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Further results on ternary complementary sequences, orthogonal designs and weighing matrices

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Abstract

A set of sequences is complementary, if the sum of their periodic or non-periodic autocorrelation function is zero. Infinite families of orthogonal designs, based on some weighing matrices of order $2n$, weight $2n-k$ and spread σ, are constructed from two circulants matrices by using complementary sequences of zero non-periodic autocorrelation function, i.e. ternary complementary pairs. Moreover, a new measure is introduced, called ζ-efficiency, for ternary complementary pairs and some of its basic properties are explored. Using the notion of ζ-efficiency, some infinite classes of weighing matrices from ternary complementary pairs are constructed. Finally, a multiplication theorem for sequences with zero periodic autocorrelation function is given and its consequences are studied. As an application, we give some more weighing matrices from the derived pairs of zero periodic autocorrelation function.

1 Introduction

An *orthogonal design* of order n and type $(s_1, s_2, \dots, s_k)$ denoted $OD(n; s_1, s_2, \dots, s_k)$ in the commuting variables $x_1, x_2, \dots, x_k$, is a square matrix D of order n with entries from the set $\{0, \pm x_1, \pm x_2, \dots, \pm x_k\}$ satisfying

$$DD^T = \sum_{i=1}^{k} (s_i x_i^2) I_n,$$

where I_n is the identity matrix of order n. An $OD(n; w)$ is equivalent to a weighing matrix $W = W(n, w)$ with entries from $\{0, \pm 1\}$ where the number w is called the weight of W, i.e. having w non-zero entries per row and column. Orthogonal designs and weighing matrices have been studied extensively, see [1], [3] and [15] and references therein, for detailed information on known and unknown weighing matrices.

A well-known necessary condition for the existence of $W(2n, w)$ matrices states that if there exists a $W(2n, w)$ matrix with n odd, then $w < 2n$ and w is the sum of two squares. In this paper we are focusing on $W(2n, w)$ constructed from two circulant matrices. The "two circulants" construction for weighing matrices is described in the Theorem below, taken from [8].

Theorem 1 *Suppose there exist two circulant matrices A_1, A_2 of order n, with entries from $\{0, \pm 1\}$, satisfying $A_1 A_1^T + A_2 A_2^T = f I_n$. Then, if f is an integer there exists a $W(2n, f)$ weighing matrix in one of the following forms,*

$$W(2n, f) = \begin{pmatrix} A_1 & A_2 \\ -A_2^T & A_1^T \end{pmatrix} \quad or \quad W(2n, f) = \begin{pmatrix} A_1 & A_2 R \\ -A_2 R & A_1 \end{pmatrix}$$

where R is the square matrix of order n with $r_{ij} = 1$ if $i + j - 1 = n$ and 0 otherwise.

The orthogonal design constructions of this paper use the Goethals-Seidel array [9]:

Theorem 2 [8, Theorem 4.49] *If there exist four circulant matrices A_1, A_2, A_3, A_4 of order n satisfying*

$$\sum_{i=1}^{4} A_i A_i^T = f I$$

where f is the quadratic form $\sum_{j=1}^{u} s_j x_j^2$, then there is an orthogonal design $OD(4n; s_1, s_2, \ldots, s_u)$.

2 The spread of two sequences with PAF zero

The concept of the spread of two sequences was introduced in [11]. We list some necessary definitions, below.

Definition 1 *Let $A = [a_1, a_2, \ldots, a_n]$ be a sequence of length n. The* periodic autocorrelation function, PAF*, $P_A(s)$ is defined as:*

$$P_A(s) = \sum_{i=1}^{n} a_i a_{i+s} \ , s = 0, 1, \ldots, n-1$$

where we consider i+s modulo n.

Definition 2 *Let $A = [a_1, a_2, \ldots, a_n]$ be a sequence of length n. The* non-periodic autocorrelation function, NPAF*, $N_A(s)$ is defined as:*

$$N_A(s) = \sum_{i=1}^{n-s} a_i a_{i+s} \ , s = 0, 1, \ldots, n-1.$$

Definition 3 *Two sequences, $A = [a_1, \cdots, a_n]$ and $B = [b_1, \cdots, b_n]$, of length n are said to have* zero PAF *(respectively* zero NPAF*), if $P_A(s) + P_B(s) = 0$ (respectively if $N_A(s) + N_B(s) = 0$) for $s = 1, \ldots, n-1$.*

Definition 4 *Two sequences, $A = [a_1, \cdots, a_n]$ and $B = [b_1, \cdots, b_n]$, of length n and elements from $\{-1, +1\}$ are called* Golay sequences *if they have zero NPAF, i.e. if $N_A(s) + N_B(s) = 0$ for $s = 1, \ldots, n-1$.*

Definition 5 *A sequence $A = [a_1, \cdots, a_n]$ of length n is said to have spread $s = s(A)$, if the largest block of consecutive zeros is of length s.*

Note that in a sequence $A = [a_1, \cdots, a_n]$ with spread s there may be more than s zero elements in total.

