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Construction of variety competition cyclic balanced designs on a square lattice

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Abstract

A class of cyclic balanced designs with ten test-hills is introduced; these designs can be repeated cyclically any number of times up to the requirement of the experimenter and subject to the availability of experimental material. The designs are balanced with respect to first order nearest neighbours and there are 26 isomorphism classes of such designs. The designs are suitable for de Wit replacement series, mixed cropping trials or two component plant breeding trials, etc.

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1 Introduction

Different varieties, when sown together, may show an increase or decrease in yield depending upon the environmental conditions such as weather, plant density, mineral resources etc. The basic plant biology proposes that plant-plant interactions are inherently local in nature. Therefore, a plant's growth depends upon the local conditions which are of much greater worth to plants than to animals; see Stoll and Weiner [13]. Intercropping is receiving increasing attention because it offers potential advantages for resource utilization, decreased inputs and increased sustainability in crop production (Andersen et al. [1]).

It is important to differentiate intercropping and competition because they differ on the basis of objectives behind them. In intercropping, the objective is to find the best technique to grow in mixtures. However, in competition, mechanism of competition is investigated, i.e. how genotypes or species in mixture tolerate the other or provide competition benefit (Mead and Riley [8]). If the lack of resources limits the growth of an individual then that individual has suffered from competition (Stoll and Weiner [13]). According to Bulson et al. [4], components of a mixture use limiting resources more efficiently than pure stands.

Better biological efficiency of mixtures compared with monocultures may result from differences in growing cycles and root and root architecture (Wilson [19], Ponce [10], Aufhammer et al. [2], Vandermeer [16]). This fact has been observed in small grains when one component of a mixture is less at risk to lodging and provides support for the second component (Sobkowicz [11]). Space between plants is the main building block in studying plant communities. Its role is fundamental and cannot be ignored (Hutching [7], and Crawley and May [5]). It is worth differentiating between behaviour of a plant to collect its share from resources and the reaction it shows when its neighbours share resources (Goldberg [6] and Tremmel and Bazzaz [15]). Sobkowicz and Tendziagolska [12] assessed the productivity of mixtures of oats and wheat, and compared two different approaches used in plant competition studies *vis-a-vis* replacement designs and additive designs.

Array θ							Array η						
–	0	0	0	0	0	–	–	1	1	1	1	1	–
0	0	0	1	0	1	1	0	0	1	0	1	1	1
–	0	0	0	0	0	–	–	1	1	1	1	1	–

Table 1: Two arrays θ and η with their respective values

Boffey and Veevers [3] constructed balanced designs on a triangular lattice. However, Veevers and Zafar-Yab [18] referred in Street and Street [14], introduced three complete families of balanced designs in equal proportion for two varieties planted on a square lattice. Langton [9] used these designs and concluded that mixture performance is better than monoculture. In this paper we investigate balanced designs on a square lattice which are cyclical in nature. In arrangement of plants, the place of plantation is called a hill plot or a *hill.* In any arrangement of hills, all except the border hills are *test-hills.* Consider a competition experiment utilizing a fifty-fifty mixture of two varieties planted on a square lattice. In such an arrangement a plant of one variety could be immediately surrounded by $0, \ldots, 4$ plants of the other variety thus providing five levels of competition. It would be desirable to have such a design (Θ) so that all levels of competition for both varieties appear equally often. A balanced design Θ, obtained by taking reflection in the minor axis (mirror image) or by taking the complement (interchanging all X's and Y's), or a combination of these operations on Θ, is said to have the same *isomorphism class* as that of the basic balanced design Θ.

	(a) Incidence matrix of design θ for variety 0								Column permutation applying $7-k+1$ of incidence matrix **(a)**						
	columns								columns						
rows	1	2	3	4	5	6	7	rows	1	2	3	4	5	6	7
1	–	1	1	1	1	1	–	1	–	1	1	1	1	1	–
2	1	1	1	0	1	0	0	2	0	0	1	0	1	1	1
3	–	1	1	1	1	1	–	3	–	1	1	1	1	1	–

Table 2: Incidence matrix of design θ and its column permutation

2 Isomorphism in Competition Designs

In competition designs, two designs θ and η are said to be isomorphic if the incidence matrix of design θ (for the same or opposite variety) can be obtained either by row or column permutation of the incidence matrix of design η (for the same or opposite variety), and vice versa.

