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Hypergraceful complete graphs

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Abstract

In this paper we prove that no signed graph on the complete graph K_p, $p \geq 6$, is graceful, and we also give a characterization of graceful signed graphs on K_p, $p \leq 5$. This implies that there is no subset A of cardinality $p \geq 6$ from the set $\{0, 1, \ldots, \binom{p}{2} - n\}$, $n \leq \frac{1}{2}\binom{p}{2}$, such that each element of the set $\{1, 2, \ldots, n\}$ occurs exactly twice and each element of the remaining set $\{n+1, n+2, \ldots, \binom{p}{2} - n\}$ occurs exactly once as an absolute value of a pair of distinct elements of A. Also, all such subsets of cardinality $p \leq 5$ are determined.

1 Introduction

For standard terminology and notation in graph theory we follow West [11] and for signed graphs we follow Chartrand [7] and Zaslavsky [12, 13]. Additional terms will be defined as and when necessary.

An $S(p, \lambda)$-*synch set* is defined as a set of p distinct nonnegative integers for which not more than λ pairs have the same common difference and for which the maximum element is as small as possible (cf. [9]). In this context, p is the total number of holes present and λ is the maximum number of holes that can be simultaneously aligned in an out-of-synch position. A synch set designates positions for the p holes so that distance from the first to the last hole is minimized under the said constraints. When $\lambda = 1$, this is rephrasing of the famous 'Golomb ruler' problem which is related to the gracefulness of the complete graph K_p of order p (Golomb [8]). Simmons [9] discovered many such sets for $\lambda > 1$. These sets could be represented either as the minimum length rulers that allow the measurements to be repeated λ times, or as distinct labelings of K_p which minimize $\theta(K_p)$, the largest vertex label, and allow λ repeats of edge numbers (cf. [5, 6, 9]).

In general, a (p, q)-complete graph $G = (V, E)$ is k-*hypergraceful* with respect to a decomposition G into *edge-induced subgraphs* $G_1, G_2, \ldots, G_k$, having sizes $m_1, m_2, \ldots, m_k$ respectively, if there exists an injective function $f : V \longrightarrow \{0, 1, \ldots, q^*\}$, where $q^* = \max\{m_i : 1 \leq i \leq k\}$ such that when each edge $uv \in E(G)$ is assigned the absolute difference $|f(u) - f(v)|$ (often called the *bandwidth* of the edge) as its label, then the set of labels received by the edges of G_i is precisely $\{1, 2, \ldots, m_i\}$ for each $i \in \{1, 2, \ldots, k\}$. Such a function f is called a k-*hypergraceful labeling* (or simply a *hypergraceful labeling*) of G and the graph which admits such a labeling is called a *hypergraceful graph*. If f is a hypergraceful labeling of a graph G, then f^* is also a hypergraceful labeling of G, called the *complement* of the hypergraceful labeling of f, defined as $f^*(u) = q^* - f(u), \forall u \in V(G)$ and we have $(f^*)^* = f$. It is immediate that the case $k = 1$ in the above definition yields the well-known notion of graceful graphs [6, 9, 10]. The case $k = 2$ corresponds to the extension of the notion of graceful graphs to the realm of signed graphs (or *sigraphs* in short), as studied in [2–4, 10].

Also, it is clear, in general, that in the above definition $G_1, G_2, \ldots, G_k$ may be replaced by $G_{\sigma(1)}, G_{\sigma(2)}, \ldots, G_{\sigma(k)}$ for any permutation σ of the set $\{1, 2, \ldots, k\}$. As such, in the more general setting, the notion of hypergraceful decomposition of graphs was first introduced by Acharya [1].

A *signed graph* (*sigraph*, for short) is an ordered pair $S = (G, s)$ where $G = (V, E)$ is a simple (p, q)-graph, called its *underlying graph* and $s : E \longrightarrow \{+, -\}$ is a function, called its *signing function* or *signature* (e.g., see [2, 7, 10, 12, 13]); an edge e of S for which $s(e) = +$ (respectively, $s(e) = -$) is said to be *positive* (*negative*). Let $E^+(S)$ and $E^-(S)$ denote respectively the set of positive and the set of negative edges of S so that $E^+(S) \cup E^-(S) = E(S)$ is the entire edge set of S. If S has p vertices, $|E^+(S)| = m$ and $|E^-(S)| = n$ so that $m + n = q$, then S is called a (p, m, n)-*sigraph*.

In this paper, k-hypergraceful complete graphs for $k = 2$ are characterized and partial results for $k \geq 3$ are obtained. The case $k = 2$ is equivalent to solving the problem for sigraphs. We prove that no sigraph on the complete graph K_p, $p \geq 6$, is graceful and also give the characterization of graceful sigraphs on K_p, $p \leq 5$. This implies that there is no subset $A \subseteq \{0, 1, \ldots \binom{p}{2} - n\}$, $n \leq \frac{1}{2}\binom{p}{2}$, of cardinality $p \geq 6$, such that each element of the set $\{1, 2, \ldots, n\}$ occurs exactly twice and each element of the remaining set $\{n+1, n+2, \ldots, \binom{p}{2} - n\}$ occurs exactly once as an absolute difference of values of the pairs of distinct elements of A. Also, we produce all such subsets of cardinality $p \leq 5$.

2 The Main Results

In the following, $\lfloor x \rfloor$ will denote the greatest integer not greater than the real number x.

Theorem 1. *A necessary condition for a (p, q)-graph $G = (V, E)$ to be k-hypergraceful with respect to the given decomposition $G_1, G_2, \ldots, G_k$ is that it is possible to partition the vertex set $V = V(G)$ into two subsets V_o and V_e such that for each integer $i \in \{1, 2, \ldots, k\}$ there are exactly $\lfloor \frac{m_i+1}{2} \rfloor$ edges of G_i each of which joins a vertex of V_o with one of V_e.*

Proof. Since G is k-hypergraceful with respect to the given decomposition $G_1, G_2, \ldots, G_k$, there must be a k-hypergraceful labeling f of G. Let $V_o = \{u \in V : f(u)$ is odd$\}$ and $V_e = V - V_o$. Now, it is easy to see that every edge which joins a vertex of V_o with one of V_e receives an odd number as a label under f. Since f is k-hypergraceful labeling, the number of edges of G_i, $i \in \{1, 2, \ldots, k\}$, across V_o and V_e is precisely $\lfloor \frac{m_i+1}{2} \rfloor$, for every i, $1 \leq i \leq k$. This completes the proof. □

The case $k = 2$ of the above theorem was established in [2]; (also, see [3, 4, 10]).