Definition 6 *Two sequences $A = [a_1, \cdots, a_n]$ and $B = [b_1, \cdots, b_n]$, of length n are said to have spread $s = s(A, B)$, if $s = \min(s(A), s(B))$.*

Two sequences of length n that have spread s, each contain at least s consecutive zero elements.

Definition 7 *Two sequences $A = [a_1, \cdots, a_n]$ and $B = [b_1, \cdots, b_n]$, of length n and spread s are called spread-normalized if they are each rotated (shifted cyclically) so that a longest set of consecutive zeros appear in positions $1, 2, \ldots, s(A)$ and $1, 2, \ldots, s(B)$ in A and B respectively.*

We also require the following definition taken from [15]:

Definition 8 *Two sequences of length n, with zero PAF or zero NPAF, are said to be of type (u, v) if the sequences are composed of two variables, say a and b, so that a and $-a$ occur a total of u times and b and $-b$ occur a total of v times, i.e. the sequence elements are taken from $\{a, -a, b, -b\}$.*

Note that two sequences of type (u, v) can be used as the first rows of two circulant matrices in Theorem 1 to obtain an $OD(2n; u, v)$. For sequences A, B, $A = [a_1, \ldots, a_n]$, let us write $A \otimes B = [a_1 B, \ldots, a_n B]$, where $a_i B$ denotes scalar multiplication and $A^* = [a_n, \ldots, a_1]$ denotes the reversed sequence of A.

3 Orthogonal designs from $W(2n, 2n-k)$ and ternary complementary pairs

We now show how to combine sequences with zero PAF and spread s with ternary complementary pairs in order to give new constructions for orthogonal designs. The sequences with zero PAF come from $W(2n, 2n-k)$ weighing matrices constructed from two circulant matrices. A ternary complementary pair $TCP(n, w)$ is made up of two $\{-1, 0, +1\}$ sequences A and B both of length n, containing w non-zero elements in total, with the property that they have NPAF zero. Recall from [4] that the support of a sequence is the set of positions in which it is nonzero. A pair of sequences is disjoint if the two sequences have disjoint support and is conjoint if the two sequences have the same support (which is possible only when w is even in a $TCP(n, w)$).

Theorem 3 *Suppose there exists a weighing matrix $W(2n, 2n-k)$ constructed from two circulant matrices, whose first rows have spread σ. Suppose there exists a disjoint ternary complementary pair of length σ and weight w, i.e. $TCP(\sigma, w)$. Then there exists an orthogonal design, $OD(4n; 2w, 2w, 4n-2k)$.*

Proof

Let C and D be the first rows of the two circulant matrices that make up the weighing matrix $W(2n, 2n-k)$. We spread-normalize C and D and call the resulting $\{0, \pm 1\}$ sequences A and B respectively. We multiply A and B by the variable α to obtain:

$$\begin{aligned} A = \ & [\underbrace{0, \cdots, 0,}_{\sigma \text{ zeros}} \ a_{\sigma+1}, \ \cdots, \ a_n] \\ B = \ & [\underbrace{0, \cdots, 0,}_{\sigma \text{ zeros}} \ b_{\sigma+1}, \ \cdots, \ b_n] \end{aligned} \qquad (1)$$

where $a_k, b_k \in \{0, \pm\alpha\}, k = \sigma+1, \ldots, n$ and either $a_{\sigma+1} \neq 0$ or $b_{\sigma+1} \neq 0$.

Now denote the two sequences of length σ, with NPAF zero of the disjoint $TCP(\sigma, w)$ by $F = [f_1, \cdots, f_\sigma]$ and $G = [g_1, \cdots, g_\sigma]$. We can construct the sequences $P' = [xF + yG]$ and $Q' = [yF^\star - xG^\star]$ of length σ which have NPAF zero and are of type (w, w). The sequences $P' = [p_1, \cdots, p_\sigma] = [xf_1 + yg_1, \cdots, xf_\sigma + yg_\sigma]$ and $Q' = [q_1, \cdots, q_\sigma] = [yf_\sigma - xg_\sigma, \cdots, yf_1 - xg_1]$ can be used in Theorem 1 to give an $OD(2\sigma; w, w)$. Then the $OD(4n; 2w, 2w, 4n-2k)$ can be constructed by forming the four circulant matrices with the given first rows P, Q, R, S below, which are then used in the Goethals-Seidel array.

