votes after Rule 2. Now, $(2,2)$ has to be a codeword since $B_1(\{(1,1)\}) \subseteq B_1(\{(-1,1),(1,-1),(2,2)\})$. Then, $(0,2)$ and $(2,0)$ are non-codewords or else $(1,1)$ has enough votes. Now, $(1,3)$ and $(2,3)$ are also codewords because the union of the neighbourhoods of $(-1,1)$, $(2,0)$ and $(1,3)$ (or $(2,3)$) covers the neighbourhood of $(1,1)$. By symmetry, $(3,1)$ and $(3,2)$ are also codewords. Thus, $(1,1)$ has at least $6\frac{1}{2}$ votes after Rule 1 since it gets in total $1\frac{1}{2}$ votes from $(2,2)$. Moreover, it gives $\frac{1}{4}$ of a vote to only one codeword in Rule 2. Hence, it has at least six votes after Rule 2.

Finally, at least five neighbours of the non-codeword $(0,1)$ in Figure 2(d) have to be codewords. Otherwise three of the non-codewords in the neighbourhood cover all codewords in the neighbourhood. If $(0,1)$ has a special codeword in the neighbourhood, then it has enough votes; so assume that this is not the case. Now, $(-1,2) \in C$ and $(1,2) \in C$ by Step 1. Moreover, $(0,3) \in C$ since $I_1(0,1) \subseteq B_1(\{(0,3),(-1,0),(1,0)\})$. However, $(0,2)$ is not a special codeword, and so $(-1,3)$ or $(1,3)$ is a non-codeword. Without loss of generality, we assume that $(-1,3) \notin C$. Moreover, $(2,1) \in C$ and $(2,2) \in C$: otherwise, the neighbourhoods of one of them, $(-1,0)$ and $(-1,3)$ cover all the codewords in the neighbourhood of $(0,1)$. Thus, $(0,1)$ has at least $6\frac{1}{2}$ votes after Rule 1 since $(1,1)$ in Figure 2(d) gives in total $1\frac{1}{2}$ votes to it. Furthermore, $(0,1)$ gives $\frac{1}{4}$ of a vote only to one codeword in Rule 2.

Hence, all vertices in Q_n have at least six votes and every vertex can get votes only from itself and the vertices within graphical distance two.[1] Then all vertices in Q_{n-2} have at least six votes from vertices in Q_n. On the other hand, the total number of votes given by the vertices in Q_n is $9 \cdot |C \cap Q_n|$. Thus we have

$$9 \cdot |C \cap Q_n| \geq 6 \cdot |Q_{n-2}| = 6 \cdot (|Q_n| - 16n + 8)$$

i.e.

$$D(C) = \limsup_{n\to\infty} \frac{|C \cap Q_n|}{|Q_n|} \geq \limsup_{n\to\infty} \left(\frac{6}{9} - \frac{96n-48}{9 \cdot |Q_n|}\right) = \frac{2}{3}.$$

□

2.2 LDB codes when $r \geq 2$

The next lemma is valid whether or not the code C is an LDB code. Later we shall nevertheless see that the code which satisfies the assumptions of the lemma is always an $(r, \leq l)$-LDB code for all l.

Lemma 6. *Let C be a code in the infinite king grid. If every non-codeword has a special codeword in its r-neighbourhood, then the density of C is at least $\frac{2r}{2r+1}$.*

Proof. Let $c = (a,b)$ be a special codeword and $v = (x,y)$ the unique non-codeword in the neighbourhood of c. Then we mark by v all the vertices

$$J_H(c) = \begin{cases} (a-r,y),(a-r+1,y),\ldots,(a,y) & \text{if } x \leq a, \\ (x-r,y),(x-r+1,y),\ldots,(x,y) & \text{if } x > a \end{cases} \tag{1}$$

and

$$J_V(c) = \begin{cases} (a,y),(a,y+1),\ldots,(a,y+r) & \text{if } y \leq b, \\ (a,b),(a,b+1),\ldots,(a,b+r) & \text{if } y > b. \end{cases} \tag{2}$$

[1]Codewords give votes to the vertices in its neighbourhood by the first rule and votes can be transferred from the vertices which give them by Rule 1 to vertices in their neighbours by the second rules. Therefore, votes stay within distance two from the codewords.