Lemma 2. *If for no integer j, $0 \leq j \leq k$, $p - 2j$ is a perfect square, then K_p is not k-hypergraceful with respect to any decomposition of K_p.*

Proof. Suppose that K_p is a k-hypergraceful graph with some decomposition $G_1, G_2, \ldots, G_k$ where $|E(G_i)| = m_i$, $1 \leq i \leq k$. Then there exists a k-hypergraceful labeling $f : V \to \{0, 1, \ldots, q^*\}$, where $q^* = \max\{m_i : 1 \leq i \leq k\}$. Hence, by Theorem 1, there exists a partition of the vertex set V of K_p into two subsets V_o and V_e of cardinalities a and b say, satisfying the conditions

$$a + b = p$$

and

$$ab = \sum_{i=1}^{k} \left\lfloor \frac{m_i + 1}{2} \right\rfloor.$$

Without loss of generality, let the first t of the m_i's be even so that the other $(k-t)$ m_i's are odd, where $0 \leq t \leq k$. Then

$$ab = \sum_{i=1}^{t} \frac{m_i}{2} + \sum_{i=t+1}^{k} \frac{m_i+1}{2}$$

$$= \frac{p(p-1)}{4} + \frac{(k-t)}{2}.$$

Therefore, $|a-b| = \sqrt{p-2(k-t)}$, which should be an integer.

This contradicts the fact that $p-2j$ is not a perfect square for any j, with $0 \leq j \leq k$. □

Lemma 2 helps us to find infinitely many values of p for which K_p is not k-hypergraceful with respect to any of its decompositions. In particular, we have the following:

Observation 3. *If $x \geq 2k$, α is an odd integer with $0 \leq \alpha \leq 2k-1$ and $p = x^2 + \alpha$, then K_p is not k-hypergraceful with respect to any of its decompositions.*

2.1 The Case $k = 2$

In this case, the study reduces to that of gracefulness of sigraphs as in [2–4, 10].

By the *negation of a sigraph S*, we mean a sigraph $\eta(S)$ which is obtained from S by changing the sign of every edge to its opposite. It is straightforward to see that if a sigraph S is graceful with a graceful labeling f, then $\eta(S)$ is also graceful under the same labeling f.

We now state and prove the main result of this paper.

Theorem 4. *(a) No sigraph on K_p, $p \geq 6$ is graceful.*

(b) Every sigraph on K_p, $p \leq 3$ is graceful.

(c) A sigraph on K_4 is graceful if and only if the number of negative edges in it is not three.

(d) A sigraph S on K_5 is graceful if and only if S satisfies one of the following statements:

(i) the number n of negative edges in S is 1,

(ii) $n = 3$ and the three negative edges in S are not incident at the same vertex,

or $\eta(S)$ satisfies similar conditions with n replaced by m, the number of positive edges in S.

We prove this theorem by invoking a series of lemmas and also by giving a recursive procedure to find a possible graceful labeling of sigraphs on K_p, if any.

The following lemma is a reduction of Lemma 2 for $k = 2$.

Lemma 5. *If any integer p is such that none of p, $p-2$ and $p-4$ is a perfect square, then no sigraph on K_p is graceful.*

Now, we shall prove a basic lemma which gives an algorithm to find a 2-hypergraceful labeling f of K_p, if one exists, starting from the *basic set* B which consists of only the two elements 0 and q^*. By the 'highest *unsaturated* edge label based on the basic set B with deficiency d' we shall mean the highest number that does not appear d times as the value of on any edge of G.

Lemma 6. *Let R_f be the range of a graceful labeling f of a graceful complete sigraph K_p with n negative edges where $n \leq m$ and $q = m + n = \binom{p}{2}$. Let B be a subset of R_f such that 0 and q^* are in B. Let $q^* - x$, $x > 0$, be the highest unsaturated edge label based on B with deficiency $d > 0$. Then the following statements hold:*

1. *If $d = 2$, then both x and $q^* - x$ are in R_f.*
2. *(a) If $d = 1$ and the representation of $q^* - x$ based on R_f is 1 and none of x and $q^* - x$ is in B, then exactly one of x and $q^* - x$ is in R_f.*

   or

   (b) If $d = 1$ and the representation of $q^* - x$ based on R_f is 2 and none of x and $q^* - x$ is in B, then exactly one of x and $q^* - x$ is in R_f .

   or

   (c) If $d = 1$ and the representation of $q^* - x$ based on R_f is 2 but one of x and $q^* - x$ is in B, then both of x and $q^* - x$ are in R_f and exactly one of x and $q^* - x$ is in B.

Proof. We first prove the following important claim:

Claim 7. *If for any i, $1 \leq i \leq x - 1$, $(q^* - i)$ and $(x - i)$ are both in R_f, then they are both in B as well.*

Proof of the claim: Under the hypothesis of the claim, suppose that one of $(q^* - i)$ and $(x - i)$ is not in B. Then, we consider the following two cases:

Case 1: $(q^* - i) \notin B$.

Since $1 \leq i \leq (x - 1)$ for some i, we have $(q^* - i) > (q^* - x)$. Then, by the definition of $(q^* - x)$, the highest unsaturated edge label based on B, $(q^* - i)$ is a saturated edge label based on B. But $q^* - i = (q^* - i) - 0$ is a representation of $(q^* - i)$ in R_f by hypothesis, since $(q^* - i) \in R_f$ and $0 \in R_f$. Therefore $(q^* - i) \in B$, which is a contradiction.

Case 2: $(x - i) \notin B$.