Incidence matrix for variety 1 of design η						
–	1	1	1	1	1	–
0	0	1	0	1	1	1
–	1	1	1	1	1	–

Table 3: Column permuted incidence matrix (a) matches with the opposite variety incidence matrix of design η

Example 2.1

Consider two arrays θ and η presented in Table 1, in which the two varieties are denoted by 0 and 1. Both the arrays are apparently different but the incidence matrix of column permutation, where $k = 1 \dots 7$, in array θ (for variety 0) matches with the incidence matrix for variety 1 of array η as shown in Table 3 . Hence both arrays θ and η are isomorphic.

3 Construction of Cyclic Balanced Designs consisting of ten Test-hills

The two varieties are denoted by X and Y. In the construction of cyclic balanced designs, considering 34-hill plots are arranged in three rows such that the first and the last rows consist of 10 similar hills and the second row consists of 12-hills. The layout of hills is shown in Figure 1(a), in which hills are shown as star (*), while the boxed hills are test-hills. To obtain a cyclic balanced design the first hill and the eleventh hill of the middle row must be similar. In a single replication of such a design the test-hills need to be arranged in such a way that every next cycle results in a replication of the same balanced design. To obtain a balanced design consisting of only ten test hills there must be equal numbers of hills for both varieties X and Y. Letter strings XXX and YYY must be a part of possible arrangements to obtain zero opposite neighbours for both varieties X and Y respectively. Similarly, letter strings YXY and XYX must be a part of possible arrangements to obtain four opposite neighbours for both varieties X and Y respectively. Since the designs are cyclic in nature, therefore, the cycle of 10 test-hills can be started from any arbitrary hill. There are three possible isomorphic ways of arranging these ten test-hills in a cycle shown in Figure 1 ((b),(c) and (d)).

3.1 Designs in the isomorphism class of Figure 1(b)

Test-hills shown in Figure 1(b) when arranged in a line give a string of $X's$ and $Y's$ and is given below with the feasible permutations for balance shown in the subscript. For example $X_{(1,3)}$ indicates that X has either one or three opposite neighbours. Among feasible permutations only 10 are isomorphic and are shown in Figure 2(b).

$$Y \quad [X_{(1,3)} \quad X_{(0,2)} \quad X_{(0,2)} \quad X_{(1,3)} \quad Y_4 \quad X_4 \quad Y_{(1,3)} \quad Y_{(0,2)} \quad Y_{(0,2)} \quad Y_{(1,3)}] \quad X$$

The subscripts of string of $X's$ and $Y's$ are allotted symbols after some re-arrangements and are presented in Table 4.

3.1.1 Isomorphism classes in designs of Figure 1(b) and Figure 1(c)

Four types of isomorphism can occur in each isomorphism class; they are: (a) basic balanced design itself; (b) rotation of (a); (c) complementation of (a); and (d) a combination of (b) and (c). It is desirable to know what changes can be observed

