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Independence and global offensive alliance in graphs

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Abstract

Let G be a simple graph with vertex set $V(G)$. A non-empty set $S \subseteq V(G)$ is a global strong offensive alliance if for every vertex v in $V(G)-S$, a strict majority of its closed neighborhood is in S. The global strong offensive alliance number $\gamma_{\hat{o}}(G)$ is the minimum cardinality of a global strong offensive alliance of G. We show that if G is a connected bipartite graph of order at least three, then $\gamma_{\hat{o}}(G) \leq \frac{3}{2}\alpha(G)$ and if G is a connected unicyclic graph, then $\gamma_{\hat{o}}(G) \leq \frac{3}{2}\alpha(G)+1$, where $\alpha(G)$ is the independence number of G. Moreover, we characterize extremal bipartite graphs achieving equality in the first upper bound.

1 Introduction

Let G be a simple graph with vertex set $V(G)$. The *order* of G is $|G| = |V(G)|$. The *neighborhood* $N_G(v) = N(v)$ of a vertex $v \in V(G)$ consists of the vertices adjacent with v, and $N_G[v] = N[v] = N(v) \cup \{v\}$ is the *closed neighborhood*. A vertex of degree one is called a *leaf* and its neighbor a *support vertex*. We denote by $L(G)$ and $S(G)$ the set of leaves and support vertices of a graph G, respectively.

For a positive integer k, a set of vertices D in a graph G is said to be a k-*dominating set* if each vertex of G not contained in D has at least k neighbors in D. The order of a smallest k-dominating set of G is called the k-*domination number*, and it is denoted by $\gamma_k(G)$. By definition, a dominating set coincides with a 1-dominating set, and $\gamma_1(G)$ is the domination number $\gamma(G)$ of G.

A set $S \subseteq V(G)$ is called a *global offensive alliance* of G if $|N[v] \cap S| \geq |N[v] - S|$ for every vertex $v \in V(G) - S$. The *global offensive number* $\gamma_o(G)$ is the minimum cardinality of a global offensive alliance of G. A set $S \subseteq V(G)$ is a *global strong offensive alliance* of G if $|N[v] \cap S| > |N[v] - S|$ for every vertex $v \in V(G) - S$. The *global strong offensive number* $\gamma_{\hat{o}}(G)$ is the minimum cardinality of a global strong offensive alliance of G. If S is a global offensive alliance or a global strong offensive alliance of G and $|S| = \gamma_o(G)$ or $|S| = \gamma_{\hat{o}}(G)$, then we say that S is a $\gamma_o(G)$-set or a $\gamma_{\hat{o}}(G)$-set. Note that a global offensive alliance as well as a global strong offensive alliance of G is a dominating set of G. Alliances in graphs were introduced by Hedetniemi, Hedetniemi and Kristiansen in [5].

A subset $I \subseteq V(G)$ of the vertex set of a graph G is called *independent* if no pair of vertices in I is adjacent. The number $\alpha(G)$ represents the cardinality of a maximum independent set of G. A *matching* in a graph G is a subset of pairwise non-incident edges. A matching is said to be *perfect* if it covers all vertices of G.

For each vertex x in a graph G, we introduce a new vertex x' and join x and x' by an edge. The resulting graph is called the *corona* of G. A graph is said to be a *corona graph* if it is the corona of some graph.

2 Main results

Theorem 1 *If G is a connected bipartite graph of order at least 3, then*

$$\gamma_{\hat{o}}(G) \leq \frac{3}{2}\alpha(G). \tag{1}$$

Equality holds in (1) if and only if G is a corona of a connected bipartite graph H with a bipartition (X, Y) such that $|X| = |Y|$ and $\gamma_o(H) = |H|/2$.

Proof. Let I be a maximum independent set of G. Since $|G| \geq 3$, we can assume, without loss of generality, that $L(G) \subseteq I$ and thus it follows that $|L(G)| \leq \alpha(G)$. Since G is bipartite, evidently $2\alpha(G) \geq |G|$.