Then, the edge label $q^* - (x-i) > (q^* - x)$ and, by the definition of $(q^* - x)$, the edge label $q^*-(x-i)$ is a saturated edge label based on B. But $q^*-x+i = q^*-(x-i)$ is a representation based on R_f by hypothesis, which is not a representation based on B, a contradiction.

Now, we are ready to prove Lemma 6.

1. First, let $d = 2$.

   Then $(q^* - x)$ has no representation based on B and, by Claim 7, there is no representation of $(q^* - x)$ based on R_f with $1 \le i \le x - 1$. As $d = 2$, the only representations of $q^* - x$ based on R_f are $q^* - x$ and $(q^* - x) - 0$, and hence $(q^* - x)$ and x are in R_f.

2. (a) Suppose that $d = 1$.

   If the representation of $q^* - x$ based on R_f is 1, then by Claim 7, there is no representation of $q^* - x$ as $(q^* - i) - (x - i)$ with $1 \le i \le x - 1$. Thus, the representation of $q^* - x$ in R_f is exactly one of $q^* - x$ and $(q^* - x) - 0$. Thus, if exactly one of $q^* - x$ and x is in R_f then none of $q^* - x$ and x is in B.

   (b) None of $q^* - x$, x is in B and the replication of $q^* - x$ based on R_f is 2. Since $d = 1$, then by Claim 7, there is exactly one i with both $q^* - i$, $x - i$ in R_f for $1 \le i \le x - 1$. Since $x \ge 1$, there is exactly one more representation of $q^* - x$ based on R_f, and that must be $q^* - x$ or $(q^* - x) - 0$. Therefore exactly one of x, $q^* - x \in R_f$.

   (c) At least one of $q^* - x$ and x is in B, say $x \in B$. Then, $q^* - x$ is a representation of $q^* - x$ based on B. Since $d = 1$, by Claim 7, there is no i with both $q^* - i$ and $x - i$ in R_f, $1 \le i \le x - 1$. Therefore, the representation of $q^* - x$ based on R_f is $(q^* - x) - 0$ and hence $q^* - x \in R_f$. Thus, both $q^* - x$ and x are in R_f. As x is already in B, and $d = 1$, we see that $q^* - x \notin B$.

□

Now, first we settle the case of sigraphs on K_p, for $p = 6$.

Throughout the proof of Lemmas 8–10, we use “Vertex labels” as sets of vertex labels and “Repetition” as repetition on the same parity edges in each of the tables.

Lemma 8. *No sigraph on K_6 is graceful.*

Proof. Let us assume that some sigraph S on K_6 is graceful. Then, the possible labels of the vertices of S can be obtained by the repeated application of Lemma 6, starting with the basic set $\{0, q^*\}$. The following seven cases arise, depending on the value of n.

Case 1: $n = 7$. In this case, we have to assign the labels to the vertices of K_6 from the set $\{0, 1, \ldots, 8\}$ such that the negative edges of S receive the labels from the set $\{1, 2, \ldots, 7\}$ and the positive edges of S receive the labels from the set $\{1, 2, \ldots, 8\}$.

Vertex labels	Edge labels	Repetition
{0,1,6,7,8}	1	3
{0,1,2,7,8}	1	3

Table 1: Vertex labeling of K_6 when $n = 7$.

Then a possible set of labels of some of the vertices of K_6 under f by Lemma 6 could be one from the two sets given in Table 1.

In each of the cases shown in the Table 1, the number in the third column violates the definition of gracefulness of a sigraph.

Case 2: $n = 6$. In this case, we have to assign the labels to the vertices of K_6 from the set $\{0, 1, \ldots, 9\}$ such that the negative edges of S receive the labels from the set $\{1, 2, \ldots, 6\}$ and the positive edges of S receive the labels from the set $\{1, 2, \ldots, 9\}$. Then a possible set of labels of some of the vertices of K_6 under f by Lemma 6 could be one from amongst the following four sets:

Vertex labels	Edge labels	Repetition
{0,3,6,7,8,9}	1	3
{0,2,4,5,6,8,9}	1	3
{0,2,3,5,8,9}	3	3
{0,2,3,4,8,9}	1	3

Table 2: Vertex labeling of K_6 when $n = 6$.

In each of the cases shown in Table 2, the number in the third column violates the definition of gracefulness of a sigraph.

Case 3: $n = 5$. In this case, we have to assign the labels to the vertices of K_6 from the set $\{0, 1, \ldots, 10\}$ such that the negative edges of S receive the labels from the set $\{1, 2, 3, 4, 5\}$ and the positive edges of S receive the labels from the set $\{1, 2, \ldots, 10\}$. Then a possible set of labels of some of the vertices of K_6 under f by Lemma 6 could be one from amongst the following four sets:

Vertex labels	Edge labels	Repetition
{0,7,8,9,10}	1	3
{0,3,5,8,9,10}	5	3
{0,2,5,6,9,10}	4	3
{0,2,4,5,9,10}	5	3

Table 3: Vertex labeling of K_6 when $n = 5$.

In each of the cases shown in the Table 3, the number in the third column violates the definition of gracefulness of a sigraph.

Case 4: $n = 4$. In this case, we have to assign the labels to the vertices of K_6 from the set $\{0, 1, \ldots, 11\}$ such that the negative edges of S receive the labels from the set $\{1, 2, 3, 4\}$ and the positive edges of S receive the labels from the set $\{1, 2, \ldots, 11\}$. Then a possible set of labels of some of the vertices of K_6 under f by Lemma 6 could be one from amongst the following seven sets:

Vertex labels	Edge labels	Repetition
{0,8,9,10,11}	1	3
{0,3,6,9,10,11}	3	3
{0,3,5,9,10,11}	6	2
{0,2,6,7,10,11}	6	2
{0,2,5,7,10,11}	5	3
{0,2,4,6,10,11}	4	3
{0,2,4,5,10,11}	6	2

Table 4: Vertex labeling of K_6 when $n = 4$.

In each of the cases shown in the Table 4, the number in the third column violates the definition of gracefulness of a sigraph.

