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A characterization of k-trees that are interval p-graphs

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Abstract

Interval p-graphs were introduced by Brown et al. in 2002 as a generalization of interval bigraphs [D.E. Brown, S.C. Flink and J.R. Lundgren, Congr. Numer. 157 (2002), 79–93]. Little work has been done towards characterizing them. For interval bigraphs the only known forbidden subgraph characterization is for trees. As it appears to be quite difficult to find a forbidden subgraph characterization, we limit our work to an extension of trees called k-trees. We characterize k-trees that are interval p-graphs as spiny interior k-lobsters and use this result to give a forbidden subgraph characterization.

1 Introduction

Let a graph G have vertex set $V(G)$ and edge set $E(G)$. If $x, y \in V(G)$ are adjacent, then we denote $xy \in E(G)$. If G is multipartite, we denote the partitions of the vertex set as $V(G) = \{X_1 \cup X_2 \cup \ldots X_p\}$. The set $N(x) = \{v \in V(G) \mid vx \in E(G)\}$ is the neighborhood of a vertex x. We will denote $N(xy)$ for $N(x) \cup N(y)$. A graph is *interval* if to every vertex $v \in V(G)$, we can assign an interval of the real line, I_v, such that $xy \in E(G)$ if and only if $I_x \cap I_y \neq \emptyset$. Interval graphs were introduced by Hajos [8], and were then characterized by the absence of induced cycles larger than 3 and asteriodal triples by Lekkerkerker and Boland [11] in 1962. An *asteroidal triple* (AT) in G is a set A of three vertices such that between any two vertices in A there is a path within G between them that avoids all neighbors of the third.

A natural extension of interval graphs, called interval bigraphs, were introduced by Harary, Kabell, and McMorris [9] in 1982. A bipartite graph $G = (X, Y)$ is an

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interval bigraph if to every vertex, $v \in V(G)$, we can assign an interval of the real line, I_v, such that $xy \in E(G)$ if and only if $I_x \cap I_y \neq \emptyset$ and $x \in X$ and $y \in Y$. Interval digraphs, which are related to interval bigraphs, were introduced by Sen et al. in [16]. Interval bigraphs have been studied by several authors ([4], [6], [9], [10], [12], and [13]). Initially it was thought that the natural extension of asteriodal triples of vertices to asteriodal triples of edges along with induced cycles larger than 4 would work for a forbidden subgraph characterization [9]. However, Müller [13] found insects and Hell and Huang [10] found edge asteriods and bugs as forbidden subgraphs, and to date a complete characterization remains elusive. Three edges a, c and e of a graph G form an *asteriodal triple of edges* (ATE) if for any two there is a path from the vertex set of one to the vertex set of the other that avoids the neighborhood of the third edge. Cycle free interval bigraphs were characterized by Brown et al. in 2001 [6] , and ATEs played a significant role in that characterization as they will in our paper.

Theorem 1.1. *[6] A cycle-free bigraph G is an interval bigraph if and only if it is a lobster.*

In 2002, Brown et al. introduced a further extension of interval bigraphs called interval k-graphs [5]. We change the name to interval p-graphs here to avoid confusion.

Definition 1.2. *[5] Let $G = \{X_1, X_2, \ldots, X_p\}$ be a multipartite graph. G is an* interval p-graph *if there exists an assignment to each vertex, $v \in V(G)$, an interval of the real line, I_v, such that $xy \in E(G)$ if and only if $I_x \cap I_y \neq \emptyset$ and $x \in X_i$, $y \in X_j$ and $i \neq j$.*

We consider the vertices of each partite set to have the same color, so adjacency results when two vertices have overlapping intervals and are different colors.

The only characterization of interval p-graphs is with a consecutive ordering of complete r-partite subgraphs in [5]. A forbidden subgraph characterization appears to be very difficult as in the case for interval bigraphs, so it seems natural to consider generalizing Theorem 1.1 to the class of graphs called k-trees.

