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The edge domination number of connected graphs

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Abstract

A subset X of edges in a graph G is called an *edge dominating set* of G if every edge not in X is adjacent to some edge in X. The *edge domination number* $\gamma'(G)$ of G is the minimum cardinality taken over all edge dominating sets of G. Let m, n and k be positive integers with $n-1 \leq m \leq \binom{n}{2}$, $\mathcal{G}(m,n)$ be the set of all non-isomorphic connected graphs of order n and size m, and

$$\mathcal{G}(m,n;k) = \{G \in \mathcal{G}(m,n) : \gamma'(G) = k\}.$$

We are able to determine all integers m, n, k for which $\mathcal{G}(m,n;k) \neq \emptyset$.

1 Introduction

We limit our discussion to graphs that are simple and finite. For the most part, our notation and terminology follows that of Chartrand and Zhang [2]. Let $G = (V, E)$ denote a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We use $|S|$ to denote the cardinality of a set S. We define $n = |V(G)|$ to be the *order* of G and $m = |E(G)|$ to be the *size* of G. We simply write $e = uv$ for the edge e that joins the vertices u and v. We say that u and v are *joined* by the edge e. The vertex u and the edge e (as well as v and e) are said to be *incident* with each other. Distinct edges incident with a common vertex are *adjacent edges*. For $X \subseteq E(G)$, the subgraph $\langle X \rangle$ of G induced by X is called an *edge induced subgraph* of G. A graph G is *complete* if every two distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n. A graph G is a *bipartite graph* if $V(G)$ can be partitioned into two subsets U and W, called *partite sets*, such that every edge of G joins a vertex of U and a vertex of W. We call G a *complete bipartite graph* if every vertex of U is adjacent to

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every vertex of W. A complete bipartite graph with $|U| = s$ and $|W| = t$ is denoted by $K_{s,t}$ or $K_{t,s}$. If either $s = 1$ or $t = 1$, then $K_{s,t}$ is a *star*. A graph G is called *acyclic* if it has no cycles. A *tree* is an acyclic connected graph. A subset M of the edge set E of a graph $G = (V, E)$ is an *independent edge set* or *matching* in G if no two distinct edges in M have a common vertex. For a graph G and $X \subseteq E(G)$, we denote by $G - X$ the graph obtained from G by removing all edges in X. If $X = \{e\}$, we write $G - e$ for $G - \{e\}$. For $X \subseteq E(\overline{G})$, $G + X$ denotes the graph obtained from G by adding all edges in X. If $X = \{e\}$, we simply write $G + e$ for $G + \{e\}$. For a proper subset X of V(G), $G - X$ is the graph obtained from G by removing all vertices in X and all edges incident with vertices in X. Necessarily, $G - X$ is an induced subgraph of G; indeed, $G - X = \langle V(G) - X \rangle$.

2 The main results on γ' in $\mathcal{G}(m, n)$

A set S of vertices of G is a *dominating set* of G if every vertex in $V(G) - S$ is adjacent to some vertex in S. A *minimum dominating set* in a graph G is a dominating set of minimum cardinality. The cardinality of a minimum dominating set is called the *domination number* of G and is denoted by $\gamma(G)$. There is an analogous concept of the edge domination which was introduced by Hedetniemi and Mitchell [3]. A subset X of edges in a graph G is called an *edge dominating set* of G if every edge not in X is adjacent to some edge in X. The *edge domination number* $\gamma'(G)$ of G is the minimum cardinality taken over all edge dominating sets of G.

Let G be a connected graph of order n. We can easily conclude that $\gamma(G) \leq \frac{n}{2}$ by the fact that if S is a dominating set of G, then $V(G) - S$ is also a dominating set of G. In order to prove that $\gamma'(G) \leq \frac{n}{2}$ for any connected graph G of order n, we first observe the following facts. Let G be a connected graph of order n and X be a minimum edge dominating set of G.

1. $\langle X \rangle$ contains no cycle, otherwise, let $C : e_1, e_2, \ldots, e_t, e_{t+1} = e_1$ be a cycle induced by t edges of X. Then for each $i = 1, 2, \ldots, t$, $X - \{e_i\}$ is an edge dominating set of G.

2. $\langle X \rangle$ contains no path of order 4, otherwise, let $P : e_1, e_2, e_3$ be a path of order 4 induced by 3 edges e_1, e_2, e_3 in X. Thus $X - \{e_2\}$ is an edge dominating set of G.

3. Suppose that there exist $e_i = uv, e_j = vw \in X$. Thus there exists an edge f of G such that f is incident to u, otherwise $X - \{e_i\}$ is an edge dominating set of G. Further, $f \notin X$ and $X' = (X - \{e_i\}) \cup \{f\}$ is a new minimum edge dominating set of G with e_i and f are not adjacent in X'.

We have the following results.