Case 5: $n = 3$. In this case, we have to assign the labels to the vertices of K_6 from the set $\{0, 1, \ldots, 12\}$ such that the negative edges of S receive the labels from the set $\{1, 2, 3\}$ and the positive edges of S receive the labels from the set $\{1, 2, \ldots, 12\}$. Then a possible set of labels of some of the vertices of K_6 under f by Lemma 6 could be one from amongst the following five sets:

Vertex labels	Edge labels	Repetition
{0, 9, 10, 11, 12}	1	3
{0, 3, 6, 10, 11, 12 }	6	2
{0, 2, 7,8, 11, 12 }	5	2
{0,2,5,8,11,12}	6	2
{0,2,4,6,11,12}	2	3

Table 5: Vertex labeling of K_6 when $n = 3$.

In each of the cases shown in the Table 5, the number in the third column violates the definition of gracefulness of a sigraph.

Case 6: $n = 2$. In this case, we have to assign the labels to the vertices of K_6 from the set $\{0, 1, \ldots, 13\}$ such that the negative edges of S receive the labels from the set $\{1, 2\}$ and the positive edges of S receive the labels from the set $\{1, 2, \ldots, 13\}$. Then a possible set of labels of some of the vertices of K_6 under f by Lemma 6 could be one from amongst the following seven sets:

In each of the cases shown in the Table 6, the number in the third column violates the definition of gracefulness of a sigraph.

Vertex labels	Edge labels	Repetition
{0,10,11,12,13}	1	3
{0,3,7,11,12,13}	4	2
{0,3,6,11,12,13}	6	2
{0,2,8,9,12,13}	4	2
{0,2,5,9,12,13}	7	2
{0,2,4,7,12,13}	5	2
{0,2,4,6,12,13}	2	3

Table 6: Vertex labeling of K_6 when $n = 2$.

Case 7: $n = 1$. In this case, we have to assign the labels to the vertices of K_6 from the set $\{0, 1, \ldots, 14\}$ such that the negative edge of S receives the label 1 and the positive edges of S receive the labels from the set $\{1, 2, \ldots, 14\}$. Then a possible set of labels of some of the vertices of K_6 under f by Lemma 6 could be one from amongst the following six sets:

Vertex labels	Edge labels	Repetition
{0,11,12,13,14}	1	3
{0,3,8,12,13,14}	5	2
{0,3,6,12,13,14}	6	2
{0,2,9,10,13,14}	4	2
{0,2,5,10,13,14}	5	2
{0,2,4,13,14}	2	2

Table 7: Vertex labeling of K_6 when $n = 1$.

In each of the cases shown in the Table 7, the number in the third column violates the definition of gracefulness of a sigraph.

Thus, we see that no sigraph on K_6 is graceful. This completes the proof. ☐

One can easily see from the illustration (as given in Figure 1), how the range R_f can be obtained starting with the basic set $B = \{0, q^*\}$, using Lemma 6. We can extend the basic set B at each stage by choosing the possible vertex label's either left number or right number (which is shown in Figure 1 using the 'choice tree' diagram) and the edge labels and their repetitions are shown inside a bracket and after the bracket respectively.

Now, we consider the sigraph on K_p, $p \geq 7$. First, we prove the following lemma.

Lemma 9. *No sigraph on K_p, $p \geq 7$, with n negative edges, $n < \frac{1}{2}\binom{p}{2} - 9$, is graceful.*

Proof. By definition of a graceful sigraph, there is an assignment f of the labels to the vertices of K_p from the set $\{0, 1, \ldots, q - n = q^*\}$ such that the negative edges of S receive the labels from the set $\{1, 2, \ldots, n\}$ and the positive edges of S receive the

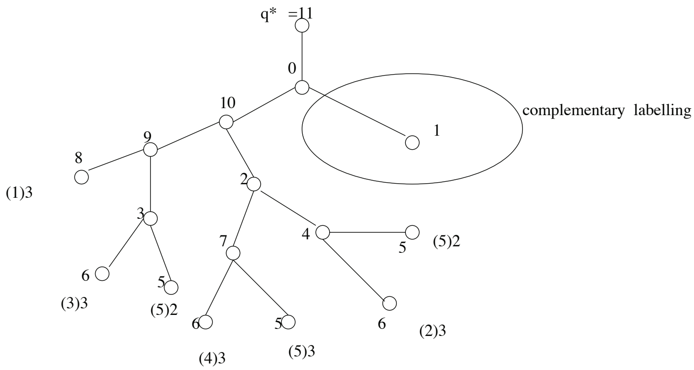

Figure 1: Illustration of using 'choice tree' diagram for obtaining an optimal labeling of a sigraph S on K_6 with $n = 4$.

labels from the set $\{1, 2, \ldots, q^* = q-n\}$, where $q = \binom{p}{2}$. Then, a possible set of labels of some of vertices of K_p could be one from amongst the seventeen sets in Table 8 by the repeated application of Lemma 6 starting with the basic set $B = \{0, q^*\}$,

In each of the cases shown in the Table 8, the number in the third column violates the definition of gracefulness of a sigraph.

Here, one can easily verify that each of the edge labels $(q^* - 6)$, $(q^* - 8)$, $(q^* - 9)$, $(q^* - 12)$, $(q^* - 18)$, which appear twice, is always greater than n (i.e., $q^* - 18 > n$), where $n < \frac{1}{2}\binom{p}{2} - 9$.

Hence no sigraph on K_p, $p \geq 7$, with the number of negative edges $n < \frac{q}{2} - 9$, is graceful. □

Lemma 10. *No sigraph on K_p, $p \geq 7$, with the number of negative edges n where $\lfloor \frac{q}{2} \rfloor - 9 \leq n \leq \lfloor \frac{q}{2} \rfloor$, is graceful.*

Proof. By the definition of a graceful sigraph, there is an assignment f of the labels to the vertices of K_p from the set $\{0, 1, \ldots, q - n = q^*\}$ such that the negative edges of S receive the labels from the set $\{1, 2, \ldots, n\}$ and the positive edges of S receive the labels from the set $\{1, 2, \ldots, q^* = q - n\}$, where $q = \binom{p}{2}$. We consider the following ten cases depending upon the value of n, and in each case by the repeated application of Lemma 6, we show that the sigraph S under consideration is not graceful by starting with the basic set $\{0, q^*\}$.