Definition 1.3. *[14] The class of* k-trees *is the set of all graphs that can be obtained by the following construction: (i) the k-complete graph, K_k, is a k-tree; (ii) to a k-tree Q' with $n-1$ vertices $(n > k)$ add a new vertex adjacent to a k-complete subgraph of Q'.*

K-trees are an extension of trees where the vertex is replaced with a k-clique, so that a 1-tree is simply a tree. K-trees have been studied extensively including [1],[14], and [7]. This paper characterizes k-trees that are interval p-graphs.

2 K-trees that are interval p-graphs

To describe the structure of k-trees, we use the generalized idea of a path introduced by Beineke and Pippert in [1].

Definition 2.1. *[1] A k-path of G is an alternating sequence of distinct $k-$ and $(k+1)$-cliques of G, $(e_o, t_1, e_1, t_2, e_2, \ldots, t_p, e_p)$, starting and ending with a k-clique and such that t_i contains exactly two of the distinct k-cliques e_i: e_{i-1} and e_i $(1 \leq i \leq n)$. Its* length *is the number, p, of $(k+1)$-cliques.*

Drawing the connection to a tree further, Proskurowski introduced the notion of a k-caterpillar in [14] and we introduce a k-lobster and a spiny interior k-lobster. A k-leaf is a vertex whose neighborhood is a clique.

Definition 2.2. *[14] A* k-caterpillar *P is a k-tree in which the deletion of all k-leaves results in a k-path, called the body of P. A k-caterpillar P is an* interior k-caterpillar *if for any k-leaf v, v is adjacent to all vertices of some k-complete subgraph e_i of every longest k-path of P.*

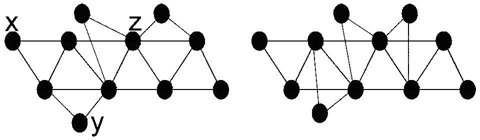

Figure 1: On the left is a 2-caterpillar, and on the right is an interior 2-caterpillar

To describe the structure of a k-lobster, we use the idea of k-distance. The *k-distance* between a k-leaf and the body is the length of the shortest k-path to a $(k+1)$-clique on the body. In Figure 2, the 2-leaf x on the 2-lobster on the right is 2-distance two from the body, although x is adjacent to y on the body.

Definition 2.3. *A* k-lobster *P is a k-tree in which the deletion of all k-leaves of k-distance 2 from the body results in a k-caterpillar. A k-lobster P is an* interior k-lobster *if the deletion of every k-leaf of k-distance 2 from the body results in an interior k-caterpillar.*

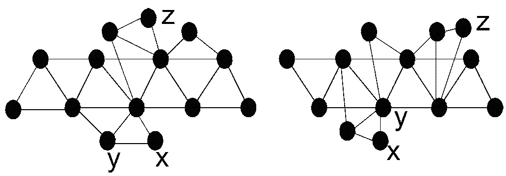

Figure 2: On the left is a 2-lobster, and on the right is an interior 2-lobster

In Figure 2, the vertices x and z are 2-distance 2 from the body on both 2-lobsters. After deleting both vertices, the 2-lobster on the right results in an interior 2-caterpillar. Thus, it is an interior 2-lobster. However, after deleting both vertices from the 2-lobster on the left, we are left with a 2-caterpillar that is not interior. This is because the neighborhood of vertex y is not an e_i on the body. Thus, it is not an interior 2-lobster.