$$\begin{aligned} P = [\ & p_1, & p_2, & \ \cdots, & p_\sigma, & \ a_{\sigma+1}, & \cdots, & \ a_n] \\ Q = [\ & -p_1, & -p_2, & \ \cdots, & -p_\sigma, & \ a_{\sigma+1}, & \cdots, & \ a_n] \\ R = [\ & q_1, & q_2, & \ \cdots, & q_\sigma, & \ b_{\sigma+1}, & \cdots, & \ b_n] \\ S = [\ & -q_1, & -q_2, & \ \cdots, & -q_\sigma, & \ b_{\sigma+1}, & \cdots, & \ b_n] \end{aligned} \qquad (2)$$

The sequences P, Q, R, S have PAF zero. This is because any products that arise in the PAF of P from elements of P' with the elements of the sequence $[a_{\sigma+1}, \cdots, a_n]$ are negated in the PAF of Q by the product of the elements of $-P'$ with the elements of the sequence $[a_{\sigma+1}, \cdots, a_n]$. We have the same for Q' and the sequence $[b_{\sigma+1}, \cdots, b_n]$ in R and S. The sum of the products from elements of the sequence $[a_{\sigma+1}, \cdots, a_n]$ with the products from elements of the sequence $[b_{\sigma+1}, \cdots, b_n]$ is equal to zero in the PAF of P, Q, R, S, since the sequences A and B have PAF zero. We have the same for the sum of the products from elements of P' with the products from elements of

Q' in the PAF of P, Q, R, S, since these sequences have NPAF zero. This gives the required $OD(4n; 2w, 2w, 4n - 2k)$. □

We illustrate the application of Theorem 3 with the following example:

Example 1 *Take $n = 11$, $k = 13$ and we begin with a spread-normalized weighing matrix $W(2 \cdot 11, 2 \cdot 11 - 13) = W(2 \cdot 11, 9)$ constructed from two circulant matrices with spread $\sigma = 2$*

```
A = [0, 0, 0, 0,  0, 1, 0, -1, 0, -1, 1]
B = [0, 0, 1, 0, -1, 0, 0,  1, 0,  1, 1]
```

Note that $\sigma = \min(s(A), s(B)) = \min(5, 2) = 2$. We multiply A and B by the variable α to obtain:

$$\begin{array}{rcl} A & = & [0,\ 0,\ 0,\ 0,\ 0,\ \alpha,\ 0,\ -\alpha,\ 0,\ -\alpha,\ \alpha] \\ B & = & [0,\ 0,\ \alpha,\ 0,\ -\alpha,\ 0,\ 0,\ \alpha,\ 0,\ \alpha,\ \alpha] \end{array}$$

We now consider the disjoint ternary complementary pair of length $\sigma = 2$, with weight $w = 2$ and NPAF zero to be:

$$F = [1, 0] \quad \textit{and} \quad G = [0, 1].$$

Then the reverse sequences $F^\star$ and $G^\star$ of F and G are:

$$F^\star = [0, 1] (= G) \quad \textit{and} \quad G^\star = [1, 0] (= F).$$

The sequences $P' = [xF + yG]$ and $Q' = [yF^\star - xG^\star]$ which have NPAF zero are:

$$P' = [x, y] \quad \textit{and} \quad Q' = [-x, y].$$

Then the sequences P, Q, R, S that will have PAF zero are:

$$\begin{array}{rl} P & = [\ x,\ y,\ 0,\ 0,\ 0,\ \alpha,\ 0,\ -\alpha,\ 0,\ -\alpha,\ \alpha] \\ Q & = [\ -x,\ -y,\ 0,\ 0,\ 0,\ \alpha,\ 0,\ -\alpha,\ 0,\ -\alpha,\ \alpha] \\ R & = [\ -x,\ y,\ \alpha,\ 0,\ -\alpha,\ 0,\ 0,\ \alpha,\ 0,\ \alpha,\ \alpha] \\ S & = [\ x,\ -y,\ \alpha,\ 0,\ -\alpha,\ 0,\ 0,\ \alpha,\ 0,\ \alpha,\ \alpha] \end{array}$$

and they can be used in the Goethals-Seidel array to obtain an $OD(4n; 2w, 2w, 4n - 2k)$, i.e. an $OD(44; 4, 4, 18)$.

Remark 1 *We note that we can strengthen the conditions of Theorem 3 by removing the disjoint property of the TCP, since we can always construct a disjoint TCP from a given one by shifting. This fact is illustrated in the following example.*

Example 2 *Take $n = 15$, $k = 21$ and we begin with a spread-normalized weighing matrix $W(2 \cdot 15, 2 \cdot 15 - 21) = W(2 \cdot 15, 9)$ constructed from two circulant matrices with spread $\sigma = 5$*

```
A = [0, 0, 0, 0, 0, 0, -1, 1, 0,  0, 0, 1, 0, 1,  1]
B = [0, 0, 0, 0, 0, 1,  0, 0, 0, -1, 0, 0, 1, 0, -1]
```