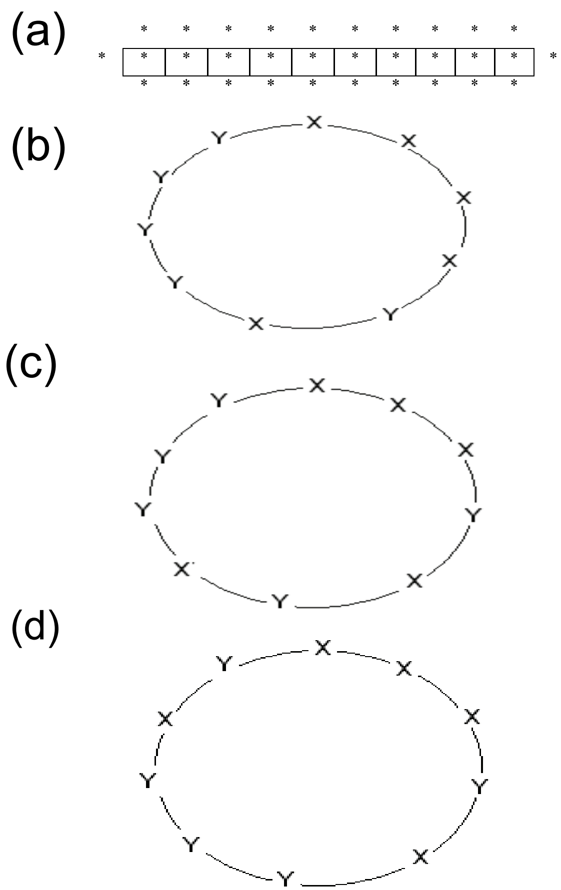
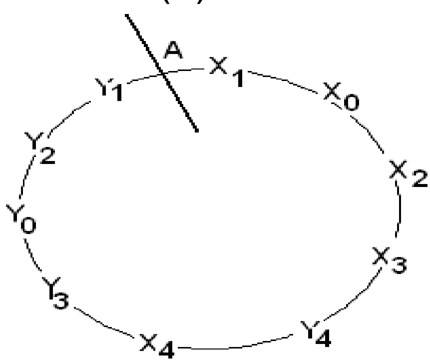


Figure 1: (a) Boxed hills are test-hills and the border hills are non-testable hills. Three possible ways of arranging ten test-hills are: (b) showing four consecutive hills of both varieties together, (c) showing three consecutive hills of both varieties together and (d) showing three consecutive hills of both varieties in isolation.

(a)



(b)

Group (i)Y [	X	X	X	X	Y	X	Y	Y	Y	Y]	X
	1	0	2	3		4					
					4		3	2	0	1	
					4		3	0	2	1	
					4		1	0	2	3	
					4		1	2	0	3	
Group (ii) Y [	X	X	X	X	Y	X	Y	Y	Y	Y]	X
	1	2	0	3		4					
					4		3	0	2	1	
					4		1	2	0	3	
					4		1	0	2	3	
Group (iii) Y [X		X	X	X	Y	X	Y	Y	Y	Y]	X
	3	0	2	1		4					
					4		1	2	0	3	
					4		1	0	2	3	
Group (iv) Y [X		X	X	X	Y	X	Y	Y	Y	Y]	X
	3	2	0	1		4					
					4		1	0	2	3	

Figure 2: (a) Ten test-hills are lying on a ring and (b) Ten basic designs — selected one from each isomorphic class. These are classified in 4 groups w.r.t. the feasible permutations of numbers of neighbours of X and Y.

number	Symbols	Feasible permutations of opposite neighbours of X	Symbols	Feasible permutations of opposite neighbours of Y
1	U_x	10234	$\acute{U}_y$	43201
2	V_x	12034	$\acute{V}_y$	43021
3	$\acute{V}_x$	30214	V_y	41203
4	$\acute{U}_x$	32014	U_y	41023

Table 4: Feasible permutations of numbers of opposite neighbours of X and those of Y which lead to balanced designs.

by applying operations (b), (c) and (d) on (a). Sixteen balanced designs can be realized by all possible combinations of arrangements of symbols allotted to possible permutations of numbers of opposite neighbours of X and Y. But these are not unique. Rotation of symbols U_x and V_x are denoted by $\acute{U}_x$ and $\acute{V}_x$ respectively. Rotation of a pair of symbols, such as U_xV_y, is $(U_xV_y)'$ which is isomorphic to $\acute{V}_y\acute{U}_x$.