Definition 2.4. *A* spiny interior k-lobster H *is a k-tree in which the deletion of all k-leaves results in a interior k-catepillar.*

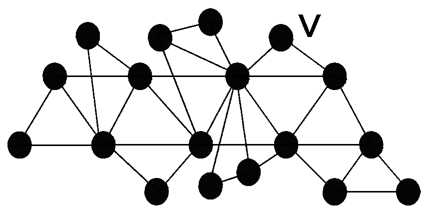

Figure 3: A spiny interior 2-lobster

Figure 3 gives an example of a 2-tree that is a spiny interior 2-lobster. Figure 5 gives two examples of 2-trees that are not spiny interior 2-lobsters. After deleting all the 2-leaves (a_1, a_7, and x_2) from the 2-tree on the left, $N(x_1) \neq e_i$ for any i on the body. Thus the resulting 2-tree after deletion is not an interior 2-caterpillar, so the original 2-tree is not a spiny interior 2-lobster. After deleting all the 2-leaves (b_1, b_8, and y_2) from the 2-tree on the right, the vertex y_1 is of 2-distance two from the body. Thus the resulting 2-tree after deletion is not an interior 2-caterpillar, so the original 2-tree is not a spiny interior 2-lobster.

If the deletion of k-leaves of k-distance 2 from the body results in an interior k-caterpillar, then the deletion of all k-leaves would also result in an interior k-caterpilar. Thus, interior k-lobsters are also spiny interior k-lobsters. However, in a spiny interior k-lobster there may also be a vertex, v, (see Figure 3) that is k-distance 1 from the body such that $N(v) \neq e_i$. Therefore, interior k-lobsters are a proper subset of spiny interior k-lobsters.

It is well known that a tree is an interval graph if and only if it is a caterpillar. However, a k-caterpillar may not be an interval graph since it could contain an asteriodal triple of vertices (x, y and z in Figure 1 are an AT). Eckhoff studied k-trees in the context of extremal interval graphs in [7]. He found that G is a $(k+1)$-extremal interval graph if and only if it is an interior k-caterpillar. Therefore, interior k-caterpillars are the class of k-trees that are interval graphs. We include a simplified proof for completeness.

Theorem 2.5. *K-trees that are not interior k-catepillars are not interval.*

Proof. Let G be a k-tree that is not an interior k-caterpillar. Let $e_0, t_1, e_1 \ldots t_n, e_n$ be the longest k-path and label it P. It follows from the definition of an interior

k-caterpillar that either there is a vertex, $v \notin P$, adjacent to some k vertices of a t_i such that $N(v) \cap P \neq e_{i-1}$ and $N(v) \cap P \neq e_i$ or there exists a k-path of length greater than or equal to 2 originating from the k vertices of some e_i.

Case 1: Assume there exists a vertex, $v \notin P$, adjacent to some k vertices of a t_i such that $N(v) \cap P \neq e_{i-1}$ and $N(v) \cap P \neq e_i$. Let $t_{i-1} = e_{i-1} + z$, $t_{i+1} = e_i + w$, and $t_i = N(v) \cap P + y$. We know t_{i-1} and t_{i+1} exist since P is the longest k-path. This implies that $z, w \in N(y)$. Label $N(v) = 1, 2, \ldots, k$ such that z is adjacent to $1, 2, \ldots, k-1$ and w is adjacent to $2, 3, \ldots, k$. The path $z1v$ avoids $N(w)$, and zyw avoids $N(v)$, and vkw avoids $N(z)$. Therefore, $\{w, v, z\}$ is an asteroidal triple, and G is not interval.

Case 2: Assume there exists a k-path of length greater than or equal to 2 originating from the k vertices of some e_i in P. Assume this path is of length 2. Let a be the first vertex in this k-path, so $N(a) \cap P = e_i$. Let b be the next vertex in this k-path, so $N(b) \subset e_i + a$ and $a \in N(b)$. Let $t_{i-1} = e_{i-1} + y$, $t_i = e_i + x$, $t_{i+1} = e_i + w$, and $t_{i+2} = e_{i+1} + z$. We know that t_{i-2} and t_{i+2} exist since P is the longest k-path. There exists a vertex of e_i not adjacent to z and similarily one not adjacent to y. Label the verticies of e_i as $1, 2, \ldots, k$ such that $1 \notin N(z)$ and $k \notin N(y)$. One of the vertices of e_i is not adjacent to b, call it h (Note that $h \in 1, 2, \ldots, k$). The path $yx1ab$ avoids $N(z)$, and $yxhwz$ avoids $N(b)$, and $zwkab$ avoids $N(y)$. Therefore, $\{y, z, b\}$ is an asteroidal triple, and G is not interval. □

Brown et al. found in [5] that although interval p-graphs can contain an asteriodal triple, they cannot contain an asteriodal triple of edges. Furthermore, Brown found in [3] that there are graphs that have no interval p-representation and do not contain an ATE (see Figure 4).