Let A and B be the partition sets of G. Define $A_1 = A - L(G)$ and $B_1 = B - L(G)$ and assume, without loss of generality, that $|A_1| \leq |B_1|$. Then $|A_1| \leq \frac{|G|-|L(G)|}{2}$. Since every vertex in B_1 has at least two neighbors in $A_1 \cup L(G)$, we see that the latter is a global strong offensive alliance of G and hence

$$\gamma_{\hat{o}}(G) \leq |A_1 \cup L(G)| \leq \frac{|G| - |L(G)|}{2} + |L(G)| = \frac{|G| + |L(G)|}{2}.$$

Combining this inequality with $|L(G)| \leq \alpha(G)$ and $|G| \leq 2\alpha(G)$, we obtain the desired bound

$$\gamma_{\hat{o}}(G) \leq \frac{|G| + |L(G)|}{2} \leq \frac{2\alpha(G) + \alpha(G)}{2} = \frac{3}{2}\alpha(G).$$

Assume now that $\gamma_{\hat{o}}(G) = \frac{3}{2}\alpha(G)$. Then it follows from the last inequality chain that $|G| = 2\alpha(G)$, $|L(G)| = \alpha(G)$ and $\gamma_{\hat{o}}(G) = \frac{|G|+|L(G)|}{2}$. The facts that $|L(G)| =$

$\alpha(G)$ and $|G| = 2\alpha(G) = 2|L(G)|$ show that G is a corona of some connected bipartite graph H with the bipartition (X, Y). The hypothesis $\gamma_{\hat{o}}(G) = \frac{3}{2}\alpha(G)$ implies that $X \cup L(G)$ and $Y \cup L(G)$ are two $\gamma_{\hat{o}}(G)$-sets and so we deduce that $|X| = |Y|$. Now let D be an arbitrary $\gamma_{\hat{o}}(G)$-set. Since $L(G) \subseteq D$ and every vertex of H has exactly one neighbor in $L(G)$, we deduce that $D - L(G)$ is a global offensive alliance of H and hence

$$\gamma_o(H) \leq \frac{3}{2}\alpha(G) - |L(G)| = |L(G)|/2 = |H|/2.$$

Now if $\gamma_o(H) < |H|/2$, then every $\gamma_o(H)$-set S can be extended to a global strong offensive alliance of G by adding the set $L(G)$ which leads to a contradiction. This yields to the desired equality $\gamma_o(G) = \frac{|H|}{2}$.

Conversely, let G be a corona of a connected bipartite graph H such that $\gamma_o(H) = |H|/2$. If S is any $\gamma_o(H)$-set, then $S \cup L(G)$ is a global strong offensive alliance of G and so $\gamma_{\hat{o}}(G) \leq |H|/2 + |L(G)| = \frac{3}{2}|L(G)| = \frac{3}{2}\alpha(G)$. If D is any $\gamma_{\hat{o}}(G)$-set with $|D| < \frac{3}{2}\alpha(G)$, then $D - L(G)$ is a global offensive alliance of H of size less than $|H|/2$, a contradiction. Thus $|D| = \frac{3}{2}\alpha(G)$, and the proof of Theorem 1 is complete. □

We note that a constructive characterization of trees T with $\gamma_o(T) = (|T| - |L(T)| + |S(T)|)/2$ has been given in [2] as follows: let $\mathcal{G}$ be the family of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ of trees, where T_1 is the path P_2, $T = T_k$, and if $k \geq 2$, T_{i+1} is obtained recursively from T_i by one of the three operations listed below. A support vertex is called *strong* if it is adjacent to at least two leaves.

- Operation $\mathcal{O}_1$: Add a vertex attached by an edge to an arbitrary support vertex of T_i.
- Operation $\mathcal{O}_2$: Add a path $P_2 = xy$ and join x by an edge to a support vertex z of T_i.
- Operation $\mathcal{O}_3$: Add $p \geq 1$ path(s) P_2 and join a vertex of each path by an edge to a leaf u of T_i adjacent to a support vertex w that is not a strong one with the condition that $p = 1$ if w has degree at least three.

Clearly trees T with $\gamma_o(T) = |T|/2$ contain no strong support vertex. Since Operation $\mathcal{O}_1$ produces strong support vertices we define $\mathcal{G}_1$ as the subfamily of $\mathcal{G}$ consisting of trees obtained from T_1 by performing Operations $\mathcal{O}_2$ and $\mathcal{O}_3$. Therefore extremal trees G achieving equality in (1) are precisely coronas of nontrivial trees $T \in \mathcal{G}_1$.

Proposition 2 *If G is a nontrivial connected bipartite graph with $\gamma_o(G) = |G|/2$, then G admits a perfect matching.*

Proof. Assume that G does not admit a perfect matching, and let I be a maximum independent set of G. Using the well-known theorems of König [6] and Gallai [4], we therefore deduce that $|I| > |G|/2$. Since G is a nontrivial connected bipartite graph, we observe that $|N[v] \cap (V(G) - I)| \geq 1 = |N[v] - (V(G) - I)|$ for every vertex $v \in I$. Thus $V(G) - I$ is a global offensive alliance of G and hence $\gamma_o(G) < |G|/2$. This contradiction to our hypothesis completes the proof. □

The converse of Proposition 2 is not true and can be seen by the following example. Let T be a tree obtained by adding an edge between the center vertices of two paths P_5. Clearly, the tree T has a perfect matching but the set $S(T)$ of support vertices is a global offensive alliance of T of size $4 < |T|/2$.