Example 3.1

Consider a design with $U_x = 10234$ and $\acute{V}_y = 43021$; the operations $(U_x\acute{V}_y)' = (\acute{V}_y)'\,\acute{U}_x = V_y\,\acute{U}_x$, where $(\acute{V}_y)' = V_y = 41203$ and $\acute{U}_x = 12034$. In other words when the pair of symbols $U_x\acute{V}_y$ is considered then there is no need to consider the pair $V_y\,\acute{U}_x$, because one design from this isomorphism class has already been constructed. Since complement of $V_y\,\acute{U}_x$ is $\acute{U}_x\,V_y$. This procedure is explained below with the help of Figure 2 (a).

Suppose the varieties are layed on a ring as displayed in Figure 2 (a). Let us make a cut at A. Straighten the ring to form a layout of varieties as under. Reading from left to right makes clear the arrangement of the first permutation of X and the second possible permutation of Y, *i.e.* $U_x\,V_y'$ in Table 4.

$$Y \quad [X_1 \quad X_0 \quad X_2 \quad X_3 \quad Y_4 \quad X_4 \quad Y_3 \quad Y_0 \quad Y_2 \quad Y_1] \quad \mathrm{X} \qquad (3.1.a)$$

However, reading from right to left, the arrangement results in the fourth possible value of X and the third possible permutation of Y, *i.e.* $V_y\,U_x'$ (See Figure 2 (a))as shown below:

$$X \quad [Y_1 \quad Y_2 \quad Y_0 \quad Y_3 \quad X_4 \quad Y_4 \quad X_3 \quad X_2 \quad X_0 \quad X_1] \quad \mathrm{Y} \qquad (3.1.b)$$

Rotation of (3.1.b) is $(V_y\acute{U}_x)' = (\acute{U}_x)'\,\acute{V}_y = U_x\,\acute{V}_y$, which is essentially (3.1.a); therefore, both these arrangements belong to the same isomorphism class. Only one of these needs to be retained. Out of 16 possible arrangements obtained by the combination of symbols of X and Y, there are only 10 isomorphism classes and these are shown in Figure 2 (b).

Number	Symbols	Feasible permutations of opposite neighbours of X	Symbols	Feasible permutations of opposite neighbours of Y
1	U_x	30142	$\acute{U}_y$	24103
2	V_x	30124	$\acute{V}_y$	42103
3	$\acute{V}_x$	10324	V_y	42301
4	$\acute{U}_x$	10342	U_y	24301

Table 5: Feasible permutations of numbers of opposite neighbours for X and Y which lead to balanced designs.

3.1.2 Construction of balanced designs.

Each of the 10 arrays given in Figure 2(b) is the central row of a design. The complete design is realized when suitable varieties are added in each of the first and the third row *i.e.* when only 10 hills for any of the first and third row are suitably assigned a variety X or Y. Consider the arrangement of Group (iv) in Figure 2 (b). Treat the digits shown below each X or Y (numbers of opposite neighbours) as their subscripts. For realizing 3 opposite neighbours of X at the left-most test hill, we must assign variety Y to hills above and below it. Similarly, for realizing 2 opposite neighbours of X at the next test hill we must assign variety Y to hills above and below it. Continuing this process, a full design is realized.

Binary digits from these designs can be obtained by replacing X and Y by 0 and 1 respectively. Veevers and Boffey [17] use octal representation of such designs to reduce the volume of tables and this is true for this paper. These designs are symmetric about the second row; therefore, octal representations of the first two rows of the designs are presented in Table 7 from $N = 1$ to $N = 10$.

3.2 Designs in the isomorphic class of Figure 1(c)

Test-hills shown in Figure 1 (c), when arranged in a line, give a string of X's and Y's; see below, with the feasible permutations for balance shown in the subscript.

$Y \quad [X_{(1,3)} \quad X_0 \quad X_{(1,3)} \quad Y_{(2,4)} \quad X_{(2,4)} \quad Y_{(2,4)} \quad X_{(2,4)} \quad Y_{(1,3)} \quad Y_0 \quad Y_{(1,3)}] \quad X$

These feasible permutations are presented in Table 5. Following the arguments of the Section 3.1, ten isomorphism classes from a total of 16 feasible permutations can be obtained. These designs are labeled from $N = 11$ to $N = 20$ in Table 7 in their octal representations.