Theorem 2.6. *[5] If a graph G has an asteriodal triple of edges then G is not an interval p-graph.*

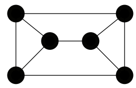

Figure 4: An example of a graph that is not p-interval ($p \geq 3$) and does not contain an ATE

Using Theorems 1.1 and 2.5 as motivation, one might hope that either k-lobsters or interior k-lobsters might work for characterizing k-trees that have a interval p-representation. However, in Figure 5 the 2-lobster on the left has an ATE (a_1a_2, a_6a_7, and x_1x_2). Interior k-lobsters are interval p-graphs, but do not completely characterize the family. We now show that spiny interior k-lobsters work.

Lemma 2.7. *If G is a k-tree that is not a spiny interior k-lobster, then G contains an asteroidal triple of edges.*

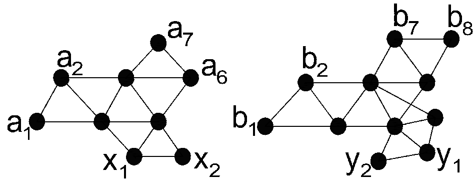

Figure 5: Examples of 2-trees that are not spiny interior 2-lobsters. The edges a_1a_2, a_6a_7, and x_1x_2 form an ATE in the 2-tree on the left. The edges b_1b_2, b_7b_8, and y_1y_2 form an ATE in the 2-tree on the right.

Proof. Let G be a k-tree that it is not a spiny interior k-lobster. Let $e_0, t_1, e_1, \ldots, t_n, e_n$ be the k and $(k+1)$ cliques of a longest k-path and label it P. It follows from the definition of a spiny interior k-lobster that either there exists a k-path of length greater than or equal to 2 originating from a k subset of some t_i of P not equal to e_{i-1} or e_i or there exists a k-path of length greater than or equal to 3 originating from the k vertices of some e_i of P.

Case 1: Assume there exists a k-path of length greater than or equal to 2 originating from a k subset of some t_i of P, not equal to e_{i-1} or e_i. Assume this path is of length 2. Label the first vertex in this k-path off P as x_1. Label the second vertex in this path as x_2, so $N(x_2) \subset (x_1 \cup N(x_1))$. Label vertices of P as follows. Let $a_1 \in t_{i-2}$, but $a_1 \notin e_{i-2}$. Let $a_2 \in t_{i-1}$, but $a_2 \notin e_{i-1}$. Let $a_3 \in t_i$, but $a_3 \notin e_i$. Let $a_4 \in t_i$, but $a_4 \notin N(x_1)$. Let $a_5 \in t_i$, but $a_5 \notin e_{i-1}$. Since $N(x_1) \cap P \neq e_{i-1}$ and $N(x_1) \cap P \neq e_i$, a_3, a_4, and a_5 are all distinct, and $a_3, a_5 \in N(x_1)$. Let $a_6 \in t_{i+1}$, but $a_6 \notin e_i$. Let $a_7 \in t_{i+2}$, but $a_7 \notin e_{i+1}$. We know that t_{i-2} and t_{i+2} exist since P is the longest k-path. We claim the edges x_1x_2, a_1a_2, and a_6a_7 are an asteriodal triple of edges (see edges x_1x_2, a_1a_2, and a_6a_7 in Figure 5 as an example of one such ATE). The path $a_2a_4a_6$ avoids $N(x_1x_2)$, since $a_4 \notin N(x_1)$ and $N(x_2) \subset N(x_1) + x_1$. Both a_2 and a_4 are in the clique t_{i-1} and a_4 and a_6 are in the clique t_{i+1}, so we know those adjacencies exist. The path $a_2a_3x_1$ avoids $N(a_6a_7)$ since $a_2, a_3 \notin e_i$. Both a_2 and a_3 are in t_{i-1}, so we know those adjacencies exist. The path $x_1a_5a_6$ avoids $N(a_1a_2)$ since $a_5 \notin e_{i-1}$. Plus, since a_5 and a_6 are in t_{i+1}, we know those adjacencies exist. Therefore, G contains an asteriodal triple of edges.