Case 1: $n = \lfloor \frac{q}{2} \rfloor$.
In this case, the following two cases arise:

(a): q is even: Then there do not exist two distinct pairs such that their absolute differences are equal. Hence, in this case no sigraph S on K_p is graceful.

(b): q is odd: Then a possible set of labels of some of the vertices of K_p is from one of the following two sets given in Table 9.

Vertex labels	Edge labels	Repetition
$\{0, q^*, q^*-1, q^*-2, q^*-3\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, q^*-6, q^*-7\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, q^*-6, 7\}$	q^*-9	2
$\{0, q^*, q^*-1, q^*-2, 3, 6, q^*-9, q^*-10\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, 6, q^*-9, 10\}$	q^*-12	2
$\{0, q^*, q^*-1, q^*-2, 3, 6, 9\}$	3	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, q^*-8\}$	4	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 8, q^*-10\}$	5	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 8, 10, q^*-16, q^*-17\}$	1	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 8, 10, q^*-16, 17\}$	q^*-18	2
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 8, 10, 16\}$	8	3
$\{0, q^*, q^*-1, 2, q^*-4, 5\}$	q^*-6	2
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, q^*-12\}$	6	3
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, 12, q^*-14\}$	q^*-18	2
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, 12, 14\}$	2	3
$\{0, q^*, q^*-1, 2, 4, q^*-6, 7\}$	q^*-8	2
$\{0, q^*, q^*-1, 2, 4, 6\}$	2	3

Table 8: Vertex labeling of K_p, $p \geq 7$ when $n < \frac{1}{2}\binom{p}{2} - 9$.

Vertex labels	Edge labels	Repetition
$\{0, q^*, q^*-1, 1, q^*-2\}$	1	3
$\{0, q^*, q^*-1, 1, 2\}$	1	3

Table 9: Vertex labels of K_p, $p \geq 7$ when q is odd.

In each of the cases shown in Table 9, the number in the third column violates the definition of gracefulness of a sigraph.

Case 2: $n = \lfloor \frac{q}{2} \rfloor - 1$. In this case, a possible set of labels of some of the vertices of K_p under f by Lemma 6 could be one from amongst the four sets in Table 10.

In each of the cases shown in Table 10, the number in the third column violates the definition of gracefulness of a sigraph.

Case 3: $n = \lfloor \frac{q}{2} \rfloor - 2$. In this case, a possible set of labels of some of the vertices of K_p under f by Lemma 6 could be one from amongst the eleven sets as given in Table 11.

In each of the cases shown in Table 11, the number in the third column violates the definition of gracefulness of a sigraph.

Case 4: $n = \lfloor \frac{q}{2} \rfloor - 3$. In this case, a possible set of labels of some of the vertices of K_p under f by Lemma 6 could one be from amongst the thirteen sets as given in Table 12.

In each of the cases shown in Table 12, the number in the third column violates

Vertex labels	Edge labels	Repetition
$\{0, q^*, q^*-1, q^*-2, q^*-3, 3\}$	1	3
$\{0, q^*, q^*-1, 2, q^*-3, q^*-4, 4\}$	2	3
$\{0, q^*, q^*-1, 2, 3, q^*-4, q^*-5, 5\}$	1	3
$\{0, q^*, q^*-1, 2, 3, 4\}$	1	3

Table 10: Vertex labels of $K_p, p \geq 7$, when $n = \lfloor \frac{q}{2} \rfloor - 1$.

Vertex labels	Edge labels	Repetition
$\{0, q^*, q^*-1, q^*-2, q^*-3\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, q^*-5, q^*-6, 6\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, 5, q^*-6, q^*-7\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, 5, q^*-6, 7\}$	2	3
$\{0, q^*, q^*-1, q^*-2, 3, 5, 6\}$	1	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 5, q^*-7\}$	3	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 5, 7, q^*-8\}$	3	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 5, 7, 8\}$	3	3
$\{0, q^*, q^*-1, 2, 4, q^*-5, q^*-6, 6\}$	2	3
$\{0, q^*, q^*-1, 2, 4, 5, q^*-6, q^*-7, 7\}$	1	3
$\{0, q^*, q^*-1, 2, 4, 5, 6\}$	1	3

Table 11: Vertex labels of $K_p, p \geq 7$, when $n = \lfloor \frac{q}{2} \rfloor - 2$.

the definition of gracefulness of a sigraph.

In this case, it is easy to check that $q^* - 6$ is greater than n.

Case 5: $n = \lfloor \frac{q}{2} \rfloor - 4$. In this case, a possible set of labels of some of the vertices of K_p under f by Lemma 6 could be one from amongst the thirteen sets as given in Table 13.

In each of the cases shown in Table 13, the number in the third column violates the definition of gracefulness of a sigraph.

In this case, it is easy to check that the edge labels $q^* - 6$, $q^* - 8$ are greater than $n = \lfloor \frac{q}{2} \rfloor - 4$.

Case 6: $n = \lfloor \frac{q}{2} \rfloor - 5$. In this case, a possible set of labels of some of the vertices of K_p under f by Lemma 6 could be one from amongst the eighteen sets as given in Table 14.

In each of the cases shown in Table 14, the number in the third column violates the definition of gracefulness of a sigraph.

In this case, it is easy to check that the edge labels $q^* - 6$, $q^* - 8$ and $q^* - 9$ are greater than $n = \lfloor \frac{q}{2} \rfloor - 5$.

Case 7: $n = \lfloor \frac{q}{2} \rfloor - 6$. In this case, a possible set of labels of some of the vertices of K_p under f by Lemma 6 could be one from amongst the seventeen sets as given in Table 15.