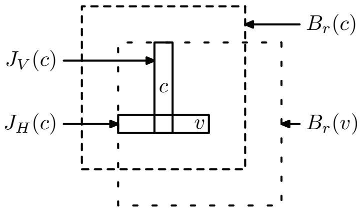


Figure 4: Sets $J_H(c)$ and $J_V(c)$, when $a \leq x$ and $b \geq y$.

Since c is in the r-neighbourhood of v i.e. $|a-x| \leq r$ and $|b-y| \leq r$, then c and v are marked by v. Moreover, both $J_H(c)$ and $J_V(c)$ contain $r+1$ vertices and exactly one of these belongs to both sets. Then $2r+1$ vertices are marked by v. If v has more special codewords in its r-neighbourhood, then there can be more than $2r+1$ vertices marked by v. Furthermore, all vertices in $J_H(c)$ and $J_V(c)$ are in the r-neighbourhood of special codeword c. Therefore v is the only non-codeword which has been marked by v.

Now, we show that a codeword cannot be marked by two non-codewords. Assume that $a \leq x$ and $b \geq y$. (Three other cases are proved in the same way.) See Figure 4. Assume to the contrary that $c' = (a', b')$ is a special codeword in the r-neighbourhood of a non-codeword $v' = (x', y') \neq v$ and w is a vertex which is marked by both v' (with c') and v (with c). Then $d(v, c') > r$ and $d(v', c) > r$ since the r-neighbourhood of c' or c contains only one non-codeword.

If $w \in J_H(c)$, then $b' \in [y-r, y+r]$. Moreover, $a' \in [x-2r, x+r]$. However, a' has to be smaller than $x-r$ since $d(v, c') > r$. Thus $J_V(c')$ cannot contain w (since the x-coordinates of vertices in $J_V(c')$ are a'). Then $J_H(c)$ and $J_H(c')$ have to intersect, but this is possible only if $y' = y$ and $\max\{x', a'\} \geq x-r$. However, this is impossible since $a' < x-r$ (observed above) and $x' < a-r \leq x-r$ because $x' \leq a'+r < x \leq a+r$ and $x' \notin [a-r, a+r]$ where the first inequality follows from the fact $d(v', c') \leq r$ and the latter condition from the fact $d(v', c) > r$.

Thus, w has to be in $J_V(c)$. Now, $x' \in [a-r, a+r]$ and $y' \in [y-r, y+2r]$. However, $y' \notin [b-r, b+r]$ since $d(v', c) > r$. Then $J_H(c')$ cannot contain w. Thus, w is in $J_V(c')$. This is possible only if $a' = a$. Then $b' \notin [y-r, y+r]$ because $d(v, c') > r$. If $b' < y-r$, then $d(c', u) > r$ for all $u \in J_V(c)$. Then $b' > y+r$, but now $J_V(c)$ and $J_V(c')$ can intersect only if $y' \leq y+r$, which is impossible since $d(v', c) > r$ and $d(v', c') \leq r$.

Hence every non-codeword in Q_n has marked at least $2r$ codewords in Q_{n+r} and every codeword has been marked by at most one non-codeword. Therefore we have

$$2r|Q_n \setminus C| \leq |C \cap Q_{n+r}|$$

and so

$$\frac{|C \cap Q_{n+r}|}{|Q_n|} \geq \frac{|C \cap Q_{n+r}|}{|C \cap Q_{n+r}| + |Q_n \setminus C|} = \frac{1}{1 + \frac{|Q_n \setminus C|}{|C \cap Q_{n+r}|}} \geq \frac{1}{1 + \frac{1}{2r}} = \frac{2r}{2r+1}.$$

Then

$$\frac{|C\cap Q_n|}{|Q_n|}\geq\frac{|C\cap Q_{n+r}|}{|Q_n|}-\frac{|Q_{n+r}|-|Q_n|}{|Q_n|}\geq\frac{2r}{2r+1}-\frac{4r(2n+r+1)}{(2n+1)^2}$$

and so

$$D(C)=\limsup_{n\to\infty}\frac{|C\cap Q_n|}{|Q_n|}\geq\frac{2r}{2r+1}.$$

□

Lemma 7. *A code is an $(r,\leq l)$-LDB code in the infinite king grid when $l>4r$ and $r\geq 2$ if and only if every non-codeword has a special codeword in its r-neighbourhood.*

Proof. First, if every non-codeword has a special codeword in its neighbourhood, then the code is clearly $(r,\leq l)$-LDB code (even for $r=1$).