Case 2: Assume there exists a k-path of length greater than or equal to 3 originating from the k vertices of some e_i of P. Assume this path is of length 3. Label the first vertex in the k-path off P as y_1. Label the second vertex as y_2, so $N(y_2) \subset (y_1 \cup N(y_1))$. Label the third vertex as y_3, so $N(y_3) \subset (y_2 \cup N(y_2))$. Label vertices of P as follows. Let $b_1 \in t_{i-2}$, but $b_1 \notin e_{i-2}$. Let $b_2 \in t_{i-1}$, but $b_2 \notin e_{i-1}$. Let $b_3 \in t_i$, but $b_3 \notin e_i$. Let $b_4 \in e_i$, but $b_4 \notin e_{i+1}$. Let $b_5 \in t_i$, but $b_5 \notin e_{i-1}$. Let $b_6 \in t_{i+1}$, but $b_6 \notin e_i$. Let $b_7 \in t_{i+2}$, but $b_7 \notin e_{i+1}$. Let $b_8 \in t_{i+3}$, but $b_8 \notin e_{i+2}$. We know that t_{i-2} and t_{i+3} exist since P is the longest k-path. If $k > 2$, then it is possible that both b_4 and b_5 are in the neighborhood of y_2. In this case, let $b_9 \in e_i$, but $b_9 \notin N(y_2)$. Notice that $b_1, b_2, b_3, b_4 \notin N(b_7b_8)$, $b_5, b_6, b_7, b_8 \notin N(b_1b_2)$, and $b_4, b_5, b_9 \in e_i$ so they are adjacent to b_3, b_6, and y_1. We claim the edges y_2y_3, b_1b_2, and b_7b_8 are an asteriodal triple of edges (see edges y_2y_3, b_1b_2, and b_7b_8 in Figure 5 as an example of one such ATE). The path $b_2b_3b_4b_6b_7$, $b_2b_3b_5b_6b_7$, or $b_2b_3b_9b_6b_7$, depending on which vertex, b_4, b_5, or b_9, is not in the neighborhood of y_2, avoids $N(y_2y_3)$. The path $b_2b_3b_4y_1y_2$ avoids $N(b_7b_8)$. The path $b_7b_6b_5y_1y_2$ avoids $N(b_1b_2)$. Therefore, G contains an asteriodal triple of edges. □

Theorem 2.8. *A k-tree is an interval p-graph if and only if it is a spiny interior k-lobster.*

Proof. Let G be a k-tree that has an interval p-representation, and assume for contradiction that it is not a spiny interior k-lobster. By Lemma 2.7, G contains an asteriodal triple of edges. By Theorem 2.6, G is not an interval p-graph, which is a contradiction to our assumption.