Vertex labels	Edge labels	Repetition
$\{0, q^*, q^*-1, q^*-2, q^*-3\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, q^*-6, q^*-7, 7\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, 6, q^*-7, q^*-8\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, 6, q^*-7, 8, q^*-9\}$	2	3
$\{0, q^*, q^*-1, q^*-2, 3, 6, q^*-7, 8, 9\}$	3	3
$\{0, q^*, q^*-1, q^*-2, 3, 6, 7\}$	1	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, q^*-7, q^*-8, 8\}$	1	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 7, q^*-8\}$	4	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 7, 8\}$	1	3
$\{0, q^*, q^*-1, 2, q^*-4, 5\}$	q^*-6	2
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, 7, q^*-9\}$	2	3
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, 7, 9\}$	2	3
$\{0, q^*, q^*-1, 2, 4, 6, \}$	2	3

Table 12: Vertex labels of $K_p, p \geq 7$, when $n = \lfloor \frac{q}{2} \rfloor - 3$.

In each of the cases shown in Table 15, the number in the third column violates the definition of gracefulness of a sigraph.

In this case, it is easy to check that the edge labels q^*-6, q^*-8, q^*-9, q^*-12 are greater than $n = \lfloor \frac{q}{2} \rfloor - 6$.

Case 8: $n = \lfloor \frac{q}{2} \rfloor - 7$. In this case, a possible set of labels of some of the vertices of K_p under f by Lemma 6 could be one from amongst the seventeen sets as given in Table 16.

In each of the cases shown in Table 16, the number in the third column violates the definition of gracefulness of a sigraph.

In this case, it is easy to check that the edge labels $q^*-6,\ q^*-8,\ q^*-9,\ q^*-12$ are greater than $n = \lfloor \frac{q}{2} \rfloor - 7$.

Case 9: $n = \lfloor \frac{q}{2} \rfloor - 8$. In this case, a possible set of labels of some of the vertices of K_p under f by Lemma 6 could be one from amongst the seventeen sets as given in Table 17.

In each of the cases shown in Table 17, the number in the third column violates the definition of gracefulness of a sigraph.

In this case, it is easy to check that each of the edge labels $q^*-6,\ q^*-8,\ q^*-9,\ q^*-12$ and q^*-16 is greater than $n = \lfloor \frac{q}{2} \rfloor - 8$.

Case 10: $n = \lfloor \frac{q}{2} \rfloor - 9$. In this case, a possible set of labels of some of the vertices of K_p under f by Lemma 6 could be one from amongst the eighteen sets as given in Table 18.

In each of the cases shown in Table 18, the number in the third column violates the definition of gracefulness of a sigraph.

In this case, it is easy to check that each of the edge labels $q^*-6,\ q^*-8,$

Vertex labels	Edge labels	Repetition
$\{0, q^*, q^*-1, q^*-2, q^*-3\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, q^*-6, q^*-7\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, q^*-6, 7, q^*-10, 10\}$	4	3
$\{0, q^*, q^*-1, q^*-2, 3, 6, q^*-9, 9\}$	3	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, q^*-8\}$	4	3
$\{0, q^*, q^*-1, q^*-2, q^*-4, q^*-5, 8, q^*-9\}$	4	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 8, 9\}$	1	3
$\{0, q^*, q^*-1, 2, q^*-4, 5\}$	q^*-6	2
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, q^*-9\}$	2	3
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, 9, q^*-11\}$	5	3
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, 9, 11\}$	2	3
$\{0, q^*, q^*-1, 2, 4, q^*-6, 7\}$	q^*-8	2
$\{0, q^*, q^*-1, 2, 4, 6\}$	2	3

Table 13: Vertex labels of $K_p, p \geq 7$, when $n = \lfloor \frac{q}{2} \rfloor - 4$.

$q^*-9,\ q^*-12,\ q^*-16,\ q^*-18$ is greater than $n = \lfloor \frac{q}{2} \rfloor - 9$.

In view of the above, the sigraph S on K_p is not graceful, where $\lfloor \frac{q}{2} \rfloor - 9 \leq n \leq \lfloor \frac{q}{2} \rfloor$. This completes the proof. □

Thus, we have proved that no sigraph on K_p, $p \geq 7$, with $n \leq \lfloor \frac{q}{2} \rfloor$ number of negative edges is graceful, enabling us to assert the following.

Theorem 11. *No sigraph on K_p, $p \geq 6$, is graceful.*

Proof. The proof follows by Lemmas 8 to 10. □

Lemma 12. *All sigraphs on K_p, $p \leq 3$, are graceful.*

The graceful labeling of sigraphs on K_p, $p \leq 3$, are given in Figure 2.

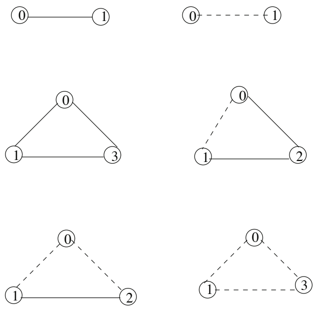

Figure 2: Graceful sigraphs on K_p when $p \leq 3$.

Vertex labels	Edge labels	Repetition
$\{0, q^*, q^* - 1, q^* - 2, q^* - 3\}$	1	3
$\{0, q^*, q^* - 1, q^* - 2, 3, q^* - 6, q^* - 7\}$	1	3
$\{0, q^*, q^* - 1, q^* - 2, 3, q^* - 6, 7\}$	$q^* - 9$	2
$\{0, q^*, q^* - 1, q^* - 2, 3, 6, q^* - 9, q^* - 10\}$	1	3
$\{0, q^*, q^* - 1, q^* - 2, 3, 6, q^* - 9, 10, q^* - 11, q^* - 13, 13\}$	3	3
$\{0, q^*, q^* - 1, q^* - 2, 3, 6, q^* - 9, 10, 11\}$	1	3
$\{0, q^*, q^* - 1, q^* - 2, 3, 6, 9\}$	3	3
$\{0, q^*, q^* - 1, 2, q^* - 4, q^* - 5, q^* - 8\}$	4	3
$\{0, q^*, q^* - 1, 2, q^* - 4, q^* - 5, 8, q^* - 10, q^* - 11, 11\}$	1	3
$\{0, q^*, q^* - 1, 2, q^* - 4, q^* - 5, 8, 10, q^* - 11, q^* - 12\}$	1	3
$\{0, q^*, q^* - 1, 2, q^* - 4, q^* - 5, 8, 10, q^* - 11, 12\}$	2	3
$\{0, q^*, q^* - 1, 2, q^* - 4, q^* - 5, 8, 10, 11\}$	1	3
$\{0, q^*, q^* - 1, 2, q^* - 4, 5\}$	$q^* - 6$	2
$\{0, q^*, q^* - 1, 2, 4, q^* - 6, q^* - 7, q^* - 11, q^* - 12, 12\}$	1	3
$\{0, q^*, q^* - 1, 2, 4, q^* - 6, q^* - 7, 11, q^* - 12\}$	6	3
$\{0, q^*, q^* - 1, 2, 4, q^* - 6, q^* - 7, 11, 12\}$	1	3
$\{0, q^*, q^* - 1, 2, 4, q^* - 6, 7\}$	$q^* - 8$	2
$\{0, q^*, q^* - 1, 2, 4, 6\}$	2	3