If S is any $\gamma_{\hat{o}}(G)$-set, then every vertex of $V(G) - S$ has at least two neighbors in S. Thus S is a 2-dominating set of G, and we obtain $\gamma_2(G) \leq |S| = \gamma_{\hat{o}}(G)$. Using this fact, Theorem 1 leads immediately to the following two corollaries.

Corollary 3 (Fujisawa, Hansberg, Kubo, Saito, Sugita, Volkmann [3] 2008) *If G is a connected bipartite graph of order at least 3, then*

$$\gamma_2(G) \leq \frac{3}{2}\alpha(G). \tag{2}$$

Corollary 4 (Blidia, Chellali, Favaron [1] 2005) *If T is a tree of order at least 3, then $\gamma_2(T) \leq \frac{3}{2}\alpha(T)$.*

Theorem 5 *If G is a connected unicyclic graph, then $\gamma_o(G) \leq (|G| + 1)/2$ and $\alpha(G) \geq \lfloor |G| /2 \rfloor$.*

Proof. Let C be the unique cycle of G. If C is even, then G is a bipartite graph and the result is valid. Thus assume that C is odd. If $G = C$, then $\gamma_o(G) = (|G| + 1)/2$, and $\alpha(G) = \lfloor |G| /2 \rfloor$. So we assume that G contains a vertex of degree at least three. Let A, B with $|A| \leq |B|$ be the unique bipartition of $G - e$, where e is any edge of C. Clearly, e has its endvertices, say u, v both in A or in B. If $|A| = |B|$, then one of A or B is a global offensive alliance of G and the other is independent and the result holds. Thus assume that $|A| < |B|$. If $u, v \in A$, then A is a global offensive alliance of G, and hence $\gamma_o(G) \leq |A| < (|G| + 1)/2$, and $\alpha(G) \geq |B| \geq \lfloor |G| /2 \rfloor$. If $u, v \in B$, then $B - \{u\}$ is independent and $A \cup \{u\}$ is a global offensive alliance of G. So $\alpha(G) \geq |B| - 1 \geq \lfloor |G| /2 \rfloor$ and $\gamma_o(G) \leq |A| + 1 \leq (|G| + 1)/2$. □

Corollary 6 *If G is a connected unicyclic graph, then $\alpha(G) \geq \gamma_o(G) - 1$.*

If C is a cycle of odd length, then $\alpha(C) = \gamma_o(C) - 1 = \lfloor |C|/2 \rfloor$, and therefore Corollary 6 is best possible.

Theorem 7 *If G is a connected unicyclic graph, then*

$$\gamma_{\hat{o}}(G) \leq \frac{|G| + \alpha(G) + 1}{2}.$$

Proof. Let I be a maximum independent set of G that contains all leaves of G. Then every vertex of $V(G) - I$ has degree at least two in G. Let A be the set of isolated vertices in the induced subgraph $G[V(G)-I]$, and define the set $B = V(G)-(I\cup A)$. It follows that the induced subgraph $G[B]$ has no isolated vertices, and it contains at most one cycle. This means that the components of $G[B]$ consist of non-trivial bipartite graphs and at most one unicyclic graph. Now let S be a minimum global offensive alliance of $G[B]$. Since for every non-trivial bipartite graph H, the inequality $\gamma_o(H) \le |H|/2$ is immediate, Theorem 5 implies that

$$|S| \le \frac{|B|+1}{2} \le \frac{|V(G)-I|+1}{2}.$$

In addition, we see that the set $I \cup S$ is a global strong offensive alliance of G and so we receive the desired result as follows:

$$\gamma_{\hat{o}}(G) \le |I| + |S| \le |I| + \frac{|V(G)-I|+1}{2} = \frac{|G|+\alpha(G)+1}{2}. \ \square$$

Corollary 8 *If G is a connected unicyclic graph, then $\gamma_{\hat{o}}(G) \le \frac{3}{2}\alpha(G)+1$.*

Proof. Theorem 5 implies $2\alpha(G) \ge |G|-1$, and hence we obtain by Theorem 7 that $\gamma_{\hat{o}}(G) \le (|G|+\alpha(G)+1)/2 \le (3\alpha(G)+2)/2$. $\square$

If G is the corona of a cycle C of odd length, then $\gamma_{\hat{o}}(G) = (3|C|+1)/2$ and $\alpha(G) = |C|$. Thus it follows that

$$\gamma_{\hat{o}}(G) = \frac{3|C|+1}{2} > \frac{3|C|}{2} = \frac{3\alpha(G)}{2},$$

and so the bound of Corollary 8 is best possible.