Suppose G is a spiny interior k-lobster. Label the cliques that create the k-path of the body as $e_0, t_1, e_1, \ldots, t_n, e_n$. For each vertex, v, in the body assign an ordered pair (x, y) such that t_x is the first clique that contains v and t_y is the last. To each vertex assign the interval $v = (x, y + \frac{1}{2})$. Assign the colors $1, 2, \ldots, k+1$ to the vertices of t_1. Assign colors to the rest of the body as follows. If $t_i = e_{i-1} + a_i$ and $t_{i-1} = e_{i-1} + b_{i-1}$, assign a_i the same color as b_{i-1}. For the $(k+1)$-clique t_i, the vertices' intervals intersect at $(i, i + \frac{1}{2})$ and are all different colors, and intervals for the vertices of each e_i intersect at $(i + \frac{1}{2}, i + 1)$ and are all different colors. For each t_i, there are $k-1$ possible unique neighborhoods for a k-leaf, $w_{i,1}, \ldots, w_{i,k-1}$, such that $N(w_{i,j}) \subset t_i$, but $N(w_{i,j}) \neq e_i$ or e_{i-1}. For each unique neighborhood, there may be many k-leaves, which all are assigned the same label. Assign the interval $w_{i,j} = (i + \frac{j-1}{2(k-1)}, i + \frac{j}{2(k-1)})$ to each k-leaf. Each k-leaf is adjacent to k of the $k+1$ vertices in the clique t_i, so let $h_{i,j} = t_i - N(w_{i,j})$. Color $w_{i,j}$ the same as $h_{i,j}$. None of the intervals for the $k-1$ k-leaves intersect, so no adjacencies result between them. Each k-leaf is the color of the vertex that is not in its neighborhood. So although each k-leaf's interval intersects all the intervals of the clique, adjacencies only result amongst each k-leaf and its k neighborhood in the clique. Furthermore, leaves with a common label are the same color, so there is no resulting adjacency from their overlapping intervals.

There may be many paths of length one or two originating from each e_i. Let there be n vertices adjacent to the k vertices of some e_i, and label them $z_{i,1} \ldots z_{i,n}$. To each assign the interval $z_{i,j} = (i + \frac{1}{2} + \frac{j-1}{2n}, i + \frac{1}{2} + \frac{j}{2n})$. Let $t_i = e_i + b_i$ and color $z_{i,j}$, $1 \leq j \leq n$ the same as b_i. For each $z_{i,j}$ there are k possible k-leaves, $m_{i,j,1}, \ldots, m_{i,j,k}$, such that $z_{i,j} \in N(m_{i,j,\ell})$ and $N(m_{i,j,\ell}) \subset e_i + z_{i,j}$. Again, there may be many different k-leaves of this type that have the same label. Assign the interval $m_{i,j,\ell} = (i + \frac{1}{2} + \frac{j-1}{2n} + \frac{\ell-1}{2nk}, i + \frac{1}{2} + \frac{j-1}{2n} + \frac{\ell}{2nk})$. If $g_{i,j,\ell} = e_i + z_{i,j} - N(m_{i,j,\ell})$, color $m_{i,j,\ell}$ the same as $g_{i,j,\ell}$. None of the intervals for the k k-leaves intersect, so no adjacenies result between the leaves. Each leaf is adjacent to k of the $k+1$ vertices of $e_i + z_{i,j}$ and is the color of the vertex that is not in its neighborhood, so the desired adjacenies result. Again, the k-leaves with the same label are the same color, so there is no resulting adjacency from their overlapping intervals. Therefore, we have a p-interval representation for the spiny interior k-lobster. □

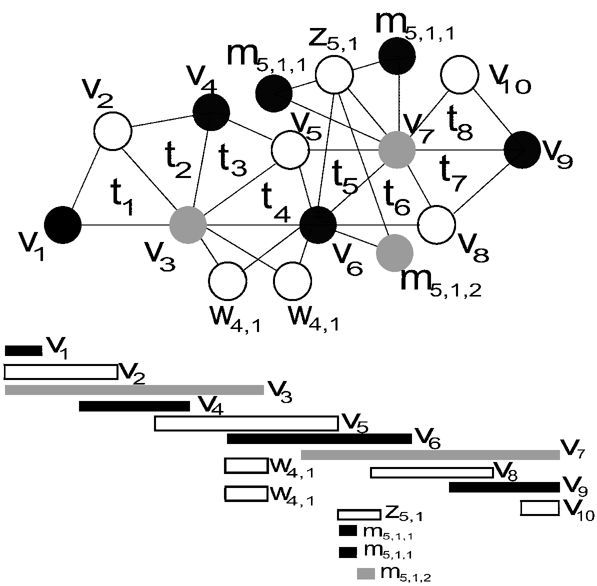

Figure 6: An example of a spiny interior 2-lobster and its interval p-representation.