Theorem 2.1 *Let G be a connected graph of order n. Then*

$$\gamma'(G) = \min\{|X| : X \textit{ is a maximal independent edge set of } G\}.$$

In particular $\gamma'(G) \le \lfloor \frac{n}{2} \rfloor$.

A characterization of graphs reaching the upper bound was obtained in [1] and can be stated as the following theorem.

Theorem 2.2 *For any connected graph G of even order n, $\gamma'(G) = \frac{n}{2}$ if and only if G is isomorphic to K_n or $K_{n/2,n/2}$.*

Using the notation $\min(\gamma'; m, n) := \min\{\gamma'(G) : G \in \mathcal{G}(m, n)\}$ and $\max(\gamma'; m, n)$ $:= \max\{\gamma'(G) : G \in \mathcal{G}(m, n)\}$, we can rephrase Theorem 2.2 as follows.

Corollary 2.3 $\max(\gamma'; m, n) = \frac{n}{2}$ *if and only if n is even and* $m \in \{\frac{n^2}{4}, \binom{n}{2}\}$.

It is easy to see that any connected graph G of order $n = 2, 3$, $\gamma'(G) = 1$. We assume from now on that $n \ge 4$.

Let G be a connected graph of order n and $\gamma'(G) = t$. Thus there exists $X \subseteq E(G)$ such that X is a maximal independent edge set of size t. Therefore $|E(G)| \le 2t(n - 2t) + \binom{2t}{2}$. Put $\varepsilon(t) := 2t(n-2t) + \binom{2t}{2}$ if $t \ge 1$ and $\varepsilon(0) = n - 2$. Then we obtain the following result.

Theorem 2.4 *Let n and t be integers with $n \ge 2t$. Then* $\min(\gamma'; m, n) = t$ *if and only if* $\varepsilon(t-1) < m \le \varepsilon(t)$.

Let G be a graph of order n. Then G is called a *split graph* if $V(G)$ can be partitioned into two set X and Y such that $\langle X \rangle$ is a complete graph and $\langle Y \rangle$ is an empty graph. A split graph G with the partitioning sets X and Y is said to be *complete* if for each pair of vertices $x \in X$ and $y \in Y$, x and y are adjacent in G. We use $CS(k, n-k)$ for the complete split graph with the partitioning sets X and Y with $|X| = k$ and $|Y| = n - k$. Thus $CS(2t, n-2t)$ is the unique split graph of order n with $\gamma' = t$ and the maximum number of edges.

Let m, n and k be positive integers with $n - 1 \le m \le \binom{n}{2}$, $\mathcal{G}(m, n)$ be the set of all non-isomorphic connected graphs of order n and size m, and

$$\mathcal{G}(m, n; k) = \{G \in \mathcal{G}(m, n) : \gamma'(G) = k\}.$$

Let G be a graph of order n and size m. If $\gamma'(G) = t$, then, by Theorem 2.4, $m \le \varepsilon(t)$. Since we deal with connected graphs, it follows that $n - 1 \le m$ and hence $n-1 \le m \le \varepsilon(t)$. This shows that if $\mathcal{G}(m, n; t) \ne \emptyset$, then $n-1 \le m \le \varepsilon(t)$. The rest of this section is devoted to proving by construction that if $n - 1 \le m \le \varepsilon(t)$, then $\mathcal{G}(m, n; t) \ne \emptyset$. Starting with the graph $CS(2t, n-2t)$ we will give an algorithm to show how edges can be removed from that graph in such a way that the resulting graph on each step has the edge domination number unchanged. The following two lemmas are simple but useful.

Lemma 2.5 *Let G be a graph of order n containing K_{2t} as its subgraph. Then* $\gamma'(G) \ge t$.

Proof. Let G be a graph of order n containing K_{2t} as its subgraph and X be a minimum edge dominating set which is also independent. Then for each pair of distinct vertices x, y of K_{2t} there exists at least one $e \in X$ such that x and e are incident or y and e are incident. This proves that $\gamma'(G) = |X| \geq t$. ∎

As a consequence of Lemma 2.5 we have the following result.

Lemma 2.6 *Let $CS(2t, n-2t)$ be the complete split graph with $X \cup Y$ as its vertex set, $|X| = 2t$ and $|Y| = n-2t$. Put $U = \{xy : x \in X, y \in Y\}$. Then for each $F \subseteq U$, $\gamma'(CS(2t, n-2t) - F) = t$.*

As a consequence of Lemma 2.6 we have that if $n - 2t + \binom{2t}{2} \leq m \leq \varepsilon(t)$, then $\mathcal{G}(m, n; t) \neq \emptyset$.

In order to proceed further we first list some more notation and observations with the following facts.