Note that $\sigma = \min(s(A), s(B)) = \min(6,5) = 5$. *We multiply* A *and* B *by the variable* α *to obtain:*

$$\begin{array}{l} A = [0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -\alpha,\ \alpha,\ 0,\ 0,\ 0,\ \alpha,\ 0,\ \alpha,\ \alpha] \\ B = [0,\ 0,\ 0,\ 0,\ 0,\ \alpha,\ 0,\ 0,\ 0,\ -\alpha,\ 0,\ 0,\ \alpha,\ 0,\ -\alpha] \end{array}$$

We now consider the ternary complementary pair of length $\sigma' = 3 < 5 = \sigma$*, with weight* $w = 4$ *and NPAF zero to be:*

$$F' = [1,0,1] \quad and \quad G' = [1,0,-1].$$

By adding one zero at the end of each sequence and cyclically shifting G' *[4], we construct the following disjoint* $TCP(4,4)$*,* $[1,0,1,0]; [0,1,0,-1]$. *We note that padding NPAF sequences with zeros does not alter the signs of the NPAF, and shifting is an equivalent transformation for TCPs ([4]). Since the construction of Theorem 3 requires that the spread of two initial sequences is equal to the length of the ternary complementary pair we pad the previously constructed TCP with one additional zero. Thus, we construct the following disjoint ternary complementary pair of length* $\sigma = 5$ *and weight* $w = 4$*:*

$$F = [1,0,1,0,0] \quad and \quad G = [0,1,0,-1,0].$$

Then the reverse sequences $F^\star$ *and* $G^\star$ *of* F *and* G *are:*

$$F^\star = [0,0,1,0,1] \quad and \quad G^\star = [0,-1,0,1,0].$$

We now consider the sequences $P' = [xF + yG]$ *and* $Q' = [yF^\star - xG^\star]$ *which have NPAF zero to be:*

$$P' = [x,y,x,-y,0] \quad and \quad Q' = [0,x,y,-x,y].$$

Then the desired sequences P, Q, R, S *that will have PAF zero are:*

$$\begin{array}{lllllllllllllllll} P & = [& x, & y, & x, & -y, & 0, & 0, & -\alpha, & \alpha, & 0, & 0, & 0, & \alpha, & 0, & \alpha, & \alpha] \\ Q & = [& -x, & -y, & -x, & y, & 0, & 0, & -\alpha, & \alpha, & 0, & 0, & 0, & \alpha, & 0, & \alpha, & \alpha] \\ R & = [& 0, & x, & y, & -x, & y, & \alpha, & 0, & 0, & 0, & -\alpha, & 0, & 0, & \alpha, & 0, & -\alpha] \\ S & = [& 0, & -x, & -y, & x, & -y, & \alpha, & 0, & 0, & 0, & -\alpha, & 0, & 0, & \alpha, & 0, & -\alpha] \end{array}$$

and they can be used in the Goethals-Seidel array to obtain an $OD(4n; 2w, 2w, 4n - 2k)$*, i.e. an* $OD(60; 8, 8, 18)$.

In order to prove similar results when comparing to those of Theorem 3, we introduce the concept of the ζ-*efficiency* of a ternary complementary pair, based on a slight modification of the definition of shifting for TCPs from [4].

Definition 9 *Shifting for a given TCP is the process of appending zeros to both ends of the sequences, and cycle permute them until a disjoint pair is found.*

Definition 10 *We define as the ζ-efficiency of a given ternary complementary pair $TCP(n, w)$ denoted by ζ, the minimum number of zeros required in order to transform the $TCP(n, w)$ to a disjoint $TCP(n', w)$ by shifting, where $n' \geq n$, i.e. $n' = n + \zeta$.*

We can now prove the following theorem.

Theorem 4 *Suppose there exists a weighing matrix $W(2n, 2n-k)$ constructed from two circulant matrices, whose first rows have spread σ. Suppose there exists a ternary complementary pair of length σ' and weight w, i.e. $TCP(\sigma', w)$ with ζ-efficiency equal to ζ. If $\sigma' + \zeta \leq \sigma$ then there exists an orthogonal design, $OD(4n; 2w, 2w, 4n-2k)$.*

Proof

We only need to construct the disjoint sequences F and G of the proof of Theorem 3. Consider the two sequences of length σ' with NPAF zero of the $TCP(\sigma', w)$. Since the $TCP(\sigma', w)$ has ζ-efficiency equal to ζ by shifting we obtain a disjoint $TCP(\sigma'+\zeta, w)$. Afterwards, we pad the sequences obtained by shifting with $\sigma - (\sigma' + \zeta)$ zeros. The rest of the proof is as of the one in Theorem 3. □

4 ζ-efficiency of ternary complementary pairs

Before continuing, we explore some basic properties of the ζ-efficiency for TCPs. In the combinatorial space of TCPs the two extreme cases, are the disjoint TCPs and the conjoint ones. Clearly, from the definition of ζ-efficiency a disjoint TCP has $\zeta = 0$. We investigate the case of conjoint TCPs below.