Combining the results from Lemma 2.7 and Theorem 2.8, we get the following corollary.

Corollary 2.9. *A k-tree is a spiny interior k-lobster if and only if it does not contain an asteriodal triple of edges.*

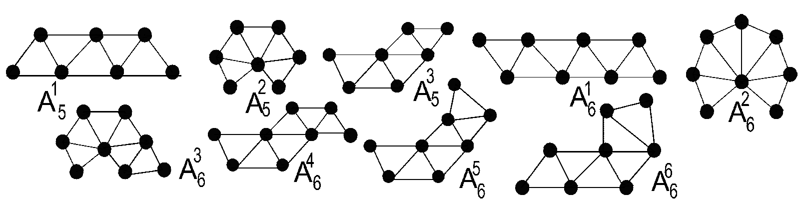

Figure 7: Non-isomorphic 2-tree paths of length 5 and 6

Next we investigate whether Theorem 2.8 can be used to find a forbidden subgraph list. For k-trees the list appears to be prohibitively long, so we restrict our consideration to 2-trees.

Theorem 2.10. *Let G be a 2-tree. G has a p-interval representation if and only if it contains no subgraph isomorphic to G_i, $1 \leq i \leq 12$.*

Proof. Let G be a p-interval 2-tree. From Theorem 2.8, we know that G is a spiny interior 2-lobster. However, it is easy to check that no G_i from Figure 9 is a spiny interior 2-lobster. Therefore, G contains no subgraph isomorphic to G_i, $1 \leq i \leq 12$.

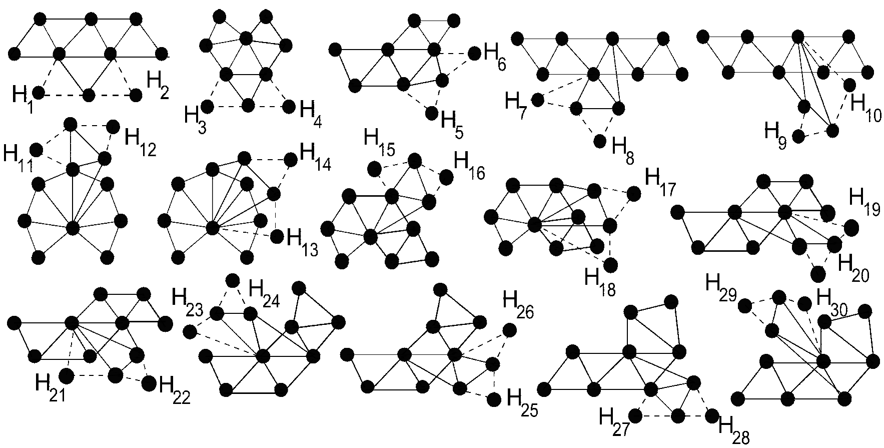

Figure 8: Possible forbidden 2-tree subgraphs

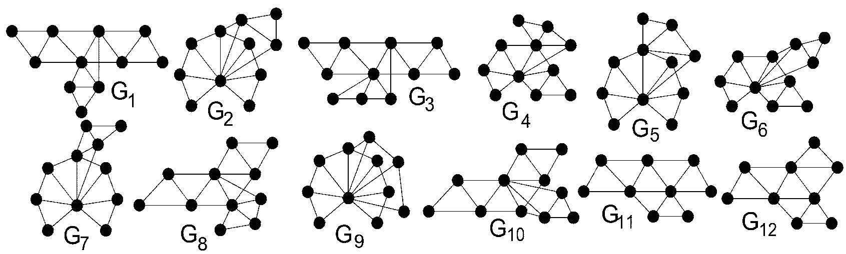

Figure 9: Forbidden subgraphs for p-interval 2-trees

Assume that G does not have a p-interval representation. From Theorem 2.8, we know that G is not a spiny interior 2-lobster. Let $e_0, t_1, e_1, \ldots, t_n, e_n$ be the longest 2-path and label it P. It follows from the definition of a spiny interior 2-lobster that either there exists a 2-path of length greater than or equal to 2 originating from two vertices of some t_i not equal to e_{i-1} or e_i or there exists a 2-path of length greater than or equal to 3 originating from the two vertices of some e_i.