3.3 Designs in the isomorphic class of Figure 1(d)

Test-hills shown in Figure 1 (d), when arranged in a line, give a string of X's and Y's; this is given below with the feasible permutations for balance shown in the subscript.

Symbols	$L_y = 23014$	$M_y = 21034$	$\acute{L}_y = 41032$	$\acute{M}_y = 43012$
$L_x = 30142$	P_{11}	P_{12}	P_{13}	P_{14}
$M_x = 10342$	P_{21}	P_{22}	P_{23}	P_{24}
$\acute{L}_x = 10324$	P_{31}	P_{32}	P_{33}	P_{34}
$\acute{M}_x = 30124$	P_{41}	P_{42}	P_{43}	P_{44}

Table 6: Feasible permutations of numbers of opposite neighbours of Y and X are shown in the first row and in the first column respectively.

Y $[X_{(1,3)}$ X_0 $X_{(1,3)}$ $Y_{(2,4)}$ $X_{(2,4)}$ $Y_{(1,3)}$ Y_0 $Y_{(1,3)}$ $X_{(2,4)}$ $Y_{(2,4)}]$ X

Symbols similar to the ones assigned in Table 5 are allotted to the feasible permutations of numbers of opposite neighbours of X and Y. Complement or rotation of L_x and $\acute{L}_x$ respectively are L_y and $\acute{L}_y$. The same is true for M_x and $\acute{M}_x$. Also let P_{ij}, where $i,j = 1, \ldots, 4$, represent the possible number of ith and jth permutation of numbers of opposite neighbours of X and Y, respectively, in Table 6. A unique property of these designs is that symbols P_{ij}'s, the test-hills, are symmetric in the sense that five consecutive test-hills are the complement of the other five.

3.3.1 Isomorphic classes in designs of Figure 1(d)

The complement of P_{32}, i.e. $(L'_x M_y)^- = L'_y M_x$, is equivalent to $M_x L'_y$, which is essentially P_{23}. Rotation of P_{32}, i.e. $(L'_x M_y)' = M'_y (L'_x)' = M'_y L_x$, is equivalent to $L_x M'_y$, which is essentially P_{41}. The complement of the rotation of P_{32}, i.e.

$$((L'_x M_y)')^- = (M'_y L_x)^- = M'_x L_y,$$

is equivalent to $L_y M'_x$, which is essentially P_{14}. Since designs P_{23}, P_{41} and P_{14} can be obtained by an operation on P_{32}, it follows that these four designs belong to the same isomorphism class and only one of these needs to be retained.

Investigations of remaining possibilities result in four isomorphic classes consisting of pairs of possibilities, namely; (P_{31}, P_{13}), (P_{33}, P_{11}), (P_{22}, P_{44}) and (P_{24}, P_{42}) and one isomorphic class that consists of four possibilities namely (P_{34}, P_{21}, P_{43}, P_{12}). As a result there are six isomorphic classes from explicit considerations. After retaining one design from each isomorphism class, the list of designs is: P_{31}, P_{32}, P_{34}, P_{33}, P_{22} and P_{24}. These designs are labeled from $N = 21$ to $N = 26$ in Table 7 in their octal representations.

A three row elongated design of any length can be obtained by horizontally repeating the cycles the desired number of times. However, none of these designs build into larger designs vertically.

N	Balanced Design		N	Balanced Design		N	Balanced Design	
1	0323	4136	2	0325	4136	3	0332	4136
4	0334	4136	5	0525	4136	6	0532	4136
7	0534	4136	8	1232	4136	9	1234	4136
10	1434	4136	11	1143	4256	12	1146	4256
13	1066	4256	14	1063	4256	15	1113	4256
16	1036	4256	17	1033	4256	18	0313	4256
19	0233	4256	20	0363	4256	21	0316	4272
22	0332	4272	23	0217	4272	24	0233	4272
25	0370	4272	26	0255	4272			

Table 7: N identifies each isomorphism class

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