Lemma 1 *Let $F; G$ be a conjoint $TCP(n, w)$. Denote by S_t the support of the conjoint $TCP(n, w)$, and let $|S_t| = t$. Then, the ζ-efficiency of the conjoint $TCP(n, w)$ is upper bounded by t, i.e. $\zeta \leq t = \frac{w}{2}$.*

Proof

Since F and G have the same support, the t nonzero entries occur in the same places. By padding both sequences with t zeros at the two ends and shifting accordingly we transform the sequences into disjoint ones. Since, the ζ-efficiency is the minimum amount of zeros to be added, the existence of suitable blocks of zeros in F and G may require less than t zeros to be added. Therefore $\zeta \leq t$. Since the weight w is the number of the nonzero entries in both sequences we have $w = 2t$, and $\zeta \leq \frac{w}{2}$. □

The previous lemmas indicate that the ζ-efficiency can be regarded as an additional criterion of how close a given TCP is to its equivalent disjoint form. Although, as noted in [4] every TCP is equivalent to a disjoint pair while a given TCP may or may not be equivalent to a conjoint pair, no closed formulae are given of how to achieve this. Therefore, better and more general bounds are needed for ζ-efficiency and restrictions when ζ is small. We can achieve this by relating the ζ-efficiency to the deficiency ([4]), δ, of a $TCP(n, w)$ defined by $\delta = 2n - w$, i.e. the number of zeros in the two sequences.

It is well known that given a disjoint $TCP(n, w)$, we can double the weight of the pair (Lemma 11, [4]) and produce a conjoint TCP. We reformulate this result using the notion of ζ-efficiency for every $TCP(n, w)$.

Lemma 2 *Let $F; G$ be a $TCP(n, w)$ with associated ζ-efficiency. Then there exists a conjoint $TCP(n + \zeta, 2w)$.*

Proof

Since the given $TCP(n, w)$ has ζ-efficiency equal to ζ, there exists a disjoint TCP of length $n + \zeta$ and weight w derived by shifting. Using Lemma 11 of [4] we construct a conjoint $TCP(n + \zeta, 2w)$. □

4.1 ζ-efficiency of ternary complementary pairs when δ is small

It is expected that when the deficiency, δ, of a TCP is small, the ζ-efficiency would be quite large. It is interesting to estimate the exact values of ζ for fixed values of δ.

We have seen earlier that one of the worst cases for ζ-efficiency occurs in conjoint TCPs. But is this the worst case scenario? If we assume the case of $\delta = 0$ then we have that the sequences composing the TCP, are in fact Golay sequences, denoted by $GS(n)$ for length n. Of course these sequences have the same support, therefore can be regarded as a conjoint TCP. The ζ-efficiency in this case is exactly the upper bound of Lemma 1 as expected.

Lemma 3 *Let $F; G$ be a $TCP(n, 2n) = GS(n)$. Then it has ζ-efficiency equal to n, i.e. $\zeta = n$.*

Proof

The given TCP is conjoint with $\delta = 0$. We have to pad the sequence 0_n of n consecutive zeros to each sequence F and G and shift along in order to produce the disjoint $TCP(n + n, 2n)$, $[F, 0_n]; [0_n, G]$. Since the amount of zeros added are n, we have that the $TCP(n, 2n)$ has ζ-efficiency equal to n. □

It is well known from [6], that $\delta = 1$ occurs only in the case of $TCP(1, 1)$ or $TCP(3, 5)$.

Lemma 4 *Let $F; G$ be a $TCP(n, 2n - 1)$. Then it has ζ-efficiency equal to ζ, where $\zeta = 0$ or $\zeta = 3$.*

Proof

We have two cases when $\delta = 1$. The first one is the trivial $TCP(1,1)$ given by $1; 0$. Since this TCP is disjoint we have that it has ζ-efficiency equal to zero. The other case is the $TCP(3,5)$ which is given up to equivalence by $[1,0,1]; [1,1,-1]$. The absence of zeros in the second sequence requires at least three zeros to be added in each sequence in order the derived TCP by shifting to have disjoint support. Therefore its ζ-efficiency is $\zeta = 3$. $\square$

4.2 ζ-efficiency of ternary complementary pairs for given δ

Since, we have a measure for the amount of zeros that occur in a TCP, the deficiency, we can derive the following bound for ζ-efficiency.