Case 1: Assume there exists a 2-path, T, of length greater than or equal to 2 originating from the two vertices of some t_i not equal to e_{i-1} or e_i. Since P was assumed to be a longest 2-path, there exists a 2-path of length greater than or equal to 2 in either direction of t_i on P. Therefore, the minimum structure for this is a 2-path, P, of length five with T originating from the middle clique, t_3, and being of length 2. Przulj and Corneil proved in [15] that there are three non-ismorphic 2-paths of length 5 (see A_5^1, A_5^2, A_5^3 in Figure 7). Since t_3 has three pairs of vertices and two are e_2 and e_3, there is only one way to add the first vertex, x_1, of T to P. Since $x_1 \in N(x_2)$, x_2's other adjacency can be either vertex from the neighborhood of x_1. This gives us six 2-tree subgraphs, H_i $1 \leq i \leq 6$ in Figure 8. However,

$H_1 \cong H_2 \cong H_3 \cong H_4 \cong H_6 \cong G_{11}$ and $H_5 \cong G_{12}$.

Case 2: Assume there exists a 2-path, S, of length greater than or equal to 3 originating from the two vertices of some e_i. Since P was assumed to be a longest 2-path, there exists a 2-path of length greater than or equal to 3 in either direction of e_i on P. Therefore, the minimum structure is a 2-path, P, of length six with S originating from the middle clique, e_3, and being of length 3. Przulj and Corneil proved in [15] that there are six non-ismorphic 2-paths of length 6 (see $A_6^i 1 \leq i \leq 6$ in Figure 7). The first vertex on S, y_1, is adjacent to all of e_i. The next vertex on S, y_2, must be adjacent to y_1 and it has two choices for the other adjacency. Similarily, y_3 has two choices of adjacency. This gives us 24 2-tree subgraphs, H_i $7 \leq i \leq 30$ in Figure 8. However, $H_8 \cong H_{28} \cong G_1$, $H_{14} \cong H_{18} \cong G_2$, $H_7 \cong H_9 \cong H_{10} \cong H_{24} \cong G_3$, $H_{15} \cong H_{20} \cong H_{22} \cong H_{26} \cong G_4$, $H_{11} \cong H_{19} \cong H_{21} \cong G_5$, $H_{16} \cong H_{17} \cong H_{30} \cong G_6$, $H_{12} \cong H_{23} \cong G_7$, $H_{25} \cong H_{27} \cong G_8$, $H_{13} \cong G_9$, and $H_{29} \cong G_{10}$. Therefore, if G does not have a p-interval representation, then it contains a subgraph isomorphic to G_i, $1 \leq i \leq 12$. □

It is interesting to note that the graphs G_i, $1 \leq i \leq 12$ are isomorphic to the graphs S_i, $1 \leq i \leq 12$ in [15]. In fact, 2-tree probe interval graphs are characterized as a subset of spiny interior 2-lobsters in [2].

3 Conclusions and future work

We have characterized k-trees that are interval p-graphs as spiny interior k-lobsters, which are precisely the k-trees that are ATE-free. However, we know that ATEs do not completely characterize interval p-graphs. It would be interesting to determine what subset of the ATE-free graphs have an interval p-representation. Figure 4 is another forbidden subgraph for p-interval graphs, but there are probably more. It would also be interesting to find the complete set of forbidden subgraphs for interval p-graphs for specific values of p.

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