Assume then that $O=(0,0)$ is a non-codeword with no special codewords in its r-neighbourhood. In particular, every vertex — codeword or not — in $B_r(O)\setminus\{O\}$ has at least two non-codewords (one of which is O) within distance r. We show that all codewords in $B_r(O)$ can always be covered by $4r$ non-codewords other than O, or fewer. Then the identifying set of these $4r$ non-codewords is the same as the identifying set of these non-codewords and O.

First, we observe that if two balls of radius r have a non-empty intersection in the infinite king grid, then at least one of the four corners must be in the intersection. Moreover, every row (column, respectively) of $B_r(O)\setminus\{O\}$ can be covered by (at most) two non-codewords none of which is O. Indeed, there exists a non-codeword $(a,b)\neq O$ covering $(-r,y),(-r+1,y),\ldots,(x,y)\in B_r(O)$, with x somewhere in $[-r,r]$. If $x=r$, we are done; if $x<r$ and if no non-codeword other than O covers $(-r,y)$ and $(x+1,y)$, then there has to be another non-codeword $(a',b')\neq O$ which covers $(x+1,y)$ (or $(x+2,y)$ if $(x+1,y)=O$). Now, (a',b') has to cover (r,y) as well since (a',b') covers $2r+1$ consecutive vertices but not $(-r,y)$.

First, we assume that a non-codeword $v\neq O$ covers $(-1,0)$ and $(1,0)$. Furthermore, we can assume without loss of generality that, among the four corners, it covers (r,r). Now, vertices in $B_r(O)$ which have not been covered by v can be divided into at most r rows and $r-1$ columns. These vertices can be covered by at most $2r+2(r-1)$ non-codewords none of which is O. Then $4r-1$ non-codewords (none of which is O) cover all codewords in $B_r(O)$.

Next, we assume that $(-1,0)$ and $(1,0)$ ($(0,-1)$ and $(0,1)$, respectively) are not covered by the same non-codeword other than O. In particular, only corners can be non-codewords in $B_r(O)\setminus\{O\}$. Assume then that a non-codeword $v\neq O$ covers $(-1,0)$ and a non-codeword $u\neq O$ covers $(1,0)$. If v and u cover adjacent corners of $B_r(O)$ (for example $(-r,r)$ and (r,r)), then $B_r(O)\setminus(B_r(u)\cup B_r(v))$ can be divided into at most one column and r rows. Thus, codewords in $B_r(O)$ can be covered by at most $2r+4$ non-codewords other than O, and $2r+4\leq 4r$ when $r\geq 2$.

Now, we can assume that every non-codeword except O which covers $(-1,0)$ ($(1,0)$, $(0,-1)$ or $(0,1)$, respectively) always covers the same corner of $B_r(O)$ and the non-codewords $v \neq O$ and $u \neq O$ ($v' \neq O$ and $u' \neq O$, respectively) which cover $(-1,0)$ and $(1,0)$ ($(0,-1)$ and $(0,1)$, respectively) cover the opposite corners of $B_r(O)$. If v and v' cover adjacent corners, then v, v', u, and u' cover all four corners and also all codewords in $B_r(O)$.