Lemma 5 *For a given $TCP(n,w)$, its ζ-efficiency is lower bounded by $w - n$, i.e. $\zeta \geq w - n$.*

Proof In order to have a pair of sequences with $A; B$ disjoint support that made up a $TCP(n,w)$ there must appear blocks $[a_i; b_i]$ in sequences $A; B$ where at least one a_i or b_i is zero. Therefore the minimum number of zeros that have to be added in both sequences and shifting must be applied afterwards, are at least $\zeta \geq n - \delta$, where $\delta = 2n - w$ the deficiency of the TCP. By substitution, we have $\zeta \geq n - (2n - w)$ and we obtain the result. $\square$

Clearly, the previous bound is not always sharp; The presence of possible blocks of zeros in the same positions $[0; 0]$ requires additional zeros to be added. This exception, can be better studied if we consider the deficiencies of the individual sequences.

Remark 2 *In case the given $TCP(n,w)$ is disjoint, therefore $\zeta = 0$ we have that $n \geq w$, i.e. sequences that have w non-zero entries must be searched for in length greater or equal to w, in order for them to be disjoint.*

4.3 Weighing matrices from ternary complementary pairs of given ζ

Now that we have acquired a bound on ζ-efficiency it would be desirable to obtain a construction for orthogonal designs or weighing matrices that involve disjoint TCPs. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ denote the set of natural numbers for the rest of the paper. It is well known, that there exist the following maps (constructions) for TCPs and weighing matrices:

1. $TCP(n,w) \to W(2n,w)$ (Use the two sequences of the TCP in Theorem 1).
2. $TCP(n,w) \to TCP(n+k,w), k \in \mathbb{N}$ (padding a TCP with zeroes does not alter the signs of NPAF).

3. $TCP(m,w) \times TCP(n,z) \to TCP(mn,wz)$, if one of the pairs is disjoint (multiplication of TCPs, Theorem 14, [4]).

Lemma 6 *Given any $TCP(n,w)$ with ζ-efficiency equal to ζ, there exist the following maps (constructions):*

(i) $TCP(n,w) \to TCP(n^2 + n\zeta, w^2)$.

(ii) $TCP(n,w) \to TCP(n^2 + n\zeta + k, w^2), k \in \mathbb{N}$.

(iii) $TCP(n,w) \to W(2n^2 + 2n\zeta + 2k, w^2), k \in \mathbb{N}$.

Proof

(i) We can construct a disjoint $TCP(n+\zeta,w)$ from the given $TCP(n,w)$. By plugging-in the two pairs in map 3 we have $TCP(n,w) \times TCP(n+\zeta,w) \to TCP(n(n+\zeta),w^2)$.

(ii) Apply map 2 to (i).

(iii) Apply map 1 to (ii).

□

Conjecture 1 *(Conjecture 2, [4]) If $TCP(n,w)$ exists and $p \mid w$, then $TCP(m,p)$ exists, for some m.*

It is intriguing to investigate the reverse of Lemma 6, which can be stated as follows, and be considered also as a partial case of Conjecture 1, for square weights.

Question 1 *Which is the minimum length m, for which a $TCP(m,w^2)$ exists?*

Answer As noted in [4], an obvious way to attempt to prove this conjecture would involve showing that, if w is composite, then a $TCP(n,w)$ must factor over multiplication of pairs. While the status of the previous conjecture in the general case is still open, results that support her were given in [5]. In our case, clearly $w = p \mid w^2$ and $m \leq n^2 + n\zeta$ from Lemma 6. Moreover, considering the lower bound on ζ-efficiency, we can adequately derive the following:

$$\zeta \geq w - n \Rightarrow n\zeta \geq nw - n^2 \Rightarrow n^2 + n\zeta \geq nw$$

This tells us that from a given $TCP(n,w)$ in order to construct a TCP having square weight, i.e. $TCP(m,w^2)$ one has to search for zero NPAF sequences having length m at least nw. □

5 A multiplication theorem for sequences with zero periodic autocorrelation function

In this Section, we give a new multiplication Theorem for sequences with zero PAF. The concept of the multiplication arises naturally, from the constructions presented for complementary sequences so far. Our product can be regarded as a "weak" version of the classical multiplication for TCPs (map 3., $TCP(m,w) \times TCP(n,z) \rightarrow TCP(mn,wz)$).

Though many multiplications exist for complementary sequences, see for example [2, 4, 8, 10, 12, 13, 14, 15] (where by the notion of multiplication we mean from a given set of sequences, there exists natural numbers (λ_1, λ_2) that multiply the length and/or the weight of the initial sequences), the majority of them involves plug-in constructions for complementary sequences of zero PAF or NPAF. The notion of a mixed product, that involves fragments of sequences of zero PAF and sequences of zero NPAF, is rare when compared with the currently known multiplication methods. Multiplying sequences adds considerable flexibility and scope to the process of building up large sequences from small ones. The following notation appears handy in the context presented below.

Notation. $DC(n,k)$ denotes two $\{0, \pm 1\}$ sequences of order n each and (total) weight k, that have PAF zero.