1. Let G and H be two vertex-disjoint graphs. We use $G * H$ to denote a graph with $V(G*H) = V(G) \cup V(H) \cup \{x\}$ and $E(G*H) = E(G) \cup E(H) \cup \{xv : v \in V(G) \cup V(H)\}$. With this notation it follows that $\gamma'(G*H) \leq \gamma'(G) + \gamma'(H) + 1$. In particular $\gamma'(K_p * K_q) = \gamma'(K_p) + \gamma'(K_q)$ if both p and q are odd and $\gamma'(K_p * K_q) = \gamma'(K_p) + \gamma'(K_q) + 1$ otherwise, where $\gamma'(K_1) = 0$.

2. Let $G_1, G_2, \ldots, G_t$ be pairwise vertex disjoint graphs with $t \geq 3$. We use $G_1 * G_2 * \cdots * G_t$ to denote a graph with $V(G_1 * G_2 * \cdots * G_t) = \{x\} \cup \bigcup_{i=1}^{t} V(G_i)$ and $E(G_1 * G_2 * \cdots * G_t) = \{xv : v \in \bigcup_{i=1}^{t} V(G_i)\} \cup \{\bigcup_{i=1}^{t} E(G_i)\}$.

3. Let X be a finite nonempty set. We use $K(X)$ to denote the complete graph with X as its vertex set.

4. Let G be a connected graph with $\gamma'(G) \geq 2$. If X is a minimum edge dominating set of G which is also independent and $e = uv \in X$, then $\gamma'(G - \{u, v\}) = \gamma'(G) - 1$.

5. Let G be a graph of order n containing K_{2t} as its subgraph and X be a minimum edge dominating set which is also independent. Then for each pair of distinct vertices x, y of K_{2t} there exists at least one $e \in X$ such that x and e are incident or y and e are incident.This proves that $\gamma'(G) = |X| \geq t$.

Lemma 2.7 *Let $P = \{u_1, v_1\}$, $Q = \{u_2, v_2, u_3, v_3, \ldots, u_t, v_t\}$ and $U = \{u_1u_i, u_1v_i, v_1u_i, v_1v_i : i = 2, 3, \ldots, t\}$. If $G = K(P) * K(Q)$, where $V(G) = P \cup Q \cup \{x\}$, and $F \subseteq U$, then $\gamma'(G + F) = t$.*

Proof. By observation 1 above, we have that $\gamma'(G) = t$. By choosing $D = \{xu_1, u_2v_2, u_3, v_3, \ldots, u_tv_t\}$ as an edge dominating set of G, it follows that for any $F \subseteq U$, D is an edge dominating set of $G + F$ and hence $\gamma'(G + F) \leq t$.

Suppose that there exists $F \subseteq U$ such that $\gamma'(G+F) = t-1$. Thus any minimum edge dominating set of $G+F$ must contain an edge in F. Let D_1 be a minimum edge dominating set of $G+F$ of cardinality $t-1$. Without loss of generality we may assume that $u_1u_2 \in E_1$. Thus $G+F-\{u_1,u_2\} = t-2$ but $G+F-\{u_1,u_2\}$ contains $K_{2(t-1)}$ as its subgraph. This contradicts Lemma 2.6. ∎

Note that graphs as described in Lemma 2.7 have order $2t+1$. And by adding $n-2t-1$ vertices and joining each vertex to x, the resulting graphs have order n and $\gamma' = t$. Therefore as a consequence of Lemma 2.7 we have that if $n-2t+2+\binom{2t-1}{2} \leq m \leq n-2t-1+\binom{2t+1}{2}$, then $\mathcal{G}(m,n;t) \neq \emptyset$.

The graph $G_1 = G = K(P) * K(Q) \equiv K_2 * K_{2(t-1)}$ with P, Q and $V(G)$ as in Lemma 2.7 satisfies $\gamma'(G) = t$. If $t \geq 3$, we can define $G_2 = K_2 * K_2 * K_{2(t-2)}$. Clearly $\gamma'(G) = t$ and by the same argument as in the proof of Lemma 2.7 we have that if $n-2t+5+\binom{2t-3}{2} \leq m \leq n-2t+2+\binom{2t-1}{2}$, then $\mathcal{G}(m,n;t) \neq \emptyset$. By continuing in this way can define $G_t = K(\{u_1,v_1\}) * K(\{u_2,v_2\}) * \cdots * K(\{u_t,v_t\})$. Since G_t has order $2t+1$ and size $3t$, it follows by adding $n-2t-1$ vertices to G_t and joining each vertex to x result a graph of order n, size $n+t-1$ and $\gamma' = t$. Put G_t^* be the resulting graph. Let $D = \{xu_i : i = 1,2,\ldots,t\}$. Then for any $F \subseteq D$, $\gamma'(G_t^* - F) = t$. Thus we can conclude the following theorem.

Theorem 2.8 *Let m,n,t be integers satisfying $1 \leq t \leq \lfloor \frac{n}{2} \rfloor$. Then $\mathcal{G}(m,n;t) \neq \emptyset$ if and only if $n-1 \leq m \leq \varepsilon(t)$.*

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