Table 14: Vertex labels of $K_p, p \geq 7$, when $n = \lfloor \frac{q}{2} \rfloor - 5$.

Lemma 13. *A sigraph on K_4 is graceful if and only if the number of negative edges is not three.*

Proof. For $q = 6$, we have to assign the labels to the vertices of K_4 from the set $\{0, 1, \ldots, q^*\}$. Then the following four cases arise:

For $n = 0$, the sigraph on K_4 is the graph K_4 which is known to be graceful (see [7]).

For $n = 1$, label the vertices of K_4 as $\{0, 1, 2, 5\}$ or $\{5, 4, 3, 0\}$ where the ends of the negative edge are labeled as $0, 1$ or $4, 5$.

For $n = 2$, label the vertices of K_4 as $\{0, 1, 2, 4\}$ or $\{4, 3, 1, 0\}$ as shown in Figure 3.

In each of these cases, it is easy to check that the given labeling of the vertices of K_4 is a graceful labeling of the corresponding sigraph on it.

For $n = 3$, $q = 6$ is even and there do not exist two distinct pairs of numbers from the set $\{0, 1, 2, 3\}$ such that their absolute difference is equal to 3. Hence, in this case no sigraph on K_4 can be graceful. □

Lemma 14. *A sigraph S on K_5 is graceful if and only if S satisfies one of the following statements:*

(i) the number n of negative edges in S is 1*,*

(ii) $n = 3$ and the three negative edges in S are not incident at the same vertex,

Vertex labels	Edge labels	Rep'n
$\{0, q^*, q^*-1, q^*-2, q^*-3\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, q^*-6, q^*-7\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, q^*-6, 7\}$	q^*-9	2
$\{0, q^*, q^*-1, q^*-2, 3, 6, q^*-9, q^*-10\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, 6, q^*-9, 10\}$	q^*-12	2
$\{0, q^*, q^*-1, q^*-2, 3, 6, 9\}$	3	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, q^*-8\}$	4	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 8, q^*-10\}$	q^*-12	2
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 8, 10, q^*-13\}$	8	2
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 8, 10, 13, q^*-15, q^*-16, 16\}$	1	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 8, 10, 13, 15\}$	2	3
$\{0, q^*, q^*-1, 2, q^*-4, 5\}$	q^*-6	2
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, q^*-12\}$	6	3
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, 12, q-13\}$	6	3
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, 12, 13\}$	1	3
$\{0, q^*, q^*-1, 2, 4, q^*-6, 7\}$	q^*-8	2
$\{0, q^*, q^*-1, 2, 4, 6\}$	2	3

Table 15: Vertex labels of $K_p, p \geq 7$, when $n = \lfloor \frac{q}{2} \rfloor - 6$.

or $\eta(S)$ satisfies similar conditions with n replaced by m, the number of positive edges in S.

Proof. By Theorem 1 and Lemma 5, it follows that if a sigraph S on K_5 is graceful then the number of negative edges in it is odd. In this case, we have to assign the labels to the vertices of K_5 from the set $\{0, 1, \ldots, q^*\}$. Then the following four cases arise:

For $n = 1$, label the vertices of K_5 as $\{0, 1, 2, 6, 9\}$ or $\{9, 8, 7, 3, 0\}$, where 0 or 8 and 1 or 9 are assigned to the end vertices of the negative edge.

For $n = 3$, the possible labels of the vertices of K_5 are from one of the sets $\{0, 1, 2, 4, 7\}$ and $\{7, 6, 5, 3, 0\}$.

In each of these cases, it is easy to check that the given labeling of the vertices of K_5 is a graceful labeling as shown in Figure 4.

In case when $n = 3$ and the three negative edges are incident at the same vertex x, by using Lemma 6 it can be shown that possible R_f are $\{0, 1, 2, 4, 7\}$ and $\{0, 3, 5, 6, 7\}$. It can be easily checked that these are not graceful labelings of sigraphs on K_5. In all other cases, the sigraphs on K_5 are graceful as shown in Figure 4.

As the number of negative edges $n = 5$ and $q = 10$ is even, there do not exist two distinct pairs of numbers from $\{0, 1, 2, 3, 4, 5\}$ such that their absolute difference are both equal to 5. Hence, in this case no sigraph on K_5 can be graceful.

Hence, by an observation made in the very beginning of this section, $\eta(S)$ exhausts the possibilities when $n \in \{7, 9\}$. Thus, the proof is seen to be complete. □

Vertex labels	Edge labels	Rep'n
$\{0, q^*, q^*-1, q^*-2, q^*-3\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, q^*-6, q^*-7\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, q^*-6, 7\}$	q^*-9	2
$\{0, q^*, q^*-1, q^*-2, 3, 6, q^*-9, q^*-10\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, 6, q^*-9, 10\}$	q^*-12	2
$\{0, q^*, q^*-1, q^*-2, 3, 6, 9\}$	3	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, q^*-8\}$	4	4
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 8, q^*-10\}$	q^*-12	2
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 8, 10, q^*-15, q^*-16, 16\}$	1	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 8, 10, 15, q^*-16, q^*-17, 17\}$	1	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 8, 10, 15, 16\}$	8	3
$\{0, q^*, q^*-1, 2, q^*-4, 5\}$	q^*-6	2
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, q^*-12\}$	6	3
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, 12, q^*-14, q^*-15, 15\}$	1	3
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, 12, 14\}$	2	3
$\{0, q^*, q^*-1, 2, 4, q^*-6, 7\}$	q^*-8	2
$\{0, q^*, q^*-1, 2, 4, 6\}$	2	3

Table 16: Vertex labels of $K_p, p \geq 7$, when $n = \lfloor \frac{q}{2} \rfloor - 7$.