Proposition 9 *If G is a a graph with minimum $\delta \ge 1$, then*

$$\gamma_{\hat{o}}(G) + \alpha(G) \le |G| + |L(G)|.$$

Proof. Let I be a maximum independent set that contains all leaves of G. Then every vertex of $I - L(G)$ has at least two neighbors in $V(G)-I$, and so $(V(G)-I)\cup L(G)$ is a global strong offensive alliance of G. Hence $\gamma_{\hat{o}}(G) \le |(V(G)-I)\cup L(G)|$, implying $\gamma_{\hat{o}}(G) + \alpha(G) \le |G| + |L(G)|$. $\square$

The next example will show that Proposition 9 is best possible.

Example 10 Let H_i consists of the vertices u_i, v_i, w_i, z_i and the edges u_iv_i, v_iw_i, w_iu_i, w_iz_i. If $k \ge 1$ is an integer, then define the graph G_k as the disjoint union $H_1 \cup H_2 \cup \ldots \cup H_k$ together with the path $w_1w_2\ldots w_k$. See Figure 1 for an example of G_3. We observe that G_k is a connected graph of order $4k$ such that

$$\gamma_{\hat{o}}(G_k) + \alpha(G_k) = 3k + 2k = 4k + k = |G_k| + |L(G_k)|.$$

Corollary 11 *If G is a graph with minimum degree $\delta \ge 2$, then $\gamma_{\hat{o}}(G) + \alpha(G) \le |G|$.*

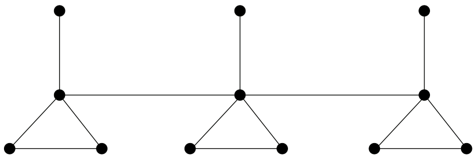

Figure 1: The graph G_3.

The following example will demonstrate that Corollary 11 is best possible.

Example 12 Let $k \geq 1$ be an integer, and let R_k consists of the vertices u, v, $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_k$ such that the vertex set $\{u, v, x_i, y_i\}$ induce a complete graph for $1 \leq i \leq k$. See Figure 2 for an example of R_3. We note that R_k is a connected graph of order $2k+2$ such that

$$\gamma_{\hat{o}}(R_k) + \alpha(R_k) = (k+2) + k = 2k + 2 = |R_k|\,.$$

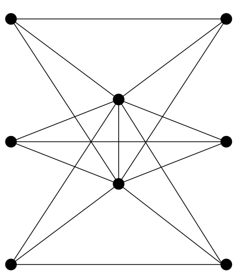

Figure 2: The graph R_3.

We finish by mentioning that Proposition 9 is not sharp for the class of nontrivial trees. Indeed if T is a nontrivial tree, then it is shown in [1] that $\alpha(T) \leq \frac{|T| + |L(T)| - 1}{2}$, and in [2] that $\gamma_{\hat{o}}(T) \leq \frac{|T| + |L(T)|}{2}$. Clearly then we have for nontrivial trees T, $\gamma_{\hat{o}}(T) + \alpha(T) < |T| + |L(T)|\,.$

References

[1] M. Blidia, M. Chellali and O. Favaron, Independence and 2-domination in trees, *Australas. J. Combin.* **33** (2005), 317–327.

[2] M. Chellali, Offensive alliances in bipartite graphs, *J. Combin. Math. Combin. Comput.* (to appear).

[3] J. Fujisawa, A. Hansberg, T. Kubo, A. Saito, M. Sugita and L. Volkmann, Independence and 2-domination in bipartite graphs, *Australas. J. Combin.* **40** (2008), 265–268.

[4] T. Gallai, Über extreme Punkt- und Kantenmengen, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **2** (1959), 133–138.

[5] S.M. Hedetniemi, S.T. Hedetniemi and P. Kristiansen, Alliances in graphs, *J. Combin. Math. Combin. Comput.* **48** (2004), 157–177.

[6] D. König, Graphs and matrices, *Mat. Fiz. Lapok* **38** (1931), 116–119. (Hungarian).

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