Theorem 5 *Suppose $A; B$ is a $DC(n,k)$, $C; D$ is a $TCP(m,w)$, and one of the pairs is disjoint. Then there exists a map (construction)*

$$DC(n,k) \times TCP(m,w) \rightarrow DC(nm,kw).$$

Proof

Construct the sequences $U; V$ of the $DC(nm,kw)$ as,

$$\begin{aligned} U &= A \otimes C + B \otimes D; \\ V &= A \otimes D^* - B \otimes C^* \end{aligned}$$

Clearly, $U; V$ are ternary sequences and the sum of their periodic autocorrelation function is zero by considering the sum of Hall and Laurent polynomials of U and V, in a similar manner just as in the proof of Theorem 14 of [4]. □

We note that the product in its given form is not commutative, i.e. for complementary sequences as above, $TCP(m,w) \times DC(n,k) \nrightarrow DC(mn,wk)$. However, if we consider an equivalent but different construction for the derived sequences $U; V$ which involves again the Kronecker product of sequences, it can be made commutative. In particular, there exists a map $TCP(m,w) \times DC(n,k) \rightarrow DC(mn,wk)$ for sequences $E = [C \otimes A + D \otimes B]; F = [D \otimes A^* - C \otimes B^*]$ where $A; B$ and $C; D$ are as in Theorem 5 and $E; F$ is an equivalent pair of $U; V$.

Furthermore, we can strengthen the application of Theorem 5 by removing the disjoint property to one of the two complementary pairs, considering the notion of ζ-efficiency for TCPs. In particular, we have the following Corollary.

Corollary 1 *Suppose $A;B$ is a $DC(n,k)$, $C;D$ is a $TCP(m,w)$ with associated ζ-efficiency equal to ζ. Then there exists a map (construction)*

$$DC(n,k) \times TCP(m+\zeta,w) \to DC(nm+n\zeta,kw).$$

Proof

Clearly, $TCP(m,w) \to TCP(m+\zeta,w)$ and name the resulting sequences of the disjoint $TCP(m+\zeta,w)$ as $E;F$. By taking the pair $(C,D)=(E,F)$ in Theorem 5 the result follows. □

It is worthwhile to note that, Corollary 1 applies to every pair of DCs or TCPs. Moreover, the appearance of a measure for TCP (ζ-efficiency) in the resulting length of a DC pair is an interesting fact that may lead to further conclusions for DCs.

5.1 Some consequences for DC pairs

We investigate some consequences for families of DC pairs that arose naturally from the new multiplication. The following map for weighing matrices, is well known:

4. $DC(n,k) \to W(2n,k)$ (Use the two sequences of the DC pair in Theorem 1).

The absence of a map of the form, $DC(n,k) \to DC(n+m,k), m \in \mathbb{N}$ (padding with zeros a DC pair does not maintain the zero correlation property), in the theory of sequences with zero PAF does not allow us to construct directly infinite families of DC pairs or weighing matrices from a given DC pair. We give below, a reformulation of a known multiplication for DC pairs ([15]) by using the multiplication Theorem for DCs.

Lemma 7 *Given any $DC(n,w)$, there exists a family of $DC(pn,w) \to W(2pn,w)$, $p \in \mathbb{N}$.*

Proof

The trivial $TCP(1,1)$ made up by $1;0$ has the disjoint property. Clearly $TCP(1,1) \to TCP(1+k,1), k \in \mathbb{N}$. From Theorem 5 we obtain $DC(n,w) \times TCP(1+k,1) \to DC(n+nk,w)$. Setting $k+1=p \in \mathbb{N}$ we obtain a family of $DC(pn,w)$ for $p \in \mathbb{N}$ and as a consequence a family of $W(2pn,w)$ for $p \in \mathbb{N}$. □

Considering known families of DC pairs or TCPs, we can build up some interesting larger families of DC pairs. A $DC(n,2n)$ is denoted by $PCS(n,2)$ in the literature [10], and it is well-known that PCS pairs give rise to a Hadamard matrix of order $2n$, $H(2n)$. Moreover, multiplicative constructions for PCS pairs are rare and a known-family of PCS is reformulated below. In addition, a map for a family of DC pairs is given, which arise from a known family of conjoint TCPs with zero deficiency, i.e. Golay sequences.

Proposition 1 *Given any DC pair with disjoint support, there exist the following maps (constructions) for families of DC pairs and weighing matrices,*

(i) $DC(n,k) \times TCP(m,2m) \rightarrow DC(nm,2km) \rightarrow W(2nm,2km)$

(ii) $DC(n,n) \times TCP(m,2m) \rightarrow DC(nm,2nm) = PCS(nm,2) \rightarrow W(2nm,2nm) = H(2nm)$

where m *is the length of a set of* $GS(m)$*, i.e.* $m \in \{2^a 10^b 26^c : a,b,c \in \mathbb{N}\}$*.*

Proof

(i) There exist $GS(m) = TCP(m,2m)$ for $m \in \{2^a 10^b 26^c : a,b,c \in \mathbb{N}\}$. Use the two pairs of $DC(n,k)$ and $TCP(m,2m)$ in Theorem 5 to obtain a $DC(nm, 2km)$. Apply map 4. to the resulting sequences to obtain a $W(2nm,2km)$.