Using Lemmas 6, 12, 13, 14 and Theorem 11, we get Theorem 4.

3 Combinatorial Result

The following combinatorial result follows from the above results:

Theorem 15. *There is no subset $S \subseteq \{0, 1, \ldots, \binom{p}{2} - r\}$ with $|S| = p \geq 6$ such that each element of the set $\{1, 2, \ldots, r\}$ repeats exactly twice and each element of its complementary set $\{r+1, r+2, \ldots, \binom{p}{2}\}$ repeats exactly once as an absolute difference of the pairs of distinct elements of S.*

For $p = 5$, there are exactly two such subsets when $r = 3$ or 7, namely $\{0, 1, 2, 4, 7\}$, $\{7, 6, 5, 3, 0\}$, and when $r = 1$ or 9, the two such subsets are $\{0, 1, 2, 6, 9\}, \{9, 8, 7, 3, 0\}$, and none when $r = 0, 2, 4, 5, 6, 8, 10$.

For $p = 4$, there are exactly two such subsets: namely, when $r = 0$ or 6, which are $\{0, 1, 4, 6\}$ and $\{6, 5, 2, 0\}$. When $r = 1$ or 5 the two such subsets are $\{0, 1, 2, 5\}, \{5, 4, 3, 0\}$. When $r = 2$ or 4 the two such subsets are $\{0, 1, 2, 4\}, \{4, 3, 1, 0\}$, and none exists when $r = 3$.

For $k \geq 3$, a characterization of k-hypergraceful complete graphs appears quite challenging.

Vertex labels	Edge labels	Rep'n
$\{0, q^*, q^*-1, q^*-2, q^*-3\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, q^*-6, q^*-7\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, q^*-6, 7\}$	q^*-9	2
$\{0, q^*, q^*-1, q^*-2, 3, 6, q^*-9, q^*-10\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, 6, q^*-9, 10\}$	q^*-12	2
$\{0, q^*, q^*-1, q^*-2, 3, 6, 9\}$	3	3
$\{o, q^*, q^*-1, 2, q^*-4, q^*-5, q^*-8\}$	4	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 8, q^*-10\}$	q^*-12	2
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 8, 10, 15, q^*-16, q^*-17, 17\}$	1	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 8, 10, 16, \}$	8	3
$\{0, q^*, q^*-1, 2, q^*-4, 5\}$	q^*-6	2
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, q^*-12\}$	6	3
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, 12, q^*-14, q^*-15\}$	1	3
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, 12, q^*-14, 15\}$	q^*-16	2
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, 12, 14\}$	2	3
$\{0, q^*, q^*-1, 2, 4, q^*-6, 7\}$	q^*-8	2
$\{0, q^*, q^*-1, 2, 4, 6\}$	2	3

Table 17: Vertex labels of $K_p, p \geq 7$, when $n = \lfloor \frac{q}{2} \rfloor - 8$.

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Vertex labels	Edge labels	Rep'n
$\{0, q^*, q^*-1, q^*-2, q^*-3\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, q^*-6, q^*-7\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, q^*-6, 7\}$	q^*-9	2
$\{0, q^*, q^*-1, q^*-2, 3, 6, q^*-9, q^*-10\}$	1	3
$\{0, q^*, q^*-1, q^*-2, 3, 6, q^*-9, 10\}$	q^*-12	2
$\{0, q^*, q^*-1, q^*-2, 3, 6, 9\}$	3	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, q^*-8\}$	4	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 8, q^*-10\}$	q^*-12	2
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 8, 10, q^*-16, q^*-17\}$	1	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 8, 10, q^*-16, 17\}$	q^*-18	3
$\{0, q^*, q^*-1, 2, q^*-4, q^*-5, 8, 10, 16, \}$	8	3
$\{0, q^*, q^*-1, 2, q^*-4, 5\}$	q^*-6	2
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, q^*-12\}$	6	3
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, 12, q^*-14, q^*-15\}$	1	3
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, 12, q^*-14, 15\}$	q^*-16	3
$\{0, q^*, q^*-1, 2, 4, q^*-6, q^*-7, 12, 14\}$	2	3
$\{0, q^*, q^*-1, 2, 4, q^*-6, 7\}$	q^*-8	2
$\{0, q^*, q^*-1, 2, 4, 6\}$	2	3

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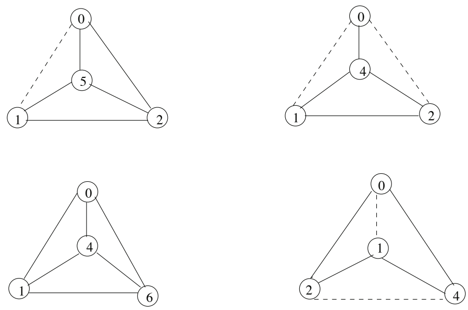

Figure 3: Graceful sigraphs on K_4.

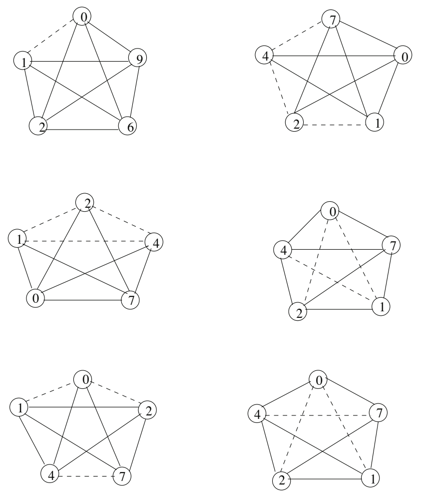

Figure 4: Illustration of graceful sigraphs on K_5.

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