Finally, we can assume without loss of generality that v and v' cover $(-r,-r)$ and u and u' cover (r,r). Let $w \neq O$ and $w' \neq O$ be two non-codewords which cover $(-1,1)$ and $(1,-1)$, respectively. If w (or w', respectively) covers $(-r,-r)$ or (r,r), then we can substitute one of v and u' to w (or one of v' and u to w', respectively). Now, there are at most $r-1$ columns or rows in $\{(a,b) \in B_r(O) \mid a \leq 0, b \geq 0\}$ (or $\{(a,b) \in B_r(O) \mid a \geq 0, b \leq 0\}$, respectively) which cannot be covered by four of v, v', u, u', w, and w'. So all in all, if both w and w' cover at least one of $(-r,-r)$ and (r,r), we have at most $2(r-1)+2(r-1)+4=4r$ non-codewords none of which is O covering $B_r(O)$.

If w (or w') covers $(-r,r)$ (or $(r,-r)$, respectively)[2], then w (or w') covers all vertices in $\{(a,b) \in B_r(O) \mid a \leq 0, b \geq 0\}$ (or $\{(a,b) \in B_r(O) \mid a \geq 0, b \leq 0\}$, respectively). Hence, $B_r(O)$ can be covered by at most $\max\{6, 5+2(r-1)\} \leq 4r$ (when $r \geq 2$) non-codewords, none of which is $O=(0,0)$. ☐

Remark 8. The previous lemma is also valid for $(1, \leq l)$-LDB codes in the infinite king grid if $l \geq 7$.

Theorem 9. *The density of an $(r, \leq l)$-LDB code in the infinite king grid is at least $\frac{2r}{2r+1}$ when $l > 4r$ and $r \geq 1$.*

Proof. The claim is proved in Theorem 5 when $r=1$. Moreover, the claim follows from Lemmas 6 and 7 when $r \geq 2$. ☐

3 Constructions

Theorem 10. *There exists an $(r, \leq l)$-LDB code with density $\frac{2r}{2r+1}$ for any $r \geq 1$ and $l \geq 1$ in the infinite king grid.*

Proof. Let $C = \{(x,y) \mid x-y \equiv 1,2,\ldots,4r \pmod{4r+2}\}$ be a code. A part of the code when $r=1$ is given in Figure 5. We show that C is an $(r, \leq l)$-LDB code for any $l \geq 1$. Clearly, the density of the code is $\frac{2r}{2r+1}$.

Now, every codeword (x,y) with $x-y \equiv 2r \pmod{4r+2}$ is a special codeword since $(x-r, y+r)$ is the only non-codeword in its neighbourhood. In the same way, codewords with $x-y = 2r+1 \pmod{4r+2}$ are special codewords. On the other hand, every non-codeword (x,y) with $x-y \equiv 0 \pmod{4r+2}$ has a special codeword $(x+r, y-r)$ in its neighbourhood. In the same way, non-codewords with $x-y \equiv 4r+1 \pmod{4r+2}$ have a special codeword in its neighbourhood.

Hence every non-codeword has a special codeword in its neighbourhood and therefore C is an $(r, \leq l)$-LDB code for all $l \geq 1$. ☐

[2] w does not cover $(r,-r)$ since then w also covers both $(-1,0)$ and $(1,0)$.

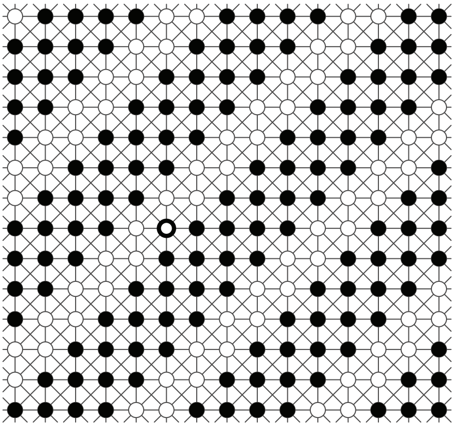


Figure 5: $(1, \leq l)$-LDB code for all $l \in \mathbb{N}_+$. The origin is the non-codeword which is surrounded with a thick circle.

Now, we have seen that the codes which were given in the previous theorem are so-called *optimal* $(r, \leq l)$-LDB codes, when $l > 4r$ or when $l = 4$ and $r = 1$. Indeed, by Theorems 5 and 9, there does not exist any $(r, \leq l)$-LDB code the density of which would be less than the density of the codes given in the previous theorem in the infinite king grid.

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