(ii) As before apply Theorem 5 and map 4. to the given pairs of sequences.

□

It is well-known that there exist Golay sequences, $GS(m)$ for lengths $m = 1,2$. Hence, the following families of DC pairs are merely a (trivial) application of the previous Proposition, but are useful in the construction of weighing matrices (see Section 5.2).

Corollary 2 *Given any* $DC(n,k)$ *with disjoint support, there exist the following maps (constructions) for DC pairs,*

(i) $DC(n,k) \times TCP(1,2) \rightarrow DC(n,2k) \rightarrow W(2n,2k)$.

(ii) $DC(n,k) \times TCP(2,4) \rightarrow DC(2n,4k) \rightarrow W(4n,4k)$.

Proof

(i) Use $TCP(1,2)$, $[1];[-1]$, in Proposition 1.

(ii) Use $TCP(2,4)$, $[1,1];[1,-1]$, in Proposition 1.

□

5.2 Some weighing matrices from DC pairs

In this Section, we present some weighing matrices, constructed from a DC pair, that are obtained via multiplication.

Lemma 8 *There exist* $DC(n,k) \rightarrow W(2n,k)$ *for :*

(i) $n = 29$ *and* $k = 36$.

(ii) $n = 74$ *and* $k = 144$.

Proof

(i) We can construct a disjoint $DC(29, 18)$ from [7]:

```
-+0-0-000-00000000000-+0000+-
00-000+0+0+---0000--000000000
```

Plug-in these sequences in Corollary 2, to obtain a $DC(29, 18) \times TCP(1, 2) \to DC(29, 36)$:

```
-++-0--0---+++0000++0-+0000+-
+-++0+-0-+-+++0000++0+-0000-+
```

Apply map 4. to obtain a $W(58, 36)$.

(ii) There exists a disjoint $DC(37, 36)$ from [15]:

```
++--0-0-++0+00++0+0+00-+0+000-0+00000
0000-0+000+0--00-0-0+-00+0++-0-0++-+0
```

Plug-in these sequences in Corollary 2, to obtain a $DC(37, 36) \times TCP(2, 4) \to DC(74, 144)$:

```
++++-----+--+---+++++-++-+-+++++-+++-++++--+--+++-+++-+--+---++++-+--++-00
-+-++-+-+++---+--+-+---+++++-+-+++-+++-+--+++--+---+----+++-++-+----++--00
```

Apply map 4. to obtain a $W(148, 144)$.

□

We note that $W(58, 36)$ and $W(148, 144)$ are listed as open in the second edition of the Handbook of Combinatorial Designs, [3]. However, a $DC(29, 36)$ pair and hence a $W(58, 36)$ can be constructed from directed sequences of length 29 and type $(18, 18)$ given in [7], directly by substitution of the variables of the directed sequences with 1 (for terminology and undefined terms, also see [7]). Moreover, a $W(148, 144)$ was constructed recently by four complementary sequences of zero NPAF in [16].

5.3 Some additional thoughts

In [4] where the multiplication of TCPs was given, the theory underlined there proved to be wonderfully compact, in the sense that all known cases of TCPs could be constructed from a handful set of primitive cases. It would be reasonable, to assume that the multiplication given by Theorem 5 for DC pairs could act as an initiative to investigate a similar notion of primitivity in the context of sequences with zero PAF; implying that we could construct all known DC pairs from a initial known set of zero PAF sequences.

Though, we can consider analogue maps (constructions) for DC pairs, similar to the ones given in Proposition 1 it is clear that we cannot derive prime weights for DCs since our approach is multiplicative on the involved weights of DCs and TCPs. On the other hand, we could regard DC pairs of prime weight as the initial primitive cases. While the presented theory gives certain arguments for a radical approach of DC pairs to be classified by weight (a direct analogue of TCPs), the absence of a map of the form $DC(n,k) \to DC(n+m,k), m \in \mathbb{N}$ is a strong argument to be taken against this point of view.

We conclude, with the following Conjecture which focuses our attention on DC pairs with prime weight. It may also be considered as a partial converse to Theorem 5.

Conjecture 2 *If $DC(n,w)$ exists and $p \mid w$ then $TCP(m,p)$ or $DC(m,p)$ exists, for some m.*

An obvious way to attempt to prove this conjecture would involve showing that, if w is composite, then a $DC(n,w)$ must factor over a mixed product of DC and TCP pairs.

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